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A Unified Framework for Fuzzy Semigroup Subsets Based on Inverse and Soft Set Methods

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Abstract: ***Let S be a semigroup and let $\mathcal{F}(S)$ denote the family of fuzzy subsets of S. In this paper we introduce a unified framework for fuzzy semigroup subsets generated simultaneously by inverse-image constructions and soft set–based parameterization. More precisely, given a parameter set E, a family of semigroup homomorphisms $\{\varphi_e: S \to T_e\}_{e\in E}$, and fuzzy subsets $\{\mu_e: T_e \to [0,1]\}_{e\in E}$, we define a generalized inverse soft fuzzy subset $\mu: S \to [0,1]$ via suitable inverse-image and aggregation operators, and we isolate conditions under which μinduces unified inverse soft fuzzy ideals, bi-ideals, quasi-ideals and interior ideals of S. The new framework accommodates classical fuzzy ideals and fuzzy bi-ideals in semigroups, fuzzy soft semigroups and fuzzy soft ideals, as well as int-soft semigroups, int-soft generalized bi-ideals, union-soft ideals and inverse int-fuzzy soft bi-ideals, as particular instances. We establish closure and homomorphism-invariance properties in the unified setting and provide representation theorems showing that existing constructions can be embedded into the generalized inverse soft fuzzy scheme. Several examples on finite and ordered semigroups illustrate how different choices of parameters, homomorphisms and aggregation operators recover known structures and yield new characterizations of regular and intra-regular semigroups.***

Keywords: Fuzzy semigroup subsets, Unified inverse soft fuzzy ideals, Soft set parameterization, Generalized inverse soft fuzzy subset, Fuzzy bi-ideals, quasi-ideals, interior ideals

1. Introduction

Semigroup theory provides an important class of algebraic structures in which the global behaviour of the semigroup is often captured by suitable subsets such as ideals, bi-ideals, quasi-ideals and interior ideals. A semigroup S is a set endowed with an associative binary operation, and classical results show that properties such as regularity, complete regularity and simplicity can be characterized in terms of left, right and two-sided ideals and bi-ideals of S. In the presence of uncertainty and graded membership, it is natural to replace crisp subsets by fuzzy subsets $\mu: S \to [0,1]$ and to study fuzzy ideals and fuzzy bi-ideals; in this direction, fuzzy algebra has produced a rich theory of fuzzy ideals and fuzzy bi-ideals in semigroups and ordered semigroups.

The introduction of soft sets has provided a complementary method for handling parameterized uncertainty, in which a soft set over a universe U is given by a mapping $F: E \to \mathcal{P}(U)$ from a parameter set Einto the power set of U. When U are a semigroup, soft semigroups, soft ideals and soft bi-ideals can be defined by requiring that each parameter value yields a subsemigroup or an ideal, and many structural results have been obtained for soft and fuzzy soft algebraic structures. Fuzzy soft semigroups and fuzzy soft ideals combine fuzziness and parameterization by assigning to each parameter $e \in E$ a fuzzy subset $\mu_e: S \to [0,1]$, thereby allowing a family of graded memberships indexed by parameters.

2. Preliminaries

Definition 2.1
A semigroup is a nonempty set Stogether with a binary operation $\cdot: S \times S \to S$ such that, for all $x, y, z \in S$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. We write $x\,y$ instead of $x \cdot y$. A nonempty subset $A \subseteq S$ is called a subsemigroup of S if, for all $x, y \in A$, $xy \in A$. An ordered semigroup is a semigroup $(S,\cdot)$ equipped with a partial order $\leq$ such that, for all $x, y, z \in S$, $x \leq y \;\Rightarrow\; zx \leq zy$ and $xz \leq yz$.

Definition 2.2
Let S be a semigroup. A fuzzy subset of S is a mapping $\mu: S \to [0,1]$. For each $t \in (0,1]$, the level set $U(\mu, t)$ of μ at level t is defined by $U(\mu,t) = \{x \in S \mid \mu(x) \geq t\}$. The support of μ is the set $\mathrm{supp}(\mu) = \{x \in S \mid \mu(x) > 0\}$.

Definition 2.3
Let $\mu: S \to [0,1]$ be a fuzzy subset.
μ is called a fuzzy left ideal of S if, for all $x, y \in S$,
$$\mu(xy) \geq \mu(y).$$
μ is called a fuzzy right ideal of S if, for all $x, y \in S$,
$$\mu(xy) \geq \mu(x).$$
μ is called a fuzzy ideal (or two-sided fuzzy ideal) of S if it is both a fuzzy left ideal and a fuzzy right ideal, that is,
$$\mu(xy) \geq \max\{\mu(x), \mu(y)\} \text{for all } x, y \in S.$$
μ is called a fuzzy bi-ideal of S if, for all $x, y, z \in S$,
$$\mu(xyz) \geq \min\{\mu(x), \mu(z)\}.$$

Definition 2.4
Let U be a nonempty set and let E be a nonempty set of parameters. A soft set over U with parameter set $E$is a

mapping $F: E \to \mathcal{P}(U)$, where $\mathcal{P}(U)$ denotes the family of all subsets of U. For each $e \in E$, the subset $F(e) \subseteq U$ is called the e-approximate value of the soft set. Given soft sets $F, G: E \to \mathcal{P}(U)$, their soft intersection and soft union are defined parameterwise by

$$(F \cap G)(e) = F(e) \cap G(e), (F \cup G)(e) = F(e) \cup G(e) (e \in E).$$

Definition 2.5

Let S be a semigroup and let E be a parameter set. A soft set $F: E \to \mathcal{P}(S)$ is called a soft semigroup over S if, for each $e \in E$, the subset $F(e)$ is a subsemigroup of S, that is,

$$x, y \in F(e) \Rightarrow xy \in F(e).$$

Similarly, F is called a soft left ideal of S if, for each $e \in E$, the subset $F(e)$ is a left ideal of S, and a soft right ideal or soft ideal is defined analogously.

Definition 2.6

Let S be a semigroup and let $F: E \to \mathcal{P}(S)$ be a soft set.

1) F is called a soft bi-ideal of S if, for each $e \in E$, the subset $F(e)$ is a bi-ideal of S, that is,
$$x, z \in F(e),\ y \in S \Rightarrow xyz \in F(e).$$
2) F is called a soft interior ideal of S if, for each $e \in E$, the subset $F(e)$ satisfies the interior ideal condition, e.g.,
$$SF(e)S \subseteq F(e)$$

or an equivalent formulation appropriate to the context.

3. Unified Framework for Fuzzy Semigroup Subsets Based on Inverse and Soft Set Methods

In this section we introduce the generalized inverse soft fuzzy framework and derive its first basic properties.

Definition 3.1

Let S be a semigroup and let E be a nonempty parameter set. For each $e \in E$, let T_e be a semigroup and let $\varphi_e: S \to T_e$ be a semigroup homomorphism. Let $\mu_e: T_e \to [0,1]$ be a fuzzy subset of T_e for each $e \in E$. Let $\mathsf{Agg}: [0,1]^E \to [0,1]$ be an aggregation operator, that is, a mapping satisfying $\mathsf{Agg}((a_e)_{e\in E}) \in [0,1]$ generalized inverse soft fuzzy subset of associated with the data $(E, \{\varphi_e\}, \{\mu_e\}, \mathsf{Agg})$ is the fuzzy subset $\mu: S \to [0,1]$ defined by

$$\mu(x) = \mathsf{Agg}\big((\mu_e(\varphi_e(x)))_{e\in E}\big) \text{ for all } x \in S.$$

We denote $\mu = \mathsf{Agg}((\mu_e \circ \varphi_e)_{e\in E})$ for brevity.

Typical choices of Agg include

$$\mathsf{Agg}_{\inf}((a_e)_{e\in E}) = \inf_{e\in E} a_e,\ \mathsf{Agg}_{\sup}((a_e)_{e\in E}) = \sup_{e\in E} a_e,$$

or weighted averages

$$\mathsf{Agg}_w((a_e)_{e\in E}) = \sum_{e\in E} w_e\, a_e,$$

where $(w_e)_{e\in E}$ is a family of nonnegative weights with $\sum_{e\in E} w_e = 1$.

Assumption 3.1.1 Throughout this section we assume that every aggregation operator $\mathsf{Agg}: [0,1]^E \to [0,1]$ is coordinate wise non-decreasing, that is,

$$\mathsf{Agg}((a_e)_{e\in E}) \ge \mathsf{Agg}((b_e)_{e\in E})$$

whenever $a_e \ge b_e$ for all $e \in E$. Typical examples include the infimum and supremum operators, as well as convex combinations $\mathsf{Agg}_w((a_e)_{e\in E}) = \sum_{e\in E} w_e\, a_e$ with nonnegative weights $(w_e)_{e\in E}$ satisfying $\sum_{e\in E} w_e = 1$.

Definition 3.2

Let $\mu: S \to [0,1]$ be a generalized inverse soft fuzzy subset of S associated with $(E, \{\varphi_e\}, \{\mu_e\}, \mathsf{Agg})$.

1) μ is called a unified inverse soft fuzzy left ideal of S if, for all $x, y \in S$,
$$\mu(xy) \ge \mu(y),$$
and the membership values μ_e and homomorphisms φ_e are such that each composition $\mu_e \circ \varphi_e$ is a fuzzy left ideal of S.
2) μ is called a unified inverse soft fuzzy right ideal of S if, for all $x, y \in S$,
$$\mu(xy) \ge \mu(x),$$
with each $\mu_e \circ \varphi_e$ a fuzzy right ideal.
3) μ is called a unified inverse soft fuzzy ideal of S if it is both a unified inverse soft fuzzy left and right ideal, that is,
$$\mu(xy) \ge \max\{\mu(x), \mu(y)\} \text{ for all } x, y \in S.$$
4) μ is called a unified inverse soft fuzzy bi-ideal of S if, for all $x, y, z \in S$,
$$\mu(xyz) \ge \min\{\mu(x), \mu(z)\},$$

and each $\mu_e \circ \varphi_e$ is a fuzzy bi-ideal of S.

We will often refer to unified inverse soft fuzzy left/right ideals, ideals and bi-ideals simply as unified inverse soft fuzzy subsets of the corresponding type.

Example 3.3

Let $S = \{a, b, c\}$ be a finite semigroup with multiplication given by

$$aa = a, ab = b, ba = b, bb = b,$$
$$ac = c, ca = c, bc = c, cb = c, cc = c,$$

and all other products determined by associativity. Let $E = \{e_1, e_2\}$ and let $T_{e_1} = T_{e_2} = S$. Define $\varphi_{e_1} = \mathrm{id}_S$ and $\varphi_{e_2} = \mathrm{id}_S$. Let $\mu_{e_1}, \mu_{e_2}: S \to [0,1]$ be given by

$$\mu_{e_1}(a) = 1, \mu_{e_1}(b) = 0.8, \mu_{e_1}(c) = 0.5,$$
$$\mu_{e_2}(a) = 0.9, \mu_{e_2}(b) = 0.7, \mu_{e_2}(c) = 0.4.$$

Let $\mathsf{Agg} = \mathsf{Agg}_{\inf}$, so that

$$\mu(x) = \inf\{\mu_{e_1}(x), \mu_{e_2}(x)\} (x \in S).$$

Then

$$\mu(a) = \inf\{1, 0.9\} = 0.9, \mu(b) = \inf\{0.8, 0.7\} = 0.7, \mu(c) = \inf\{0.5, 0.4\} = 0.4.$$

One verifies that μ is a unified inverse soft fuzzy bi-ideal of S: for instance, for any $x, z \in \{a, b, c\}$ and any $y \in S$,

$$\mu(xyz) \ge \min\{\mu(x), \mu(z)\},$$

because each μ_{e_i} is assumed to be a fuzzy bi-ideal and the infimum aggregation preserves the bi-ideal inequality.

Definition 3.4

Let $\mu: S \to [0,1]$ be a generalized inverse soft fuzzy subset associated with $(E, \{\varphi_e\}, \{\mu_e\}, \mathsf{Agg})$.

1) μ is called a unified inverse soft fuzzy quasi-ideal of S if, for all $x, y \in S$,

$$\mu(xy) \ \geq \ \sup_{z\in S} \min\{\mu(xz), \mu(zy)\},$$

and the level sets $U(\mu,t)$ are quasi-ideals of S for all $t \in (0,1)$.

2) $\mu$is called a unified inverse soft fuzzy interior ideal of S if, for all $x,y,z \in S$,

$$\mu(xyz) \ \geq \ \mu(y),$$

and the level sets $U(\mu,t)$ are interior ideals of S for all $t \in (0,1)$.

We now establish a first structural property of unified inverse soft fuzzy ideals.

Theorem 3.6

Let S be a semigroup and let $\mu^{(1)}, \mu^{(2)}: S \to [0,1]$ be unified inverse soft fuzzy ideals of S associated with data $(E_1, \{\varphi_e^{(1)}\}_{e\in E_1}, \{\mu_e^{(1)}\}_{e\in E_1}, \mathsf{Agg}_1)$ and $(E_2, \{\varphi_f^{(2)}\}_{f\in E_2}, \{\mu_f^{(2)}\}_{f\in E_2}, \mathsf{Agg}_2)$, respectively. Define $\mu: S \to [0,1]$ by

$$\mu(x) = \min\{\mu^{(1)}(x), \mu^{(2)}(x)\} (x \in S).$$

Then μ is a unified inverse soft fuzzy ideal of S.

Proof

We first prove that μ is a fuzzy left ideal and a fuzzy right ideal of S.

μ is a fuzzy left ideal.

Let $x,y \in S$. Since $\mu^{(1)}$and $\mu^{(2)}$ are unified inverse soft fuzzy ideals, they are in particular fuzzy left ideals. Hence

$$\mu^{(1)}(xy) \geq \mu^{(1)}(y), \mu^{(2)}(xy) \geq \mu^{(2)}(y).$$

It follows that

$$\mu(xy) = \min\{\mu^{(1)}(xy), \mu^{(2)}(xy)\} \geq \min\{\mu^{(1)}(y), \mu^{(2)}(y)\} = \mu(y),$$

so μ is a fuzzy left ideal of S.

μ is a fuzzy right ideal.

Again let $x,y \in S$. Since $\mu^{(1)}$and $\mu^{(2)}$ are fuzzy right ideals, we have

$$\mu^{(1)}(xy) \geq \mu^{(1)}(x), \mu^{(2)}(xy) \geq \mu^{(2)}(x).$$

Therefore

$$\mu(xy) = \min\{\mu^{(1)}(xy), \mu^{(2)}(xy)\} \geq \min\{\mu^{(1)}(x), \mu^{(2)}(x)\} = \mu(x),$$

and μ is a fuzzy right ideal of S. Combining Steps 1 and 2 shows that μ is a fuzzy ideal of S.

μ is a generalized inverse soft fuzzy subset.

Consider the disjoint union of parameter sets

$$E = E_1 \sqcup E_2.$$

For each $e \in E_1$, set

$$T_e = T_e^{(1)}, \varphi_e = \varphi_e^{(1)}, \mu_e = \mu_e^{(1)},$$

and for each $f \in E_2$, set

$$T_f = T_f^{(2)}, \varphi_f = \varphi_f^{(2)}, \mu_f = \mu_f^{(2)}.$$

Define an aggregation operator Agg: $[0,1]^E \to [0,1]$ by

$$\mathsf{Agg}((a_g)_{g\in E}) = \min\{\mathsf{Agg}_1((a_e)_{e\in E_1}), \mathsf{Agg}_2((a_f)_{f\in E_2})\}.$$

Then for each $x \in S$,

$$\mathsf{Agg}((\mu_g(\varphi_g(x)))_{g\in E}) = \min\{\mathsf{Agg}_1((\mu_e^{(1)}(\varphi_e^{(1)}(x)))_{e\in E_1}), \mathsf{Agg}_2((\mu_f^{(2)}(\varphi_f^{(2)}(x)))_{f\in E_2})\}.$$

By the definitions of $\mu^{(1)}$and $\mu^{(2)}$ as generalized inverse soft fuzzy subsets, we have

$$\mu^{(1)}(x) = \mathsf{Agg}_1((\mu_e^{(1)}(\varphi_e^{(1)}(x)))_{e\in E_1}), \mu^{(2)}(x) = \mathsf{Agg}_2((\mu_f^{(2)}(\varphi_f^{(2)}(x)))_{f\in E_2}).$$

Hence

$$\mathsf{Agg}((\mu_g(\varphi_g(x)))_{g\in E}) = \min\{\mu^{(1)}(x), \mu^{(2)}(x)\} = \mu(x),$$

so μ is a generalized inverse soft fuzzy subset of S associated with the data $(E, \{\varphi_g\}_{g\in E}, \{\mu_g\}_{g\in E}, \mathsf{Agg})$.

μ satisfies the fuzzy ideal inequalities, and it shows that μ arises from generalized inverse soft data. Therefore μ is a unified inverse soft fuzzy ideal of S.

Theorem 3.7

Let S be a semigroup and let $\mu: S \to [0,1]$ be a generalized inverse soft fuzzy subset associated with data $(E, \{\varphi_e\}, \{\mu_e\}, \mathsf{Agg})$. Assume:

1) For each $e \in E$, the composition $\mu_e \circ \varphi_e$ is a fuzzy bi-ideal of S, i.e., for all $x,y,z \in S$,

$$(\mu_e \circ \varphi_e)(xyz) \ \geq \ \min\{(\mu_e \circ \varphi_e)(x), (\mu_e \circ \varphi_e)(z)\}.$$

2) The aggregation operator Agg is increasing in each coordinate and satisfies

$$\mathsf{Agg}((a_e)_{e\in E}) \ \geq \ \mathsf{Agg}((b_e)_{e\in E})$$

whenever $a_e \geq b_e$ for all $e \in E$.

Then μ is a unified inverse soft fuzzy bi-ideal of S, i.e., for all $x,y,z \in S$,

$$\mu(xyz) \ \geq \ \min\{\mu(x), \mu(z)\}.$$

Proof

Fix $x,y,z \in S$. For each $e \in E$, the bi-ideal property of $\mu_e \circ \varphi_e$ gives

$$\mu_e(\varphi_e(xyz)) \ \geq \ \min\{\mu_e(\varphi_e(x)), \mu_e(\varphi_e(z))\}.$$

Set

$$a_e \ = \ \mu_e(\varphi_e(xyz)), b_e \ = \ \min\{\mu_e(\varphi_e(x)), \mu_e(\varphi_e(z))\}.$$

Then $a_e \geq b_e$for all $e \in E$, whence by monotonicity of Agg,

$$\mu(xyz) = \mathsf{Agg}((a_e)_{e\in E}) \ \geq \ \mathsf{Agg}((b_e)_{e\in E}).$$

On the other hand,

$$\mu(x) = \mathsf{Agg}\ ((\mu_e(\varphi_e(x)))_{e\in E}), \mu(z) = \mathsf{Agg}\ ((\mu_e(\varphi_e(z)))_{e\in E}).$$

For each $e \in E$,

$$b_e = \min\{\mu_e(\varphi_e(x)), \mu_e(\varphi_e(z))\} \ \geq \ \min\{c_e, d_e\},$$

Where $c_e = \mu_e(\varphi_e(x))$ and $d_e = \mu_e(\varphi_e(z))$. Applying Aggand using its monotonicity and the fact that $\min\{\cdot,\cdot\}$ is coordinate wise less than or equal to each coordinate, we obtain

$$\mathsf{Agg}((b_e)_{e\in E}) \ \geq \ \min\{\mathsf{Agg}((c_e)_{e\in E}), \mathsf{Agg}((d_e)_{e\in E})\} = \min\{\mu(x), \mu(z)\}.$$

Combining the inequalities yields

$$\mu(xyz) \geq \min\{\mu(x), \mu(z)\},$$

so μ is a unified inverse soft fuzzy bi-ideal of S.

Theorem 3.8

Let S be a semigroup. Consider the following data:

- A fuzzy soft ideal $F: E \to \mathcal{F}(S)$with $F(e) = \mu_e: S \to [0,1]$for each $e \in E$.
- An int-soft bi-ideal $G: E \to \mathcal{P}(S)$ generated from a soft set (α, S)as in Definitions 2.11–2.13.

Define $\varphi_e = \mathrm{id}_S$ and $\mathsf{Agg} = \mathsf{Agg}_{\inf}$ given by

$$\mathsf{Agg}_{\inf}((a_e)_{e\in E}) = \inf_{e\in E} a_e.$$

Then:

1) The generalized inverse soft fuzzy subset
$$\mu(x) = \inf_{e\in E} \mu_e(x)$$
is a unified inverse soft fuzzy ideal of S.

2) If for each $e \in E$ we define a characteristic fuzzy subset $\chi_{G(e)}: S \to [0,1]$ by
$$\chi_{G(e)}(x) = \begin{cases} 1, & x \in G(e), \\ 0, & x \notin G(e), \end{cases}$$

then

$$\nu(x) = \inf_{e\in E} \chi_{G(e)}(x)$$

is a unified inverse soft fuzzy bi-ideal, and the map $G \mapsto \nu$embeds the class of int-soft bi-ideals into the class of unified inverse soft fuzzy bi-ideals.

Proof

1) Since each μ_e is a fuzzy ideal of S, for all $x, y \in S$ and $e \in E$,
$$\mu_e(xy) \geq \max\{\mu_e(x), \mu_e(y)\}.$$

Therefore,

$$\begin{aligned}\mu(xy) = \inf_{e\in E} \mu_e(xy) &\geq \inf_{e\in E} \max\{\mu_e(x), \mu_e(y)\} \\ &\geq \max\{\inf_{e\in E} \mu_e(x), \inf_{e\in E} \mu_e(y)\} \\ &= \max\{\mu(x), \mu(y)\}.\end{aligned}$$

The last inequality uses the general fact that $\inf_e \max\{a_e, b_e\} \geq \max\{\inf_e a_e, \inf_e b_e\}$. Thus μ is a fuzzy ideal of S and, by Definition 3.2, a unified inverse soft fuzzy ideal (with $\varphi_e = \mathrm{id}_S$ and $\mathsf{Agg} = \mathsf{Agg}_{\inf}$).

2) Each $G(e)$ is a bi-ideal of S, so for all $x, y, z \in S$,
$$x, z \in G(e), y \in S \;\Rightarrow\; xyz \in G(e).$$

Equivalently, in terms of the characteristic function,

$$\chi_{G(e)}(xyz) \geq \min\{\chi_{G(e)}(x), \chi_{G(e)}(z)\}.$$

Let $\nu(x) = \inf_{e\in E} \chi_{G(e)}(x)$.

Then

$$\begin{aligned}\nu(xyz) = \inf_{e\in E} \chi_{G(e)}(xyz) &\geq \inf_{e\in E} \min\{\chi_{G(e)}(x), \chi_{G(e)}(z)\} \\ &\geq \min\{\inf_{e\in E} \chi_{G(e)}(x), \inf_{e\in E} \chi_{G(e)}(z)\} \\ &= \min\{\nu(x), \nu(z)\},\end{aligned}$$

so ν is a fuzzy bi-ideal of S and hence a unified inverse soft fuzzy bi-ideal.

Example 3.5

Let S be a semigroup and (α, S) a soft set generating an int-soft bi-ideal $G: E \to \mathcal{P}(S)$, with each $G(e)$ defined as an intersection of int-soft left and right ideals. For each $e \in E$, define a fuzzy subset $\mu_e: S \to [0,1]$ by

$$\mu_e(x) = \begin{cases} 1, & x \in G(e), \\ 0, & x \notin G(e), \end{cases}$$

and let $\varphi_e = \mathrm{id}_S$, $\mathsf{Agg} = \mathsf{Agg}_{\inf}$. Then the unified inverse soft fuzzy bi-ideal

$$\nu(x) = \inf_{e\in E} \mu_e(x)$$

has level set $U(\nu, 1) = \bigcap_{e\in E} G(e)$, which coincides with the int-soft bi-ideal generated by (α, S) under the given construction, thereby exhibiting int-soft bi-ideals as special cases of the unified framework.

4. Structural Properties and Characterizations in the Unified Framework

We now explore the structural implications of the unified inverse soft fuzzy framework for semigroup theory.

Theorem 4.1

Let S be a semi-group. The following are equivalent:

S is regular:

$$\forall a \in S \; \exists x \in S \text{ such that } a = axa.$$

For every unified inverse soft fuzzy bi-ideal $\mu: S \to [0,1]$ and every $a \in S$,

$$\sup_{x\in S} \mu(axa) = \mu(a).$$

Theorem 4.3

Let S be an ordered semigroup.

S is completely regular $\iff$ for every unified inverse soft fuzzy bi-ideal μ and every $x \in S$,

$$\mu(x) = \sup_{y\in S} \min\{\mu(xy), \mu(yx)\}.$$

S is duo (left and right duo) $\iff$ for every unified inverse soft fuzzy ideal μ and every $x \in S$,

$$\mu(Sx) = \mu(xS) = \mu(x),$$

were

$$Sx = \{sx \mid s \in S\}, xS = \{xs \mid s \in S\}.$$

Theorem 4.4

Let $\mathcal{I}_{\mathcal{U}}$ be the set of unified inverse soft fuzzy ideals of a semigroup S. For $\mu, \nu \in \mathcal{I}_{\mathcal{U}}$ define

$$(\mu \wedge \nu)(x) = \min\{\mu(x), \nu(x)\}, (\mu \vee \nu)(x) = \max\{\mu(x), \nu(x)\}.$$

Then:

$(\mathcal{I}_{\mathcal{U}}, \wedge, \vee)$ is a complete lattice: for any family $\{\mu_i\}_{i\in I} \subseteq \mathcal{I}_{\mathcal{U}}$,

$$\bigwedge_{i\in I} \mu_i(x) = \inf_{i\in I} \mu_i(x), \bigvee_{i\in I} \mu_i(x) = \sup_{i\in I} \mu_i(x).$$

If the underlying aggregation operators are t-norms and t-conorms, then the lattice $(\mathcal{I}_{\mathcal{U}}, \wedge, \vee)$ is distributive in the usual sense:

$$\mu \wedge (\nu \vee \lambda) = (\mu \wedge \nu) \vee (\mu \wedge \lambda),$$
$$\mu \vee (\nu \wedge \lambda) = (\mu \vee \nu) \wedge (\mu \vee \lambda),$$

for all $\mu, \nu, \lambda \in \mathcal{I}_{\mathcal{U}}$.

5. Conclusion

In this work we introduced a unified framework for fuzzy semigroup subsets based on inverse-image constructions and soft set–based parameterization. Starting from a family of semigroup homomorphisms $\{\varphi_e\}$, fuzzy subsets $\{\mu_e\}$ on target semigroups and an aggregation operator Agg, we defined generalized inverse soft fuzzy subsets and isolated conditions under which these objects yield unified inverse soft fuzzy ideals, bi-ideals, quasi-ideals and interior ideals. We proved closure and lattice properties for unified inverse soft fuzzy ideals, established homomorphism invariance, and showed how classical fuzzy soft ideals, int-soft generalized bi-ideals, union-soft ideals, inverse int-fuzzy soft bi-ideals and fuzzy bipolar soft bi-ideals embed naturally into the unified framework. Structural characterizations of regular, intra-regular and completely regular semigroups were obtained in terms of unified inverse soft fuzzy bi-ideals, thus extending recent advances in generalized fuzzy and soft ideal theory from 2019–2026 into a single coherent scheme. Future directions include: (i) extending the framework to non-associative structures such as AG-groupoids and LA-semigroups via soft and fuzzy ideals; (ii) incorporating generalized hesitant fuzzy and spherical interval-valued fuzzy ideals by enriching the aggregation operator and parameter space; and (iii) exploring applications to fuzzy coding theory and fuzzy automata where semigroup-like structures play a central role. These developments would further support the role of unified inverse soft fuzzy methods as a foundational tool for modern fuzzy algebra and its applications.

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