International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Impact Factor (2012): 3.358

Signed Roman Domination Number of Generalized Petersen Graph

Shailaja S. Shirkol[1], Manjula C. Gudgeri [2]

[1]Department of Mathematics, SDM College of Engineering and Technology, Dharwad. Karnataka, India.

[2]Department of Mathematics, KLES Dr. M. S. Sheshagiri College of Engineering and Technology, Belgaum, Karnataka, India.

Abstract: ***A signed Roman Dominating Function (SRDF) on a graph G is a function f : V(G) → {-1, 1, 2} such that*** $\sum_{u\in N[v]} f(u) \geq 1$ ***for every v Є V(G) and every vertex u Є V(G) for which f(u) = -1 is adjacent to at least one vertex w for which f(w) = 2. The weight of SRDF is the sum of its function values over all vertices. The signed Roman domination number of G is the minimum weight of a SRDF in G. For natural number n and k, where n > 2k, a generalized Petersen graph P(n, k) is obtained by letting its vertex set to be {u1, u2, …, un, v1, v2, …, vn} and its edge set to be {ui ui+1, ui vi, vi vi+k}; where i = 1, 2,..., n and subscripts are reduced modulo n. In this paper we determine the signed Roman domination number of generalized Petersen graph P(n, k) for k = 1 & 3. We characterize generalized Petersen graph which have efficient signed Roman domination number.***

Keywords: Generalized Petersen graph, Roman domination, Signed domination, signed Roman dominating function, signed Roman domination number & efficient signed Roman domination.

AMS subject classification: 05C69.

1. Introduction

In this paper, *G* is a simple graph with vertex set *V = V(G)* and edge set *E = E(G)*. The order of $|V|$ of *G* is denoted by *n*. For every vertex *v Є V*, the open neighborhood *N (v)* is the set $\{u \in V(G)/uv \in E(G)\}$ and the closed neighborhood of *v* is the $N[v] = N(v) \cup \{v\}$. The degree of the vertex $v \in V$ is *d (v)* = $|N(v)|$. The minimum and maximum degree of a graph *G* are denoted by δ and Δ respectively. A graph *G* is *r* - regular if *d (v) = r* for each v of *G* [3]. A set *D* of vertices of a graph *G = (V, E)* is called dominating set, if each vertex in $V - D$ is adjacent to at least one vertex in *D*. The domination number of G denoted by $\gamma(G)$ is the cardinality of minimum dominating set of *G* [9].

Cockayne et al. (2004) defined a Roman Dominating Function (RDF) on a graph *G = (V, E)* to be a function *f : V → {0, 1, 2}* satisfying the condition that every vertex *u* for which *f (u) = 0* is adjacent to atleast one vertex *v* for which *f (v) = 2*. For a real valued function *f : V → R* the weight of f is *w(f)* = $\sum_{v\in V} f(v)$, and for *S* $\subseteq$ *V* we defined *f (S)* = $\sum_{v\in S} f(v)$, and so *w (f) = f (V)*. The Roman domination number, denoted by $\gamma_R(G)$ is the minimum weight of an RDF in *G*. That is $\gamma_R(G) = \min\{w(f)/f \text{ is a RDF in } G\}$. An RDF of weight $\gamma_R(G)$ is called a $\gamma_R(G)$ – function [2].

The definition of Roman dominating function was motivated by an article in Scientific American by Ian Stewart entitled “Defend the Roman Empire” (Stewart 1999) [14] and suggested even earlier by Re Velle (1997). Each vertex in our graph represents a location in the Roman Empire. A location (vertex v) is considered unsecured if no legions are stationed there (i.e., *f(v) = 0*) and secured otherwise (i.e., *f*(*v*) *Є {1, 2}*). An unsecured location (vertex *v*) can be secured by sending a legion to *v* from an adjacent location (an adjacent vertex *u*). But Constantine the Great (Emperor of Rome) issued a decree in the 4th century A. D. for the defense of a his cities. He decreed that a legion cannot be sent from a secured location to an unsecured location if doing so leaves that location unsecured. Thus two legions must be stationed at a location (*f*(*v*) = *2*) before one of the legions can be sent to an adjacent location. In this way Emperor Constantine the Great can defend the Roman Empire. Since it is expensive to maintain a legion at a location the Emperor would like to station as few legions as possible, while still defending the Roman Empire. A Roman dominating function of weight $\gamma_R(G)$ corresponds to such an optimal assignment of legions to locations. However Constantine's model did not achieve the desired goal of being both cost effective and of defending the Roman Empire. In this paper we explore an alternative model which would save the Emperor substantial cost of maintaining legions, while still defending the Roman Empire. The 4th century A.D. saw a very large number of new, small legions created, a process which began under Constantine II. In particular, auxiliary cohorts (about a tenth the size of the legion) and auxilia palatina were formed. Auxiliary troops were mainly recruited from the peregrini, i.e., free provincial subjects of the Roman Empire who did not hold Roman citizenship, in contrast to the legions, which only admitted Roman citizens. Auxiliary troops were considered second class soldiers and were look down on by the elite troops of the comitatensis who were paid regularly and were much equipped. As a cost effective way of securely defending the Roman Empire, Emperor Constantine's strategy would be to minimize the number of legions stationed by placing auxiliary troops at every unsecured location provided that the number of legions stationed at a location and its neighboring locations always exceeded the number of auxiliary troop stationed there for every location in the Roman Empire.

In graph theoretic terms, we define a signed dominating function (SDF) on a graph *G = (V, E)* is a function *f : v → { -1, 1}* such that $f(N[v] \geq 1$ for every vertex *v Є V*. Thus combining properties of both a roman dominating function

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and a signed dominating function we define signed Roman dominating function (SRDF) on a graph $G = (V, E)$ to be a function $f : V \rightarrow (-1, 1, 2)$ satisfying the condition that f is a dominating function (that is, the sum of the values assigned to a vertex and its neighbors is at least 1 for every vertex) and every vertex u for which $f(u) = -1$ is adjacent to at least one vertex v for which $f(v) = 2$. The signed Roman domination number denoted $\gamma_{sR}(G)$, is the minimum weight of a SRDF in G ; that is, $\gamma_{sR}(G) = \min\{w(f) is\ a\ SRDF\ in\ G$. A SRDF of weight $\gamma_{sR}(G)$, we call a $\gamma_{sR}(G)-$function. [7] A signed Roman dominating function $f : v \rightarrow (-1, 1, 2)$ can be represented by the ordered partition $\{V_{-1}, V_1, V_2\}$ of V, where $V_i = \{v \in V / f(v) = i\}$.

2. Preliminary Results and Observations

Observation 2.1 [7] : Let $f = (V_{-1}, V_1, V_2)$ be a SRDF in a graph G. Then the following holds.
a) Every vertex in V_{-1} is dominated by a vertex in V_2
b) $w(f) = |V_1| + 2|V_2| - |V_{-1}|$.
c) $V_1 \cup V_2$ is a dominating set in G.

Proposition 2.2 [7] : $(2\Delta + 1)|V_2| + \Delta|V_1| \geq (\delta + 2)|V_{-1}|$

Corollary 2.3 [7]: For a cubic graph $|V_{-1}| \leq \frac{1}{5}[7|V_2| + 3V1.$

Corollary 2.4 [7] : For r ≥ 1, if G is a r-regular graph order n, then $\gamma_{sR}(G) \geq \frac{n}{(r+1)}$.

Observation 2.5 [7] : For $n \geq 2$,
$\gamma_{sR}(K_{1,n-1}) = 1$ for n even
2 for n odd

Proposition 2.6 [7]: For $n \geq 1$, $\gamma_{sR}(K_n) = 1$ unless $n = 3$, in which case $\gamma_{sR}(K_n) = 2$
in which case $\gamma_{sR}(K_n) = 2$.

Proposition 2.7 [7] : For $n \geq 3$, $\gamma_{sR}(C_n) = \left\lceil \frac{2n}{3} \right\rceil$.

Proposition 2.8 [7] : For $n \geq 1$, $\gamma_{sR}(P_n) = \left\lceil \frac{2n}{3} \right\rceil$.

Proposition 2.9 [7]: Let G be a graph of order n., then the following holds
a). $\gamma_{sR}(G) \leq n$ with equality if and only if $G = \overline{K}_n$
b). $\gamma_{sR}(G) \geq 2\gamma(G) - n$, with equality if and only if $G = \overline{K}_n$.

3. Some exact values of $\gamma_{sR}(Pn, k)$

Theorem 3.1 : For $G = P(n, 1)$, a generalized Petersen graph then
i). $\gamma_{sR}(P(4m, 1)) = 2m$
ii) $\gamma_{sR}(P(4m + 1, 1)) = \gamma_{sR}(P(4m, 1)) + 3$.
iii). $\gamma_{sR}(P(4m + 2, 1)) = \gamma_{sR}(P(4m, 1)) + 4$
iv). $\gamma_{sR}(P(4m + 3, 1)) = \gamma_{sR}(P(4m, 1)) + 2$

Proof : Let $G = P(n, 1)$, $n \geq 3$ be a generalized Petersen graph with $2n$ vertices whose vertex set be $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and a edge set $\{u_i u_{i+1}, u_i v_i, v_i v_{i+k}, i = 0, 1, 2, \ldots, n-1\}$.
Let $f : V \rightarrow \{-1, 1, 2\}$ be a signed Roman domination function.
For $r \geq 1$, if G is r - regular graph of order n then $\gamma_{sR}(G) \geq \frac{n}{r+1}$.
As G is cubic graph of order n then, $\gamma_{sR}(G) \geq 2m$
Case 1. If $n=4m$, $m \geq 1$
Let $g : v \rightarrow \{-1, 1, 2\}$ be a function defined as follows.
Let $g(u_i) = -1$ if $i \equiv 5 \bmod (4)$ and $i \equiv 2 \bmod (4)$
Let $g(u_i) = 1$ if $i \equiv -1 \bmod (4)$
Let $g(u_i) = 2$ if $i \equiv 4 \bmod (4)$ and
Let $g(v_i) = -1$ if $i \equiv 3 \bmod (4)$ and $i \equiv 4 \bmod (4)$
Let $g(v_i) = 1$ if $i \equiv 1 \bmod (4)$
Let $g(v_i) = 2$ if $i \equiv 2 \bmod (4)$
The assignment is shown in following figure

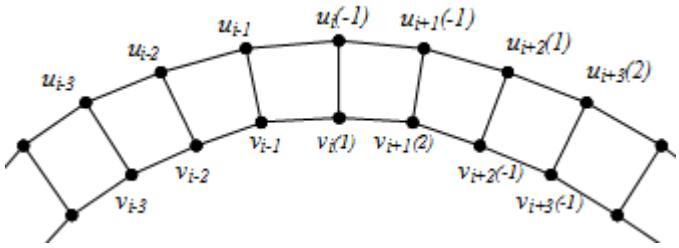


Figure 1

With this assignment the first copy of m will have weight exactly two. As we have m such copies,
$\gamma_{sR}(P(4m, 1)) \leq 2m$
Consequently $\gamma_{sR}(P(4m, 1)) = 2m$

Case 2. If $n=4m+1$, $m \geq 1$
The proof is same as case 1 for first 4m-vertices whose minimum weight is $2m$. The remaining two vertices namely u_{4m+1} and v_{4m+1} are to be assigned with 2 and 1 respectively. Otherwise the vertex u_1 will loose the SRDF property. Thus g is a SRDF of G. Hence the total weight of the graph is $2m+3$.
Thus $\gamma_{sR}(P(4m + 1, 1)) = \gamma_{sR}(P(4m, 1)) + 3$.

Case 3. If $n=4m+2$
The proof is same as case 1 for first $4m$-vertices. Here we are left with 4 vertices which are to be assigned as follows. Vertices namely u_{4m+1}, u_{4m+2}, v_{4m+1}, v_{4m+2} as $-1, 2, 1, 2$ respectively to get the minimum weight. Thus g is a SRDF of G .Hence total weight of the graph is $2m+4$.
Thus $\gamma_{sR}(P(4m + 2, 1)) = \gamma_{sR}(P(4m, 1)) + 4$.

Case 4. If $n=4m+3$
The proof is same as case 1 for first $4m$-vertices. Here we are left with six vertices namely u_{4m+1}, u_{4m+2}, u_{4m+3}, v_{4m+1}, v_{4m+2}, v_{4m+3} which are to be assigned as $-1, -1, 2, 1, 2, -1$ respectively to get the minimum weight. Thus g is a SRDF of G and Hence total weight of the graph is $2m+2$.
$\gamma_{sR}(P(4m + 3, 1)) = \gamma_{sR}(P(4m, 1)) + 2$

Observation 3.2 : The set of vertices assigned 2 forms a dominating set in $P(n, k)$, where k is odd. Therefore $\sum N[V_2] = 2n$.

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Lemma 3.3: For a graph *G = P (n, k)*
$|V_{-1}| = 2m$ for *n = 3m, 3m+1*
$|V_{-1}| = 2m + 1$ for *n = 3m+2*

Proposition 3.4: For *G = P(n, 3)* a generalized Petersen graph then,
i). $\gamma_{sR}(P(4m,1)) = 2m$ for all m ≥ 2
ii) $\gamma_{sR}(P(4m+1,3)) = \gamma_{sR}(P(4m,3)) + 3$
iii). $\gamma_{sR}(P(4m+2,3)) = \gamma_{sR}(P(4m,3)) + 6$
iv). $\gamma_{sR}(P(4m+3,3)) = \gamma_{sR}(P(4m,3)) + 5$
Proof : Let *G=P (n, 3)* be a generalized Petersen graph with *2n* vertices whose vertex set is *{u_1, u_2,...,u_n, v_1, v_2,...,v_n}* and edge set *{ $u_i u_{i+1}$, $u_i v_i$, v_i, v_{i+k}, i=0,1,2,3----n-1}* and
Let *f: V → { -1,1,2}* be a signed Roman dominating function.

For r ≥ 1, if *G* is r-regular graph of order n then $\gamma_{sR}(G) \geq \frac{n}{r+1}$
Since G is cubic graph of order *2n* then $\gamma_{sR}(G) \geq 2m$

Case 1. If *n = 4m, m ≥ 2*
Let *g : V → {-1, 1, 2}* be a function defined as follows
Let $g(u_i) = -1$ *if* $i \equiv 5 \bmod (4)$ and $i \equiv 0 \bmod (4)$
Let $g(u_i) = 1$ *if* $i \equiv -2 \bmod (4)$
Let $g(u_i) = 2$ *if* $i \equiv -1 \bmod (4)$
Similarly Let $g(v_i) = -1$ *if* $i \equiv 2 \bmod (4)$ and $i \equiv 3 \bmod (4)$
Let $g(v_i) = 1$ *if* $i \equiv 4 \bmod (4)$
Let $g(v_i) = 2$ *if* $i \equiv 1 \bmod (4)$
The assignment is as shown below

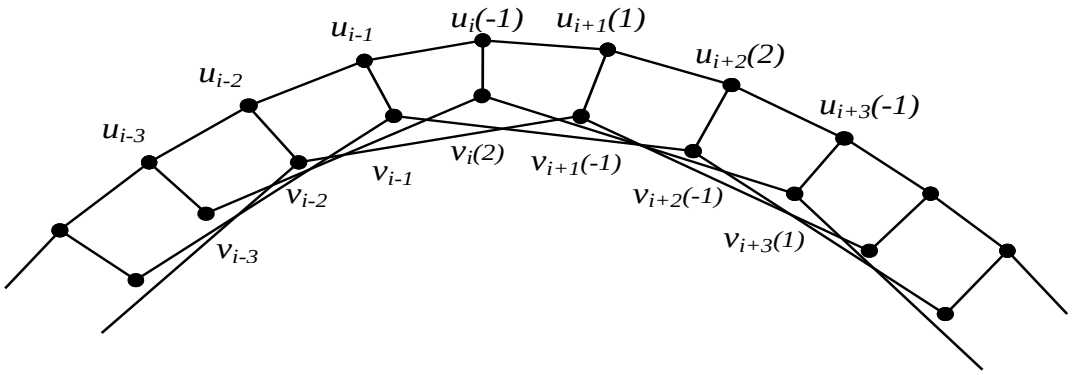


Figure 2

With this assignment the first copy of m will have weight exactly *4.*
As we have m such copies $\gamma_{sR}(P(n,3)) \leq w(g) \leq 2m = \frac{n}{2}$
Consequently $\gamma_{sR}(P(4m,3)) = 2m$
Case 2. If *n = 4m + 1, m ≥ 2*
The proof is same as case 1 for first 4m-vertices whose minimum weight is $\frac{n}{2}$.
There are two vertices left namely u_{4m+1}, v_{4m+1} which are to be assigned as *1* and *2* respectively to have minimum weight.
Hence total weight of a graph is $\frac{n}{2} + 3$.
Thus $\gamma_{sR}(P(4m+1,3)) = \gamma_{sR}(P(4m,3)) + 3$

Case 3. If *n = 4m + 2*
The proof is same as case 1 for first *4m*-vertices whose minimum weight is $\frac{n}{2} = 2m$.
Here we are left with u_{4m+1}, u_{4m+2}, v_{4m+1}, v_{4m+2}
which are to be assigned as *1, 2, 2, 1* respectively to get minimum weight.
Hence total weight of a graph $\frac{n}{2} + 6$
Thus $\gamma_{sR}(P(4m+2,3)) = \gamma_{sR}(P(4m,3)) + 6$.

Case 4. If *n = 4m + 3*
The proof is same as case 1 for first *4m*-vertices whose minimum weight is $\frac{n}{2}$.
Here we are left with six vertices namely u_{4m+1}, u_{4m+2}, u_{4m+3}, v_{4m+1}, v_{4m+2}, v_{4m+3} which are to be assigned as *-1, 2, -1, 2, 1,2* respectively to get the minimum weight.
Hence total weight of a graph is $\frac{n}{2} + 5$
Thus $\gamma_{sR}(P(4m+3,3)) = \gamma_{sR}(P(4m,3)) + 5$.

4. Efficient signed Roman domination

A signed Roman domination function (SRDF) on a graph *G = (V, E)* is a function *f : V → {-1, 1, 2}* satisfying the condition that,
i) The sum of its function values over closed neighborhood is atleast 1 and
ii) For every vertex *u* for which *f(u) = -1* is adjacent to atleast one vertex *v* for which *f(v)=2.*

The weight of SRDF is sum of its function values over all vertices. The singed Roman domination number of *G* is a minimum weight of SRDF in *G*. The SRDF is said to be an efficient signed Roman domination if *f[v] = 1*, for all *v Є V (G).*

Theorem.4.1 If *G = P(n, k)* has an efficient signed Roman function
Then $\gamma_{sR}(G) = \gamma(G)$.
In the following lemma a useful necessary condition for *P(n, k)* to have an efficient signed Roman dominating function is given.

Lemma 4.2: If *P(n, k)* where *k* is odd has an efficient signed Roman dominating function, then $\gamma_{sR}(P(n,k) = \gamma(P(n,k)) = \frac{n}{2}$ *and* $4/n$

Lemma 4.3: If *k* is odd number and *4/n* then $\gamma_{sR}(P(n,k) = \frac{n}{2}$ and there fore *P (n, k)* has an efficient signed Roman dominating function.

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Theorem 4.4: A generalized Petersen graph *P(n, k)* has an efficient signed Roman domination function if and only if $n \equiv 0 \bmod (4)$ and *k* is odd.

Proof : Sufficiency of the statement follows from lemma 4.2. For necessity, suppose *f* is an efficient signed Roman dominating function in *P(n, k)*, a graph must have efficient dominating set in *P(n, k)*. As in Lemma 4.1, we have $|S| = \frac{n}{2} = 2m$. Where m is a number of *u* vertices in *S*, which is equal to the number of *v* vertices in *S*.

Each *u*-vertex dominates three *u*-vertices (including itself) and one *v*-vertex. So there are *3m*, *u*-vertices dominated by *u*-vertices and *m* of them dominated by *v*- vertices. Let u_i and u_j be two *u*-vertices in *S*, such that on one of the *u*-paths from u_i to u_j there is no other *u*-vertex in *S*. Then there are exactly five *u*-vertices on the *u*-path from u_i to u_j , including u_i and u_j . For, since *S* is an efficient signed Roman dominating set and by lemma 4.2 the number of vertices on that path dominated by *v*-vertex is atmost *1*, and also since there are *m v*- vertices in *S*, there must be atleast one vertex of that path dominated by a *v*-vertex. So there is a unique pattern for the *u*-vertices in *S*, say $\{u_{i-1}, u_{i+3}\} \subseteq S$ and similarly $\{v_{i+1}, v_{i+5}\} \subseteq S$, see figure for the pattern. By this unique pattern, it is clear that *P(n, k)* does not have an efficient signed Roman dominating function for even values of *k*. See figure 3 for a generalized Petersen graph *P(16, 5)* and an efficient signed Roman domination.

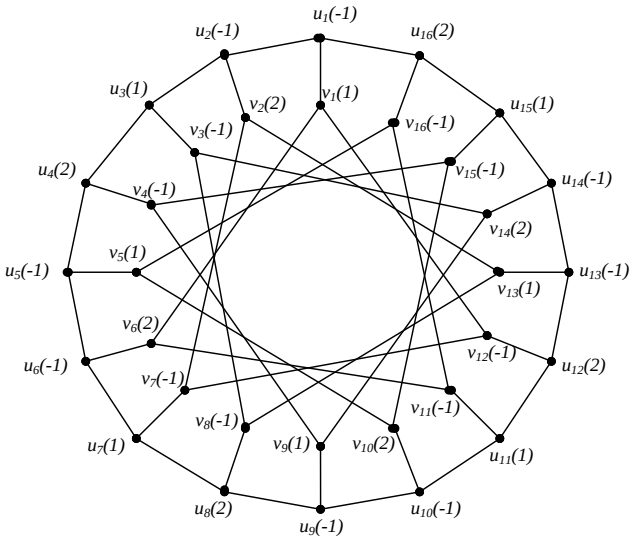


Figure 3

5. Scope and Conclusion

5.1 Scope

1) We can find signed Roman domination number for other classes of graphs.
2) We can determine the signed Roman domination number of any grid graph $G_{m,n}$.

5.2 Conclusion

We found signed Roman domination number of a generalized Petersen graph *P(n, k)* for *k = 1* and *3* and also found condition for a *P(n, k)* to have efficient SRDF for Generalized Petersen graph of odd *k*.

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