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# On the Existence of Semi-Parallel Vector Fields in a Finsler Space

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**Abstract:** ***This paper is concerned with the existence of semi parallel vector field in a Finsler space. Semi paralleled vector field in a Riemannian manifold was studied by C.M. Fulton and M. Hashiguchi, while semi parallel vector field and normalized semi parallel vector field in a Finsler space were defined and studied by Singh and Prasad. In this chapter while studying the existence of semi parallel vector field and normalized semi parallel vector field analogous to the study of the existence of concurrent vector field, it has also been shown that the definition given by Singh and Prasad, of semi parallel vector field and normalized semi parallel vector field. In a Finsler space in its present form is untenable. We have given a modified definition of semi-parallel vector field and normalized semi-parallel vector field in a Finsler space and studied some of their properties vis-a-vis other will know as Finsler space such as C2-like, P2-like, C-reducible, R-reducible, P-reducible and S3-like etc.***

**Keywords:** vector field, Finsler space, Semi-parallel vector field, Finsler connection, Curvature tensor, Differential geometry

## 1. Introduction

Semi-parallel vector field in a Riemannian manifold was studied by Fulton [1] and Hashiguchi [2] while Semi-parallel vector field and normalized semi-parallel vector field in a Finsler space were defined and studied by Singh and Prasad [12]. In this paper while studying the existence of semi-parallel vector field and normalized semi-parallel vector field analogous to the study of the existence of concurrent vector field [9], it has been shown that the definition given by Singh and Prasad [12], of semi-parallel vector fields and normalized semi-parallel vector fields in a Finsler space in its present form is untenable. We have given a modified definition of semi-parallel vector field and normalized semi-parallel vector field in a Finsler space and studied some of their properties vis-a-vis other well known Finsler spaces such as C2-like [8], P2-like [6], C-reducible [7], P-reducible [10] and S3-like [6] etc.

Let $F_n$ be an n-dimensional Finsler space with metric function $L(x, y)$, metric tensor $g_{ij}(x, y)$, angular metric tensor $h_{ij}$ and torsion tensor $C_{ijk}$. The h- and v-covariant derivatives of a vector field $X_i$ are defined as Rund [11].

$$\text{(a) } X^i{}_{|j} = \partial_j X^i - N^r_j \Delta_r X^i + X^r F^i_{rj},$$
$$\text{(b) } X^i{}_{|j} = \Delta_j X^i + X^r C^i_{rj}, \tag{1.1}$$

where $N^r_j = F^r_{oj}, \partial_j$ and $\Delta_j$ respectively denote partial differentiation with respect to $x^j$ and $y^j$, such that an index $o$ means contraction by $y$.

The semi-parallel and normalized semi-parallel vector fields in a Finsler space were defined by Singh and Prasad [12] as follows:

**Definition 1.1 -** A non-zero vector field $V^j(x)$, in a Finsler space is said to be semi-parallel if there exist scalars $\lambda(x) = 0$ and $\varepsilon(x)$ satisfying the conditions

$$\text{(a) } C_{jhk} V^j = 0$$
$$\text{(b) } V^j{}_{|k} = \lambda(\delta^j_k + \varepsilon V^j V_k), \tag{1.2}$$

where $C_{jhk}$ is the (h)hv- torsion tensor.

**Definition 1.2 -** A non-zero unit vector field $X^j(x)$, in a Finsler space is said to be normalized semi-parallel if there exist scalars $\rho(x) \neq 0$ satisfying the conditions

$$\text{(a) } C_{jhk} X^j = 0,$$
$$\text{(b) } X^j{}_{|k} = \rho(\delta^j_k - X^j X_k) \tag{1.3}$$

## 2. Existence of Semi-Parallel Vector Fields

To study the existence of semi-parallel vector fields, we shall consider some special cases:

**Case I. Two-dimensional Finsler spaces $F_2$:** Any vector field in $F_2$ is expressed as

$$V^j l^j \cos\theta + m^j \sin\theta \tag{2.1}$$

Where $V^j$ makes an angle θ with $l^j$. Assume that in $F_2$ there exists a non-zero vector field $V^j(x)$ satisfying $V^j C_{jhk} = 0$, which implies, $V^j C_j = 0$. This from equation (2.1) gives $C \sin\theta$, i.e., either $C=0$, implying the given space is a Riemannian space or Sin $\theta = 0$, implying $V^j = l^j$, which is a contradiction. From Definition 1.1, we can observe that $V^j(x)$ is a semi-parallel vector field satisfying $V^j C_{jhk} = 0$, therefore we have:

**Theorem 2.1:** In a two-dimensional non-Riemannian Finsler space $F_2$, a semi-parallel vector field satisfying Definition 1, does not exist.

Furthermore, if we take h-covariant derivative of $C_j = C\, m_j$ in a Finsler space of two-dimensions and use $m_{j|k} = 0$, we get $C_{j|k} = C_{|k} m_j$. Differentiating $V^j C_j = 0$ and using Definition 1.1, we get $\lambda(\delta^j_k + \varepsilon V^j V_k) C_j + V^j C_{|k} m_j = 0$, which easily leads to $C\, C_k \lambda = 0$. As $\lambda$ is a non-zero scalar by choice, therefore we shall have either $C=0$ or $C_j=0$, which is true only if the space reduces to a Riemannian space. Hence we have :

**Theorem 2.2:** If $V^j$ is a semi-parallel vector field in $F_2$, satisfying Definition 1.1, the Finsler space reduces to a Riemannian space.

Therefore, we can have $C_{j|k} = C_{|k} m_j + C\, n_j h_k$. Differentiating $V^j C_j = 0$, we get $V^j (C_{|k} m_j + C n_j h_k) + \lambda(\delta^j_k + \varepsilon V^j V_k) C_j = 0$, which implies $C(C V^j n_j h_k + \lambda C_k) = 0$. Further multiplication by $y^k$ gives $C^2 V^j n_j h_0 = 0$, which implies:

1) Either C = 0 i.e., the space is Riemannian, or
2) $V^j n_j = 0$ i.e. $\cos\psi = 0$, which means $V^j = l^j \cos\theta$, a contradiction, or
3) $h_0 = 0$.

To analyze the third condition, we need to take $V^j P_{jkh} = 0$, where $P_{jkh} = C_{jkh|0}$, Rastogi and Kawaguchi [10]. This on simplification leads to $L(V^j n_j)[C_{(2)|0}(n_k n_h - m_k m_h) + C_{(3)|0}(m_k n_h + m_h n_k)] = 0$, which on multiplication by $m_k m_h$ and $n_k n_h$ respectively gives $(V^j n_j) C_{(2)|0} = 0$ and $(V^j n_j) C_{(3)|0} = 0$. As $C_{(2)|0}$ and $C_{(3)|0}$ are not zero, we get $(V^j n_j) = 0$, which is nothing but the second condition. Hence we have:

**Theorem 2.3:** In a three-dimensional non-Riemannian Finsler space $F_3$, semi-parallel vector field satisfying Definition 1.1, does not exist.

**Case II. Three-dimensional Finsler space $F_3$:** Any vector field $V^j$ in a three-dimensional Finsler space $F_3$ is expressed as

$$V^j = l^j \cos\theta + m^j \cos\phi + n^j \cos\psi \qquad (2.2)$$

where cos θ, cos ϕ and cos ψ are direction cosines of this vector.

As from Definition 1.1, $V^j C_{jhk} = 0$, equation (2.2) gives cos $\phi=0$, i.e., $V^j = l^j \cos\theta + n^j \cos\psi$. Since we know that [10] in $F_3$.

$$C_{rjk} = C_{(1)} m_r m_j m_k - C_2(m_r m_j n_k + m_j m_k n_r + m_k m_r n_j - n_r n_j n_k) + C_{(3)}(m_r n_j n_k + m_j n_k n_r + m_k n_r n_j), \qquad (2.3)$$

In a C-reducible Finsler space $F_n$, Matsumoto [7], $C_{jkh}$ is expressed as

$$C_{jkh} = (C_j h_{kh} + C_k h_{hj} + C_h h_{jk})/(n+1) \qquad (2.4)$$

If $V^j$ is a semi-parallel vector field, multiplying (2.4) by $V^j$ and using Definition 1.1, we get on simplification

$$C_h V_k + C_k V_h - (C_h l_k + C_k l_h) V^j l_j = 0 \qquad (2.5)$$

which gives $V_k = V^j l_j l_k$. Differentiating this result with the help of equation (1.1) a and applying Definition 1.1, we get $h_{jk} = 0$. Hence we have:

**Theorem 2.4:** A C-reducible Finsler space of n-dimensions $F_n$, with a semi-parallel vector field $V^j$ reduces to a Riemannian space.

**Remark:** Theorems of this section are also true for normalized semi-parallel vector field.

## 3. Some Curvature Properties

If we differentiate $V^r C_{rjk} = 0$, by Cartan's covariant derivative it gives

$$V^r C_{rjk|m} = -\lambda C_{jkm} \qquad (3.1)$$

which implies

$$V^r P_{rjk} = 0 \qquad (3.2)$$

Using Ricci-identities [11]

$$X^r{}_{|k|m} - X^r{}_{|m|k} = X^h R_h{}^r{}_{km} - X^r{}_{|h} R^h{}_{km}, \qquad (3.3)$$

for a semi-parallel vector field, we obtain

$$V^h R_h{}^r{}_{km} = \varsigma_{(k,m)} \{\delta^r_k(\lambda_{|m} - \lambda^2 \varepsilon V_m) + \varepsilon V^r \lambda_{|m} V_k\}, \qquad (3.4)$$

where $\varsigma_{(k,m)}$ means change of index $k$ and $m$ and subtraction.

From equation (3.4), on contraction in *r* and *m*, we get

$$V^h R_{hk} = (1-n)(\lambda_{|k} - \lambda^2 \varepsilon V_k) + \varepsilon V^m (\lambda_{|m} V_k - \lambda_{|k} V_m) \quad (3.5)$$

Similarly from the Ricci-identities [11]

$$X^j{}_{|k}\big|_m - X^j\big|_{m|k} = X^h P_h{}^j{}_{km} - X^j{}_{|h} A^h{}_{km} - X^j\big|_h P^h{}_{km}, \quad (3.6)$$

we can obtain on simplification

$$V^h P_h{}^j{}_{km} = \lambda A^j_{km}, \; V^h V^k P_h{}^j{}_{km} = 0 \quad (3.7)$$

Using T-tensor defined independently by Matsumoto [8] and Kawaguchi [4]

$$T_{hijk} = L C_{hij}\big|_k + l_h C_{ijk} + l_i C_{hjk} + l_j C_{hik} + l_k C_{hij}, \quad (3.8)$$

we can easily obtain $V^i V^h T_{hijk} = 0$, which implies Matsumoto [8]

**Theorem 3.1:** An n-dimensional Finsler space $F_n$, admitting a semi-parallel vector field is a Riemannian space.

## 4. Modified Definition of Semi-Parallel Vector Fields in a Finsler Space

**Definition 4.1:** A non-zero vector field $V^j(x)$, in a Finsler space is said to be semi-parallel if there exist non-zero scalars $\lambda(x), \alpha(x,y)$ and $\varepsilon(x)$ satisfying the conditions

(a) $V^r A_{rjk} = \alpha h_{jk}$

(b) $V^j{}_{|k} = \lambda(\delta^j_k + \varepsilon V^j V_k)$ (4.1)

such that $A_{rjk} = L C_{rjk}$.

**Remark:** If we assume that □ in Definition 4.1 is a function of *x* alone, equation (4.1)(a), implies that the vector $V_h$ is an h-vector defined and studied by Izumi [3], while studying conformal transformations of Finsler spaces.

**Definition 4.2:** A non-zero unit vector field $X^j(x)$, in a Finsler space is said to be normalized semi-parallel if there exist non-zero scalars $\rho(x)$ and $\alpha(x,y)$ satisfying the conditions

(a) $X^r A_{rjk} = \alpha h_{jk}$

(b) $X^j{}_{|k} = \rho(\delta^j_k - X^j X_k)$ (4.2)

In a two-dimensional Finsler space $A_{rjk}$ is expressed as [11]

$$A_{rjk} = LC m_r m_j m_k \quad (4.3)$$

therefore equation (4.3) with the help of (2.1) and Definition 4.1, easily gives

**Theorem 4.1:** In a two-dimensional Finsler space $F_2$, admitting a semi-parallel vector field satisfying Definition 4.1, α=LC sin θ.

Applying Definition 4.1 to equation (2.2), and using equation (2.3), we get LC cos ϕ = 2α. Hence we have:

**Theorem 4.2:** In a three-dimensional Finsler space $F_3$, admitting a semi parallel vector field satisfying Definition 4.1, LC cos ϕ = 2α.

Using $V^r A_r = LC \cos\phi = 2\alpha$, on applying Cartan's covariant differentiation and contraction, we get on simplification

$$h_0 \cos\psi = \lambda V_0 \cos\phi + (\cos\phi)_{|0} \quad (4.4)$$

Hence we have :

**Theorem 4.3:** In a three-dimensional Finsler space $F_3$, admitting a semi-parallel vector field satisfying Definition 4.1, angles ϕ and ψ are related by (4.4).

**Remark:** If in Definition 4.1, λ=-1 and ε=0, a semi-parallel vector field reduces to a concurrent vector field Rastogi and Dwivedi [9].

Assuming $V^j_{|k} = 0$, equation (4.1)(b) gives $V^j_{|j} = 0$ if and only if $V_k V^k = -n/\varepsilon$, but this in turn implies $V^j(n-1) = 0$, i.e., either V=0 or n=1. Since V is non-zero, therefore we have:

**Theorem 4.4:** In an n-dimensional Finsler space $F_n$($n \geq 2$), admitting a semi-parallel vector field V satisfying Definition 4.1, $V^j_{|k}$ can not vanish.

## 5. Some More Curvature Properties

With the help of equation (3.3) and Definition 4.1, we can obtain

$$V^r R_r{}^j{}_{kh} - L^{-1}\alpha(R^j_{kh} - y^j y_m R^m{}_{kh}) = \varsigma_{(k,h)}\{\lambda_{|h}(\delta^j_k + \varepsilon V^j V_k) + \varepsilon V^j_{|h} V_k + \lambda V^j \varepsilon_{|h} V_k\}, \quad (5.1)$$

which leads to

$$V^r R_{rk} - L^{-1}\alpha(R^j_{kj} - y_m R^m_{k0}) = (n-1)(\varepsilon\lambda V_k - \lambda_{|k}) + V^j \varsigma_{(kj)}\{V_k(\varepsilon\lambda)_{|j}\}. \quad (5.2)$$

Hence we have:

**Theorem 5.1:** In an n-dimensional Finsler space $F_n$ admitting a semi-parallel vector field satisfying Definition 4.1, $R_r{}^j{}_{kh}$ and $R_{rk}$ satisfy (5.1) and (5.2) respectively.

**Remark:** If we take normalized semi-parallel vector field $X^j$, equations (5.1) and (5.2) respectively reduce to

$$X^r R_{r\,kh}^{j} - L^{-1}\alpha(R_{kh}^{j} - y^j y_m R_{kh}^{m}) = \varsigma_{(k,h)}\{\rho_{|h}(\delta_k^j - X^j X_k) - X_{|h}^{j} X_k\} \quad (5.3)$$

and

$$X^r R_{rk} - L^{-1}\alpha(R_{kj}^{j} - y_m R_{k0}^{m}) = X^j \varsigma_{(kj)}\{X_j \rho_{|k}\} - (n-1)(\rho X_k + \partial_k) \quad (5.4)$$

Differentiating equation $V^j A_{jkm} = \alpha h_{km}$, we obtain on simplification

$$\varsigma_{(p,m)}\{V^j A_{jkm|p} - h_{km}(\alpha_{|p} - \lambda\varepsilon\alpha V_p)\} = 0. \quad (5.5)$$

From equation (5.5), we can easily obtain

**Theorem 5.2:** In an n-dimensional Finsler space $F^n$ admitting a semi-parallel vector field $V^r$, the necessary sufficient condition for the symmetry of $V^j A_{jkm|p}$ in $m$ and $p$ is that $\alpha_{|p} = \lambda\,\varepsilon\,\alpha\,V_p$.

If we assume $\lambda\,\varepsilon\,\alpha\,V_p = \alpha_{|p}$, from equation $V^j A_j = (n-1)\alpha$, we can obtain $V^j A_{j|h} = -\lambda A_h$, which by virtue of (3.3) leads to

$$V^r(A_m R_{r\,hk}^{m} - A_r\big|_p R_{hk}^{p} = \varsigma_{(h,k)}\{A_k(\lambda_{|h} - \lambda^2\varepsilon V_h)\}. \quad (5.6)$$

In a special cases where λ is proportional to □, we can observe that $\lambda^{-1}\lambda_{|h} = \alpha^{-1}\alpha_{|h}$, therefore equation (5.6) reduces to

$$V^r(A_m R_{r\,hk}^{m} - A_r\big|_p R_{hk}^{p}) = 0 \quad (5.7)$$

Hence we have:

**Theorem 5.3:** In an n-dimensional Finsler space $F_n$ admitting a semi-parallel vector field $V^r$, the necessary and sufficient condition for the curvature tensor $R_{r\,hk}^{m}$ to satisfy equation (5.7) is that scalar λ is proportional to □ and $\alpha_{|p} = \lambda\,\varepsilon\,\alpha\,V_p$.

Since we know that the curvature tensor is $S_{h\,jk}^{\ r}$ expressed as [8]

$$S_{h\,jk}^{\ r} = \varsigma_{(j,k)}\{A_{hk}^{m} A_{mj}^{r}\}, \quad (5.8)$$

therefore multiplying (5.8) by $V^h$ and applying Definition 4.1 a semi-parallel vector field, we get

$$V^h S_{h\,jk}^{\ r} = 0 \quad (5.9)$$

Differentiating (5.9) with respect to $y^m$ by Cartan's covariant derivative and applying Definition 4.1, we get

$$V^h S_{h\,jk|m}^{\ r} = -\lambda S_{m\,jk}^{\ r},\ V^h S_{h\,jk|0}^{\ r} = 0. \quad (5.10)$$

Hence, we have:

**Theorem 5.4:** In an n-dimensional Finsler space $F_n$ admitting a semi-parallel vector field $V^r$, curvature tensor $S_{h\,jk}^{\ r}$ satisfies (5.9) and (5.10).

Since we know that Rund [11]

$$P_{h\,jk}^{\ r} - P_{h\,kj}^{\ r} = -S_{h\,jk|0}^{\ r}, \quad (5.11)$$

therefore from (5.10) we can obtain

$$V^h(P_{h\,jk}^{\ r} - P_{h\,kj}^{\ r}) = 0. \quad (5.12)$$

Hence, we have:

**Theorem 5.5:** In an n-dimensional Finsler space $F_n$ admitting a semi-parallel vector field $V^r$, curvature tensor $P_{h\,jk}^{\ r}$ satisfies (5.12).

Since we know that [8], $V^h\big|_m\big|_k - V^h\big|_k\big|_m = V^r S_{r\,km}^{\ h} + \alpha(l_k\delta_m^h - l_m\delta_k^h)$, therefore using (5.9), we can observe that $V^h\big|_m\big|_k - V^h\big|_k\big|_m = \alpha(l_k\delta_m^h - l_m\delta_k^h)$, which on substituting $V^h\big|_m = \alpha h_m^h$ and simplification leads to $\alpha\big|_0 = L\,\alpha$. Hence, we have

**Theorem 5.6:** In an n-dimensional Finsler space $F_n$ admitting a semi-parallel vector field $V^r$, scalar α satisfies $\alpha\big|_0 = L\,\alpha$.

Since we know that [8] $V^{j}{}_{|k}\big|_m - V^{j}{}_{|m}\big|_k = V^h P_{h\,km}^{\ j}$, $-V^j\big|_h A^h{}_{km} - V^j\big|_h P^h{}_{km}$, therefore substituting from Definition 4.1, we obtain

$$V^h P_{h\,km}^{j} = \lambda\varepsilon\alpha(V_k h_m^j - 2V^j h_{km}) + \lambda A_{km}^{j} + \alpha P_{km}^{j} - \alpha\big|_k h_m^j$$

, which by virtue of equation (5.12) gives

$$\varsigma_{(m,k)}\{h^j{}_m(\lambda\,\varepsilon\,\alpha\,V_k - \alpha\big|_k)\} = 0. \quad (5.13)$$

Hence we have:

**Theorem 5.7:** An n-dimensional Finsler space $F_n$ admitting a semi-parallel vector field $V^r$, satisfies (5.13).

**Remark:** If we contract equation (5.13) for $j$ and $m$, it reduces to $\alpha|_0 = \lambda\,\varepsilon\,\alpha\,V_0$.

## 6. Some Special Cases

**Case I. C2-like Finsler Space:** In a C2-like Finsler space [8], $C_{jkh}$ is expressed as

$$C_{jkh} = C^{-2} C_j C_k C_h. \tag{6.1}$$

Applying Definition 4.1 and using equation (3.3) in (6.1) we obtain after a lengthy calculation for a semi-parallel vector field $V^r$

$$V^j(C_r R^r_{jkh} + C_{j|r} R^r_{kh}) = \varsigma_{(h,k)}[\lambda_{|h}\{C_k + (n-1)\varepsilon\lambda V_k\} - \lambda V_h\{(n-1)\alpha\varepsilon_{|k} + \lambda\varepsilon C_k\}]. \tag{6.2}$$

For a normalized semi-parallel vector field equation (6.2) reduces to

$$X^j(C_r R^r_{jkh} + C_{j|r} R^r_{kh}) = \varsigma_{(h,k)}[\rho_{|h}\{C_k - (n-1)\rho V_k\} + \rho^2 V_h C_k] \tag{6.3}$$

Hence, we have:

**Theorem 6.1:** An n-dimensional C2-like Finsler space $F_n$ admitting a normalized semi-parallel vector field $X^r$, satisfies (6.3).

**Case II: P2-like Finslr Spaces:** In a P2-like Finsler space curvature tensor $P_{hrjk}$ is expressed as [6]

$$P_{hrjk} = P_h C_{rjk} - P_r C_{hjk} \tag{6.4}$$

From equation (6.4), we can obtain

$$V^h P_{hrjk} = (n-1)L\, C_{rjk}(\alpha_{|0} - \lambda\,\varepsilon\,\alpha\,V_0) - \alpha\, P_r\, h_{jk}, \tag{6.5}$$

which for $*V_j = V_j - V^h l_j l_k$, implies

$$V^h V^k P_{hrjk} = (n-1)\,L\,\alpha\, h_{rj}(\alpha_{|0} - \lambda\,\varepsilon\,\alpha V_0) - \alpha\, P_r * V_j \tag{6.6}$$

Further from (6.4) we can get

$$V^h V^k (P_{hrjk} - P_{hjrk}) = \alpha(P_j * V_r - P_r * V_j). \tag{6.7}$$

Since we can easily find

$$P_{jrhk} V^h V^k = \alpha(P_j * V_r - P_r * V_j), \tag{6.8}$$

Therefore from (6.7) and (6.8), we can easily obtain following curvature identity

$$(P_{hrjk} + P_{rjhk} + P_{jhrk}) V^h V^k = 0 \tag{6.9}$$

Hence, we have:

**Theorem 6.2:** An n-dimensional P2-like Finsler space $F_n$ admitting a semi-parallel vector field $X^r$, satisfies the curvature identity (6.9).

**Case III. C-reducible Finsler space:** In a C-reducible Finsler space $F_n$ [7], with the help of equation (2.3), we obtain

$$C_h * V_k + C_k * V_h = 2\alpha h_{hk}, \tag{6.10}$$

Which in $F_3$ gives $C * V_k = 0$. Since $C \neq 0$, therefore $* V_k = 0$, i.e., $\alpha h_{hk} = 0$. Since □ is not zero, we must have $h_{hk} = 0$, i.e., the given space is Riemannian. Hence, we have:

**Theorem 6.3:** A three-dimensional C-reducible Finsler space $F_3$, admitting a semi-parallel vector field reduces to a Riemannian space.

Multiplying equation (6.10) by $C^h C^k$, we get $(n-2)\alpha = 0$, which gives

**Theorem 6.4:** An n-dinensional C-reducible Finsler space $F_n$, admitting a semi-parallel vector field does not exist.

**Case IV. P-reducible Finsler space:** In a P-reducible Finsler space $F_n$, $P_{rjk}$ is expressed as [10]

$$P_{rjk} = (n+1)^{-1}(A_{r|0} h_{jk} + A_{j|0} h_{kr} + A_{k|0} h_{rj}) \tag{6.11}$$

From equation (6.11), we can obtain by virtue of Definition 4.1,

$$2(n+1)L\, h_{jk}(\alpha_{|0} - \lambda\varepsilon\alpha V_0) = A_{k|0} * V_j + A_{j|0} * V_k$$

,

which on multiplication by $g^{jk}$ leads to

$$\alpha_{|0} = n(n-1)^{-1}\,\lambda\,\varepsilon\,\alpha\,V_0. \tag{6.12}$$

Hence, we have:

**Theorem 6.5:** An n-dimensional P-reducible Finsler space $F_n$, admitting a semi-parallel vector field satisfies (6.12).

**Remark:** In case of a three-dimensional P-reducible Finsler space $F_3$, admitting a semi-parallel vector field,

$C_{|0} = C\lambda\varepsilon V_0$, $\alpha_{|0} = (3/2)\lambda\varepsilon\alpha V_0$ and $V_j n^j = 2C_{|0}\alpha / C^2 h_0$.

Hence, we have:

**Theorem 6.6:** In a three- dimensional P-reducible Finsler space $F_3$, admitting a semi-parallel vector field, scalar □ is proportional to $C^{3/2}$.

**Case V. P-reducible Finsler space of R-K type:** It is known that in a P-reducible Finsler space $F_3$ of R-K type [10], $A_{r|0} = 4L(C_{(3)|0}m_r + C_{(3)}h_0 n_r)$, which for a semi-parallel vector field gives $V^r A_{r|0} = 8\alpha LC^{-2}(CC_{(3)})_{|0}$. Direct calculation gives $V^r A_{r|0} = 4\alpha LC^{-1}C_{|0}$, therefore on comparison we get

$$C_{|0} = 2C^{-1}(C\,C_{(3)})_{|0} \quad (6.13)$$

which for $C_{(3)|0} = 0$, i.e., $P_{rjkh}$ is symmetric in *k* and *h*, gives

**Theorem 6.7:** In a P-reducible Finsler space $F_3$, admitting a semi-parallel vector field, with second curvature tensor $P_{rjkh}$ symmetric in *k* and *h*, $C_{|0} = 0$.

**Case VI. S3-like Finsler space:** In a three-dimensional space $F_3$, v-curvature tensor is expressed as [6]

$$S^r_{hjk} = L^2(2C^2_{(2)} + C^2_{(3)} - C_{(1)}C_{(3)})(h_{rj}h^r_k - h_{hk}h^r_j), \quad (6.14)$$

which on application of equation (5.9) gives

**Theorem 6.8:** In a three-dimensional Finsler space $F_3$ admitting a semi-parallel vector field $2C^2_{(2)} + C^2_{(3)} - C_{(1)}C_{(3)} = 0$.

In an S3-like Finsler space $F_n$, v-curvature tensor is given by

$$S^r_{h\,jk} = S(h_{rj}h^r_k - h_{hk}h^r_j), \quad (6.15)$$

which on application of (5.9) gives on simplification *S=0*, i.e. $S^r_{h\,jk} = 0$.

Hence, we have:

**Theorem 6.9:** In an n-dimensional S3-like Finsler space $F_n$, a semi-parallel vector field does not exist.

**Case VII. h-isotropic Finsler space:** In an n-dimensional h-isotropic Finsler space $F_n$ curvature tensor R $R^r_{jkh}$ is expressed as [5]

$$R^r_{jkh} = R(g_{jk}\delta^r_h - g_{jh}\delta^r_k) \quad (6.16)$$

Using equation (6.16) in (5.4), we obtain

$$R = (1 + L^{-a}\alpha X_0)^{-1}(\rho_{|k}X^k - \rho) \quad (6.17)$$

Since in general $(1 + L^{-1}\alpha X_0)$ is non-zero, therefore we get

**Theorem 6.10:** In an h-isotropic Finsler space $F_n$ admitting a normalized semi-parallel vector field, the necessary and sufficient condition for the curvature scalar *R* to vanish is given by $\rho_{|k}X^k = \rho$.

**Case VIII. C^h-recurrent Finsler space:** It is known that in a $C^h$-recurrent Finsler space $F_n$, torsion tensor $C_{jkh}$ satisfies [5]

$$C_{hjk|m} = C_{hjk}K_m, \quad (6.18)$$

Where $K_m$ is a function of *x* and *y*.

Also it is known that in a $C^h$-recurrent Finsler space [5], $S_{hjkl|m} = 2K_m S_{hjkl}$, which by virtue of equations (5.9) and (5.10) gives $\lambda S^r_{m\,jk} = 0$, i.e., either $\lambda = 0$ or $S^r_{m\,jk}$ vanishes. Since λ is non-zero, therefore $S^r_{m\,jk} = 0$. Hence we have:

**Theorem 6.11:** A $C^h$-recurrent non-Riemannian Finsler space $F_n$, admitting a semi-parallel vector field $V^r$ shall have vanishing v-curvature tensor.

Substituting $S^r_{m\,jk} = 0$, in (5.11), we get $P^r_{m\,kj} = P^r_{m\,kj}$.

Hence, we have:

**Corollary 6.1:** A $C^h$-recurrent non-Riemannian Finsler space $F_n$ admitting a semi-parallel vector field $V^r$ shall have a curvature tensor $P^r_{m\,kj}$, which is symmetric in *j* and *k* and also $V^j{}_{|k}\big|_m = V^j\big|_{m|k}$.

Substituting $S^r_{m\,jk} = 0$ and $R^h_{j\,mr} = 0$ in the Bianchi-identities [5]

$$\varsigma_{(l,m)}\{P^h_{j\,kl}\big|_m - P^h_{j\,rm}C^r_{kl} - S^h_{j\,rm}P^r_{kl}\} + S^h_{j\,lm|k} = 0,$$

and

$$\varsigma_{(l,m)}\{P^h_{jlk|m} + P^h_{jmr}P^r{}_{kl} + R^h_{jmr}C^r_{kl}\} + R^h_{jmk}\big|_l + S^h_{jkr}R^r_{ml} = 0,$$

respectively, we get

**Corollary 6.2:** A $C^h$-recurrent non-Riemannian Finsler space $F_n$ admitting a semi-parallel vector field $V^r$ satisfies Bianchi-identities

$$\varsigma_{(l,m)}\{P^h_{j\,kl}|_m - P^h_{j\,rm}C^r_{kl}\} = 0,\ \varsigma_{(l,m)}\{P^h_{j\,lk|m} + P^h_{j\,mr}P^r_{lk}\} = 0$$

**Case IX: h-isotropic and $C^h$-recurrent Finsler space:** It is known [5] that in an n-dimensional Finsler space $F_n$, which is h-isotropic and $C^h$-recurrent $K_{i|j} = K_{j|i}$, $R_{ijkl} = 0$. Using $R_{ijkl} = 0$, in (5.4), we get

$$(n-2)\rho_{|h} + X_h\{X^j{}_{\rho|j} + \rho(n-1)\} = 0\ .\ (6.19)$$

Hence we have:

**Theorem 6.12:** In an n-dimensional $C^h$-recurrent, h-isotropic Finsler space $F_n$, admitting a normalized semi-parallel vector field $X^r$, scalar ρ satisfies equation (6.19)

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