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Numerical Solution of the First-Order Hyperbolic Partial Differential Equation with Point-Wise Advance

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Abstract: ***In this paper, we construct an explicit numerical scheme based on Lax-Friedrichs finite difference approximation to find the numerical solution of first-order hyperbolic partial differential equation with point-wise advance. The differential equation involving point-wise delay and advance models the distribution of the time intervals between successive neuronal firings. We construct higher order numerical approximation and discuss their consistency, stability and convergence. Analysis shows that numerical scheme is conditionally stable, consistent and convergent in discrete*** L^{∞} ***norm. We also extend our method to the higher space dimensions. Some test examples are included to illustrate our approach. These examples verify the theoretical results and show the effect of point-wise advance on the solution.***

Keywords: hyperbolic partial differential equation, transport equation, point-wise advance, finite difference method, Lax-Friedrichs scheme**.**

1. Introduction

Hyperbolic partial differential-difference equations provide a tool to simulate several realistic physical and biological phenomena. Several biological phenomena can be modeled by time dependent first-order partial differential difference equations of hyperbolic type which contains point-wise advance or shifts in space.

We consider the following first-order hyperbolic partial differential equation having point-wise advance with an initial data u^0 on the domain $\Omega := (0, \mathrm{X})$. In general it reads

$$\begin{aligned} u_t + a u_x &= b u(x+\tau, t), && x \in \Omega,\ t > 0, \\ u(x,0) &= u^0(x), && x \in \overline{\Omega}, \\ u(0,t) &= \phi(t), && for\ a > 0, \\ u(s,t) &= \psi(s,t), && \forall s \in [\mathrm{X}, \mathrm{X}+\tau],\ for\ a < 0, \end{aligned} \tag{1}$$

where $a = a(x,t)$ and $b = b(x,t)$ are sufficiently smooth and bounded functions of x and t in the entire domain and does not change its sign in the entire domain, τ is the value of point-wise advance which is nonzero fixed real number. Let $|a(x,t)| < A$ and $|b(x,t)| < B, \forall (x,t)$. The unknown function u is defined in the underlying domain and also in the interval $[X, X+\tau]$ due to the presence of point-wise advance. So our domain is $[0, X] \cup [X, X+\tau]$ and $t > 0$. The coefficients are sufficiently smooth functions in these intervals and the unknown function u is as smooth as the initial data. Due to the presence of point-wise advance in equation (1), we need a boundary-interval condition in the right side of domain, i.e., in the interval $[X, X+\tau]$. The equation (1) is first-order hyperbolic with advance terms, so it requires one boundary condition according to the direction of characteristics, see [8]. Due to the presence of point-wise advance and non-constant coefficients, it is not difficult but impossible to find analytical solution of such type of partial differential equations by using the usual methods to find the exact solution of partial differential equations, see [4].

If delay and advance arguments are sufficiently small, the author used the Taylor series approximation for the term containing shift arguments. The Taylor series approximation may lead to a bad approximation when the size of shift arguments is large. Therefore, the numerical scheme presented in [9] does not work for the problem with large shift argument. To overcome this difficulty, we generate a special type of mesh so that the difference term lies on the nodal point in the discretize domain and present a numerical scheme that works nicely in both the cases. We construct the numerical scheme to find the approximate solution of problem (1) in Section 2 and discuss the stability, consistency and convergence. In Section 3, we discuss the extension of numerical scheme in higher space dimensions. In Section 4, we include some test examples for numerical illustration. Finally, in Section 5, we make conclusions that illustrate the effect of advance arguments on the solution behavior.

2. Numerical Approximation

In this section, we construct numerical scheme based on the finite difference method [8]. We discuss first and second order explicit numerical approximations for the given equation (1) based on Lax-Friedrichs finite difference approximations. For space time approximations based on finite differences, the (x,t) plane is discretize by taking mesh width Δx and time step Δt, and defining the gird points (x_j, t_n) by

$$x_j = j\Delta x, \qquad j = 0, 1, \ldots, J-1, J;$$

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$t_n = n\Delta t, \qquad n = 0,1,2,...$

Now we look for discrete solution U_j^n that approximate $u(x_j, t_n)$, $\forall j, n$. We write the closure of $\Omega_{\Delta x}$ as $\overline{\Omega}_{\Delta x}$ and $\overline{\Omega}_{\Delta x} = (x_j = j\Delta x, \qquad j = 0,1,2,..., J)$.

2.1 Construction of the numerical scheme

In this approximation, we approximate the time derivative by forward difference and space derivative by centered difference and then we replace U_j^n by the mean value between U_{j+1}^n and U_{j-1}^n for stability purpose. Numerical scheme is given by

$$\frac{U_j^{n+1} - \frac{U_{j+1}^n + U_{j-1}^n}{2}}{\Delta t} + a_j^n \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = b_j^n u(x_j + \tau, t) \qquad (2)$$

To tackle the point-wise advance in the numerical scheme (2), we discretize the domain in such a way that $(x_j + \tau)$ is a grid point, $\forall j = 0,1,..., J$; i.e., we choose Δx such that $\tau = m_0 \Delta x,\ m_0 \in \square$ and we take total number of points in $x-$ direction s.t.

$$J = \frac{X}{\Delta x} = lX\,\frac{mantissa(\tau)}{\tau}, \qquad l \in \square,$$

where mantissa of any real number is defined as positive fractional part of that number.

The term containing point-wise delay $(\forall\ j = 0,1,..., J)$ can be written as

$$u(x_j + \tau, t_n) = u(j\Delta x + m_0\Delta x, t_n) = u((j + m_0)\Delta x, t_n) \approx \mathrm{U}_{j+m_0}^n$$

Therefore the numerical approximation is given by

$$\frac{U_j^{n+1} - \frac{U_{j+1}^n + U_{j-1}^n}{2}}{\Delta t} + a_j^n \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = b_j^n U_{j+m_0}^{n,}, \qquad \forall j = 1,2,..., J-1 \qquad (3)$$

together with initial and boundary-interval conditions are given by

$$\begin{aligned} U_j^0 &= u^0(x_j), && j = 1,..., J-1, \\ U_0^n &= \phi(0, t_n), && n = 1,2,... \\ U_J^n &= \psi(\mathrm{s}, t_n), && \forall s \in [X, X+\tau] \quad n = 1,2,... \end{aligned} \qquad (4)$$

2.2 Stability Analysis

Definition: The finite difference method is called stable in the certain norm $\|.\|$ if there exists constant $C > 0$, independent of the space step and possibly depending on the time step such that

$$\left\|U^n\right\| \le C\left\|U^0\right\|, \qquad \forall\ n = 1,2,...$$

now consider the finite difference scheme as given equation (3) i.e.

$$U_j^{n+1} = \frac{1}{2}\left(1 - a_j^n \frac{\Delta t}{\Delta x}\right) U_{j+1}^n + \frac{1}{2}\left(1 + a_j^n \frac{\Delta t}{\Delta x}\right) U_{j-1}^n + b_j^n \Delta t U_{j+m_0}^n$$

$$\left|U_j^{n+1}\right| \le \frac{1}{2}\left|\left(1 - a_j^n \frac{\Delta t}{\Delta x}\right)\right|\left|U_{j+1}^n\right| + \frac{1}{2}\left|\left(1 + a_j^n \frac{\Delta t}{\Delta x}\right)\right|\left|U_{j-1}^n\right| + \Delta t\left|b_j^n\right|\left|U_{j+m_0}^n\right|$$

taking the sup norm, we get

$$\left\|U^{n+1}\right\|_{L^\infty} = \sup_j \left|U_j^{n+1}\right|$$

$$\le \frac{1}{2}\sup_j\left|\left(1 - a_j^n \frac{\Delta t}{\Delta x}\right)\right|\left|U_{j+1}^n\right| + \frac{1}{2}\sup_j\left|\left(1 + a_j^n \frac{\Delta t}{\Delta x}\right)\right|\left|U_{j-1}^n\right| + \Delta t \sup_j\left|b_j^n\right|\left|U_{j+m_0}^n\right|$$

Using CFL condition $A\frac{\Delta t}{\Delta x} \le 1$, (where A is the bound of $a(x,t),\ \forall (x,\mathrm{t})$), first two terms in the above inequality can be combined and we get

$$\left\|U^{n+1}\right\|_{L^\infty} \le (1 + B\Delta t)\left\|U^n\right\|_{L^\infty}$$

where $|b(x,t)| < B,\ \forall (x,t)$. The term $B\Delta t$ can be controlled by Δt from which we can predict that the effect of the term $B\Delta t$ is of the form $O(\Delta t)$. Using these values, we get

$$\left\|U^{n+1}\right\|_{L^\infty} \le (1 + O(\Delta t))\left\|U^n\right\|_{L^\infty}$$

i.e.,

$$\left\|U^{n+1}\right\|_{L^\infty} \le C\left\|U^n\right\|_{L^\infty}$$

which implies the stability of the numerical scheme where stability constant C is of the form $C = (1 + O(\Delta t))$.

2.3 Consistency of numerical scheme

The consistency error of the numerical scheme is the difference between both sides of the equation when the approximate solution U_j^n is replaced by exact solution $u(x_j, t_n)$ in the numerical scheme. If u is sufficiently smooth, the consistency error T_j^n of this finite difference scheme is given by

$$T_j^n = \frac{u(x_j, t_{n+1}) - \frac{u(x_{j+1}, t_n) + u(x_{j-1}, t_n)}{2}}{\Delta t} + a_j^n \frac{u(x_{j+1}, t_n) - u(x_{j-1}, t_n)}{2\Delta x} - b_j^n u_{j+m_0}^n$$

now using Taylor series approximation for the term $u(x_j, t_{n+1})$ w.r.t. to t and for the terms $u(x_{j-1}, t_n)$ and $u(x_{j+1}, t_n)$ w.r.t. to x, we get

$$T_j^n = \left[u_t + \frac{1}{2}\Delta t u_{tt} - \frac{\Delta x^2}{2\Delta t}u_{xx} + O(\Delta t^2) + O\left(\frac{\Delta x^4}{\Delta t}\right)\right]_j^n + \left[a\left(u_x + \frac{1}{6}\Delta x^2 u_{xxx}\right) + O(\Delta x^4)\right]_j^n - b_j^n u_{j+m_0}^n$$

$$= [u_t + a u_x]_j^n - b_j^n u_{j+m_0}^n + \frac{1}{2}\Delta t u_{tt} - \frac{\Delta x^2}{2\Delta t}u_{xx}$$

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$$+\frac{\Delta x^2}{6}a_j^n u_{xxx}+O(\Delta x^4+\Delta t^{-1}\Delta x^4+\Delta t^2).$$

Since u is exact solution, we get

$$[u_t+au_x]_j^n-b_j^n u_{j+m_0}^n=0$$

hence

$$T_j^n=\frac{1}{2}\Delta t u_{tt}-\frac{\Delta x^2}{2\Delta t}u_{xx}+\frac{\Delta x^2}{6}a_j^n u_{xxx}$$
$$+O(\Delta x^4+\Delta t^{-1}\Delta x^4+\Delta t^2).$$

Therefore, $T_j^n\to 0$ while $(\Delta x,\Delta t)\to(0,0)$, which shows that the numerical scheme is consistent of order 2 in space and of order 1 in time as long as $\Delta t^{-1}\Delta x^2\to 0$.

2.4 Convergence of the scheme

Definition: A finite difference scheme is said to be convergent if for any fixed point (x^*,t^*) in a given domain $(0,\mathrm{X})\times(0,t_n)$,

$$x_j\to x^*,\ \ t_n\to t^*\quad\Rightarrow\quad U_j^n\to u(x^*,t^*)$$

the error in the approximation is given by

$$e_j^n=U_j^n-u(x_j,t_n).$$

Now U_j^n satisfies the finite difference scheme (3) exactly, while $u(x_j,t_n)$ leaves the remainder $T_j^n\Delta t$. Therefore the error in the approximation is given by

$$e_j^{n+1}=\frac{1}{2}\left(1-a_j^n\frac{\Delta t}{\Delta x}\right)e_{j+1}^n+\frac{1}{2}\left(1+a_j^n\frac{\Delta t}{\Delta x}\right)e_{j-1}^n$$
$$+b_j^n\Delta t e_{j+m_0}^n-\Delta t T_j^n$$

and $e_0^n=0$.

Let $E^n=\max\left\{\left|e_j^n\right|,\ j=0,1,...,J\right\}$

Hence for $\left|a_j^n\frac{\Delta t}{\Delta x}\right|\le 1$,

$$E^{n+1}=\max_j\left|e_j^{n+1}\right|$$
$$\le E^n+\left|b_j^n\right|\Delta t E^n+\Delta t\max_j\left|T_j^n\right|$$
$$\le(1+B\Delta t)E^n+\Delta t\max_j\left|T_j^n\right|$$

using the given initial value for U_j^n, so $E^0=0$ and if we suppose that the consistency error is bounded i.e. $\left|T_j^n\right|\le T_{\max}$, then by induction method

$$E^n\le n\Delta t T_{\max}\le t_n T_{\max},$$

which shows that the method has first-order convergent provided that the solution has bounded derivatives up to second order.

3. Extension to Higher Spatial Dimensions

Now we consider the extensions of the numerical schemes to the higher spatial dimensions. For the sake of simplicity, we consider the problem in two spatial dimensions. The extension to three spatial dimensions can be done in similar fashion. The natural generalization of the one dimensional model problem (1) is the following equation together with the initial data and boundary interval conditions in the rectangular domain $\Omega=(0,\mathrm{X})\times(0,\mathrm{Y})$.

$$u_t+au_x+bu_y=cu(x+\delta,\mathrm{y}+\tau,t),\qquad(5)$$

where a,b,c are functions of x,y and t. δ and τ are the values of point-wise delay in x and y-direction respectively. For numerical approximations, we discretize the domain by taking uniform gird points with the spacing Δx in the x-direction and Δy in the y-direction. The gird points (x_j,y_k,t_n) are defined as follows

$$x_j=j\Delta x,\quad j=0,1,...,J_x;$$
$$y_k=k\Delta y,\quad k=0,1,...,J_y;$$
$$t_n=n\Delta t,\quad n=0,1,...$$

Now we write the extension of Lax-Friedrichs scheme (5). The approximate solution at the gird point (x_j,y_k,t_n) is denoted by $U_{j,k}^n$. Thus the numerical scheme is given by

$$U_{j,k}^{n+1}=\frac{1}{4}\left(U_{j+1,k}^n+U_{j-1,k}^n+U_{j,k+1}^n+U_{j,k-1}^n\right)-$$
$$\frac{\Delta t}{2\Delta x}a_{j,k}^n\left(U_{j+1,k}^n-U_{j-1,k}^n\right)-\frac{\Delta t}{2\Delta y}b_{j,k}^n\left(U_{j,k+1}^n-U_{j,k-1}^n\right)$$
$$+\Delta t c_{j,k}^n U_{j+m_0,k+q_0}^n,\qquad(6)$$

together with appropriate initial and boundary-interval conditions. We take the grid points in the both directions (x and y) in such a way that the term containing point-wise advance is also belong to discrete set of grid points which can be done as we did in the one dimensional case. We take total number of points in both x and y direction such that corresponding delays are on m_0 and q_0 nodal points and total number of points in the both directions is given by

$$J_x=\frac{X}{\Delta x}=lX\frac{mantissa(\delta)}{\delta},\quad l\in\square,$$
$$J_y=\frac{Y}{\Delta y}=rY\frac{mantissa(\tau)}{\tau},\quad r\in\square,$$

Table 1: The maximum absolute error for example 1

$\Delta t\downarrow\ \Delta x\to$	1/100	1/200	1/400	1/800
$\Delta x/2$	0.05449	0.02733	0.01365	0.00682
$\Delta x/4$	0.02724	0.01366	0.00682	0.00340
$\Delta x/8$	0.01362	0.00683	0.00341	0.00170
$\Delta x/16$	0.00681	0.00341	0.00170	0.00084

Table 2: The maximum absolute error for example 2

$\Delta t\downarrow\ \Delta x\to$	1/100	1/200	1/400	1/800
$\Delta x/2$	0.05199	0.02608	0.01303	0.00651
$\Delta x/4$	0.02599	0.01304	0.00651	0.00325
$\Delta x/8$	0.01299	0.00652	0.00325	0.00162
$\Delta x/16$	0.00649	0.00326	0.00162	0.00081

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Most of the analysis of the numerical approximation in one dimension is easily extended to the two dimensional case [8]. Truncation error of this approximation (6) will remain as in the one dimensional case except some additions due to the presence of y variable, see [8]. Usual analysis will give the CFL condition for stability in the following form

$$\frac{A\Delta t}{\Delta x}+\frac{B\Delta t}{\Delta x}\le 1,$$

where A and B are the bounds of a and b respectively.

The proof of convergence follows in similar way, leading to error in the approximation

$$E^n \le n\Delta t T_{\max} \le t_n T_{\max},$$

provided that the CFL condition is satisfied and u has bounded derivatives up second order.

4. Numerical Illustration

The purpose of this section is to include some numerical examples to validate the predicted results established in the paper and to illustrate the effect of point-wise advance on the solution behavior. We perform numerical computations using MATLAB. The maximum absolute errors for the considered examples are calculated using half mesh principle as the exact solution for the considered examples are not available [5]. We calculate the errors by refining the grid points. The error in the numerical approximation is given by

$$E(\Delta x,\Delta t)=\max_{0\le j\le J,0\le n\le Nt}\left|U^{\Delta t}_{\Delta x}(j,n)-U^{\Delta t/2}_{\Delta x/2}(2j,2n)\right|$$

We consider $\Omega=(0,1)$, $\Delta x=\Delta t=.001$ for example 1 and 2. The numerical solution is plotted for different values of τ at the time $t=0.5$ in figure 1 and 3 and for various values of time t in figure 2 and 4.

Example1. Consider the problem (1) with the following coefficients and initial- boundary conditions:

$$a(x,t)=\frac{1+x^2}{1+2xt+2x^2+x^4};\quad b(x,t)=0.5;$$

$$u(x,0)=\exp[-10(4x-1)^2];\quad u(s,t)=0,\quad \forall s\in[1,1+\tau].$$

Example2. Consider the problem (1) with the following coefficients and initial- boundary conditions:

$$a(x,t)=\frac{1+x^2}{1+2xt+2x^2+x^4};\quad b(x,t)=\frac{1}{1+2x^2t^2+x^4};$$

$$u(x,0)=\exp[-10(4x-1)^2];\quad u(s,t)=0,\quad \forall s\in[1,1+\tau].$$

We consider the two dimensional problem (5) with variable coefficients. We consider $\Omega=(0,1)\times(0,1)$, $\Delta x=\Delta y=.01$ and time step $\Delta t=.001$. The approximate solution is plotted with $\delta=0.5$ and $\tau=0.5$ at time $t=0.5$ in figure 5.

Example3. Consider the 2-D problem (5) with the following coefficients and initial- boundary conditions:

$$a(x,y,t)=\frac{1+x^2+y^2}{1+2(x+y)t+2(x^2+y^2)+x^4};$$

$$b(x,y,t)=\frac{1}{1+2(x^2+y^2)t^2+x^4};$$

$$u(x,y,0)=\exp[-10(4x+4y-1)^2];$$

$$u(s_1,s_2,t)=0,\quad \forall s_1\in[1,1+\delta]\quad\text{and}\quad \forall s_2\in[1,1+\tau].$$

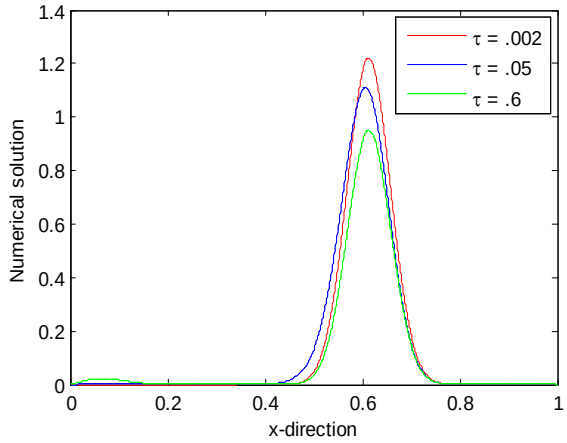


Figure 1: The effect of the point-wise advance on solution at $t=0.5$ for example 1.

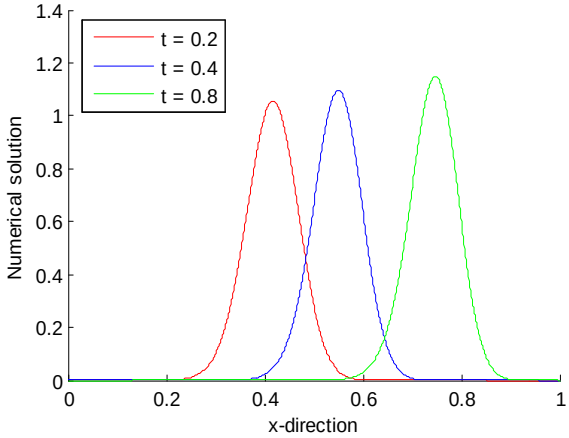


Figure 2: The numerical solution of Example 2 for different time levels for $\tau=.05$.

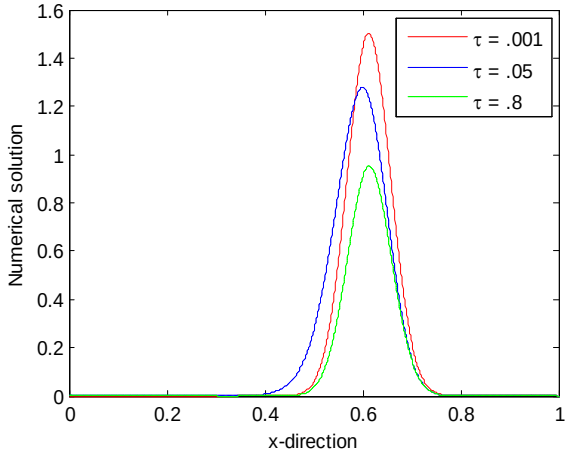


Figure 3: The effect of the point-wise advance on solution at $t=0.5$ for example 2.

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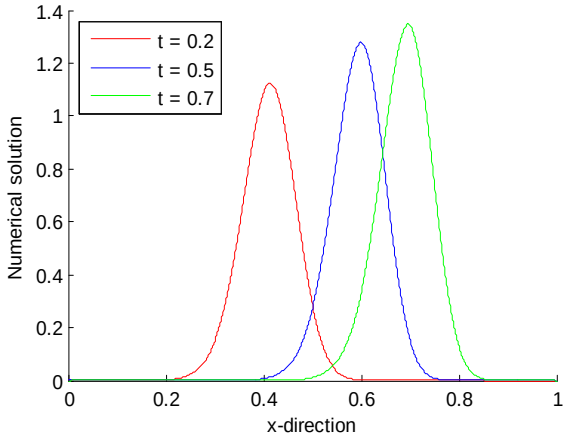


Figure 4: The numerical solution of Example 2 at different time levels for $\tau = .05$.

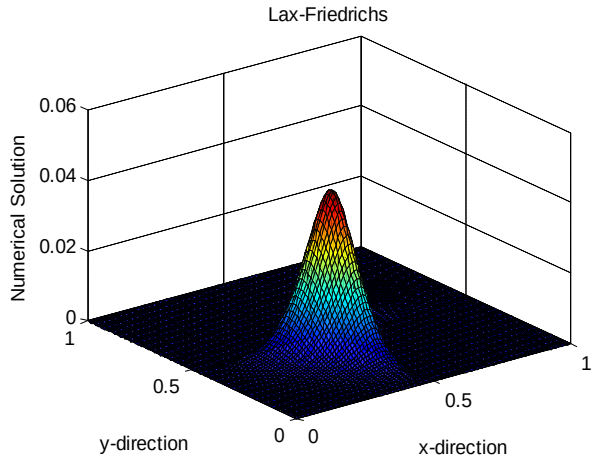


Figure 5: The numerical solution of Example 3 for $\delta = 0.5$ and $\tau = 0.5$ at $t = 0.5$.

5. Conclusion

In this paper we propose a numerical scheme based on Lax-Friedrichs finite difference approximations of order greater than one in space to solve hyperbolic partial differential equation with point-wise advance. The consistency, stability and convergence analysis prove that the proposed numerical schemes are consistent, stable with CFL condition and convergent in both space and time. This second order numerical scheme in space maintains the height and width better than a first-order scheme as author discussed in paper [9]. The effect of point-wise advance on the solution behavior is shown by the some test examples. Error tables illustrate the fact that the methods are convergent in space and time. The solutions are plotted in graphs which shown in figures 1-5. Also we extend our ideas in higher space dimensions and include numerical experiment to show the behavior of solution in two space dimension.

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