International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2013): 6.14 | Impact Factor (2013): 4.438

Solution of One Dimensional Wave Equation using Laplace Transform

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Abstract: ***In this paper the equation of motion for the string under certain assumption has been derived which is in the form of second order partial differential equation. The governing partial differential equation represents transverse vibrating of an elastic string which is known as one dimensional wave equation. The analytical solution has been obtained using Laplace Transform.***

Keywords: Partial Differential Equation, Wave equation, Laplace Transform, Transverse

1. Introduction

In this paper, we deal with initial boundary value problem that is a second order Partial Differential Equation that occurs frequently in many physical phenomena which is known as one dimensional wave equation. The governing equation represents transverse vibrating of an elastic string. An analytical solution obtained by using Laplace Transform.

The solution of wave equation was one of the major mathematical problems of the mid eighteenth century. The wave equation was first derived and studied by D'Alembert in 1746. It also attracted the attention of Euler (1748), Bernoulli (1753) and Lagrange (1759). Solution was obtained in several different forms in series of papers. The major points at issue concerned the nature of a function and the kind of functions that can be represented by trigonometry.

2. Statement of the Problem

Consider a uniform elastic string of length *l* stretched tightly between two fixed points *O* and *A*, and displayed slightly from its equilibrium position *OA*. The line *OA* joining the points origin *O* and *A(l, O)* is taken as the *x* – axis and a perpendicular line through *O* as the *y* – axis. The problem is to determine the vibrations of the string, that is, to find its deflection *y(x, t)* at any point *x* and at any time $t > 0$ (see figure 1)

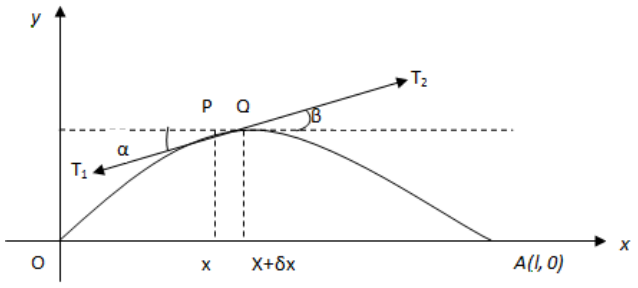


Figure 1

3. Physical Assumptions

We assume the following.

1) The vibrations are lateral and take place in a plane.
2) The string is perfectly elastic and does not offer any resistance to bending.
3) The tension in the string is so large that the action of the gravitational force on the string can be neglected.
4) The displacement *y* and the slope $\frac{\partial y}{\partial x}$ are small, so that their higher powers may be neglected.

4. Mathematical Formulation

Let *m* be the mass per unit length of the string. Consider the motion of an infinitesimal elements *PQ* of length *δs*. The mass of this element is *mδs*, its acceleration is $\ddot{y}$ and the forces acting on it are the tensions T_1 and T_2 as shown in Figure 1.

Since there is no motion in the horizontal direction, we have

$$T_1 cos\alpha = T_2 cos\beta = T = constant \quad (1)$$

By Newton's second law of making the equation of motion in the vertical direction is

$$m\delta s \frac{\partial^2 y}{\partial t^2} = T_2 sin\beta - T_1 sin\alpha \quad (2)$$

$$\frac{m\delta s}{T}\frac{\partial^2 y}{\partial t^2} = \frac{T_2 sin\beta}{T_2 cos\beta} - \frac{T_1 sin\alpha}{T_1 cos\alpha} \quad (3)$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m\delta s}(tan\beta - tan\alpha) \quad (4)$$

Since $\delta s = \delta x$ to a first approximation and

$$tan\alpha = \left(\frac{\partial y}{\partial x}\right)_x , tan\beta = \left(\frac{\partial y}{\partial x}\right)_{x+\delta x} \quad (5)$$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\left[\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\delta x}\right] \quad (6)$$

As $Q \to P, \delta x \to 0$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\lim_{\delta x \to 0}\left[\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\delta x}\right] \quad (7)$$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) \quad (8)$$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\left(\frac{\partial^2 y}{\partial x^2}\right) \quad (9)$$

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Putting $\frac{T}{m} = c^2$, we get

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad (10)$$

This is the partial differential equation giving the transverse vibration of the string. It is also called the one dimensional wave equation. To find the displacement $y(x,t)$ of semi infinite elastic string, consider the following conditions:

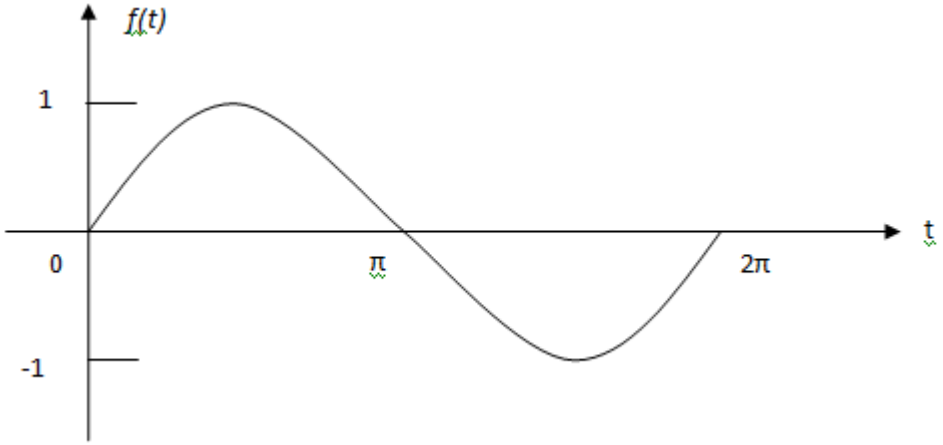


Figure 2

(i) The string is initially at rest on the x – axis from $x = 0$ to ∞ ("Semi-infinite string")

(ii) For time $t > 0$, the displacement is $y(0,t) = f(t) = sint \; if \; 0 \le t \le 2\pi$
$= 0 \; otherwise$

(iii) Furthermore,

$$\lim_{x\to\infty} y(x,t) = 0 \; for \; t \ge 0$$

Of course there is no infinite string, but our model describes a long string or rope (of negligible weight) with its right end fixed far out on the x – axis.

5. Solution of the Problem

To solve equation (10) for positive x and t, subject to the "boundary conditions"

$$y(0,t) = f(t), \lim_{x\to\infty} y(x,t) = 0 \;\; t \ge 0 \quad (11)$$

with f as given above and the initial conditions

$$y(x,0) = 0 \quad (12)$$

$$\frac{\partial y}{\partial t} = 0 \; at \; t = 0 \quad (13)$$

Now, taking the Laplace transform on both sides of equation (10), we get

$$L\left\{\frac{\partial^2 y}{\partial t^2}\right\} = c^2 L\left\{\frac{\partial^2 y}{\partial x^2}\right\}$$

$$\Rightarrow s^2 L\{y\} - sy(x,0) - \frac{\partial y}{\partial t} = c^2 L\left\{\frac{\partial^2 y}{\partial x^2}\right\}$$

Now dropping out two terms by using (12) and (13),

$$s^2 L\{y\} = c^2 L\left\{\frac{\partial^2 y}{\partial x^2}\right\}$$

By definition of Laplace Transform,

$$L\left\{\frac{\partial^2 y}{\partial x^2}\right\} = \int_0^\infty e^{-st} \frac{\partial^2 y}{\partial x^2} dt$$

On the right side, we assume that we may interchange integration and differentiation,

$$L\left\{\frac{\partial^2 y}{\partial x^2}\right\} = \frac{\partial^2}{\partial x^2}\int_0^\infty e^{-st} y(x,t)dt = \frac{\partial^2}{\partial x^2} L\{y(x,t)\}$$

Writing $Y(x,s) = L\{y(x,t)\}$,

Thus we obtain

$$s^2 Y = c^2 \frac{\partial^2 Y}{\partial x^2}$$

Therefore,

$$\frac{\partial^2 Y}{\partial x^2} - \frac{s^2}{c^2} Y = 0$$

Since this equation contains only a derivative with respect to x, it may be regarded as an ordinary differential equation for $Y(x, s)$ considered as a function of x. A general solution is

$$Y(x,s) = A(s)e^{sx/c} + B(s)e^{-sx/c} \quad (14)$$

From (11) we obtain, writing $F(s) = L\{f(t)\}$.

$$Y(0,s) = L\{y(0,t)\} = L\{f(t)\} = F(s)$$

Assuming that we can interchange integration and taking the limit, we have

$$\lim_{x\to\infty} Y(x,s) = \lim_{x\to\infty} \int_0^\infty e^{-st} y(x,t)dt$$
$$= \int_0^\infty e^{-st} \lim_{x\to\infty} y(x,t)dt = 0.$$

This implies $A(s) = 0$ in (5) because c > 0, so that for every fixed positive s the function $e^{sx/c}$ increases as x increases. Note that we may assume s > 0 since a Laplace transform generally exits for all s greater than some fixed k. Hence we have

$Y(0,s) = B(s) = F(s)$,
so that (5) becomes
$Y(x,s) = F(s)e^{-sx/c}$.

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From the second shifting theorem with $a = x/c$ we obtain the inverse transform

$$y(x,t) = f\left(t - \frac{x}{c}\right) u\left(t - \frac{x}{c}\right) \quad (15)$$

$$i.e. y(x,t) = sin\left(t - \frac{x}{c}\right) \; if \; \frac{x}{c} < t < \frac{x}{c} + 2\pi \; or \; ct > x > (t - 2\pi)c$$

and zero otherwise.

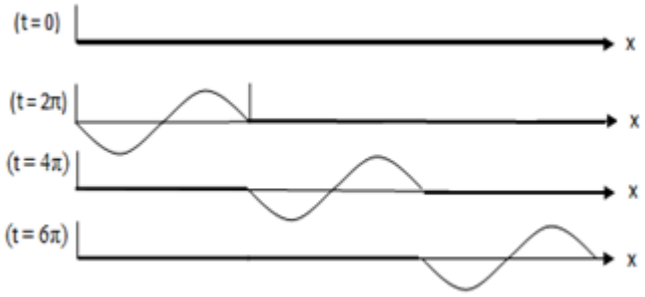


Figure 3

6. Conclusion

The solution obtained in (15) is a single sine wave traveling to the right with speed c. Note that a point x remains at rest until $t = x/c$, the time needed to reach that x if one starts at $t = 0$ (start of the motion of the left end) and travels with speed c. The result is consistent with physical nature.

References

[1] Erwin Kreyszig, Advance Engineering Mathematics, John wiley and sons, 8th edition
[2] Lawrence C. Evans, Partial Differential Equations. AMS, 1998
[3] G. James, Advanced modern Engineering Mathematics (3rd edition), Pearson education limited; New york 1999
[4] Sourav Kar / Subrata Karmakar, Engineering Mathematics – III (WBUT) Tata McGraw Hill
[5] Murray R. Spiegel, Schaum's outline of theory and problems of Laplace Transforms Mc Graw Hill
[6] N. Asmar, Partial Differential Equations, Prentice Hall, New Jersey, (2005).

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