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Eccentricity-Based Bounds for the Spectral Radius of Graph Matrices

Sunilgar Gusai

Faculty of Computer Applications, Marwadi University, Rajkot, India
Email: *sunilgargusai[at]gmail.com*

Abstract: ***This article presents a framework for bounding the spectral radius of classical graph matrices, specifically the adjacency and signless Laplacian matrices, using vertex eccentricity as a global structural parameter. By applying the Collatz–Wielandt characterization and Rayleigh quotient methods with the eccentricity vector as a test input, the study derives both upper and lower bounds that incorporate global distance distributions. The proposed bounds are shown to be sharp for regular and self-centered graph families and remain useful for irregular structures. Through these results, eccentricity emerges as a complementary control parameter to degree-based approaches, providing enhanced insight into how both local and global structures shape spectral behaviour.***

Keywords: Spectral radius; vertex eccentricity; adjacency matrix; signless Laplacian matrix; Collatz–Wielandt inequality

1. Introduction

The study of eigenvalues of graph associated matrices forms a central part of spectral graph theory. Among these eigenvalues, the spectral radius, defined as the largest eigenvalue of a real symmetric graph matrix, plays a decisive role in understanding structural, extremal, and dynamical properties of graphs. For the adjacency matrix and the signless Laplacian matrix, the spectral radius governs connectivity behaviour, growth of walks, and the stability of processes on networks, and has therefore been studied extensively.

Classical bounds for the spectral radius are predominantly degree based. Inequalities involving the maximum degree, minimum degree, or average degree have been developed and refined over several decades and remain fundamental tools in spectral analysis. These bounds are often sharp for regular or dense graphs and effectively capture local connectivity patterns. However, degree-based parameters are inherently local in nature and do not reflect global distance properties of a graph. As a result, graphs with similar degree distributions but substantially different global structures may satisfy identical degree-based bounds while exhibiting distinct spectral behaviour.

Global distance parameters provide complementary structural information. Vertex eccentricity, defined as the maximum distance from a vertex to all other vertices of the graph, quantifies how distant a vertex is from the graph's periphery. Parameters derived from eccentricity, such as radius and diameter, measure global remoteness and the extent of graph expansion or stretch. These distance-based invariants have long been studied in structural graph theory and have played an important role in chemical graph theory through distance dependent indices. Despite this, their direct influence on the spectral radius of classical graph matrices has not been examined in a systematic manner.

Recent spectral investigations involving eccentricity have primarily focused on matrices constructed explicitly from eccentricity values, such as eccentricity matrices and their associated energies. Although these approaches introduce new spectral objects with interesting properties, they do not directly address how eccentricity influences the spectral behaviour of the standard adjacency or signless Laplacian matrices. Consequently, the role of eccentricity as a controlling parameter, rather than as a matrix defining ingredient, has not been thoroughly examined.

This study addresses this gap by retaining the classical adjacency and signless Laplacian matrices and incorporating vertex eccentricity into their spectral analysis through established variational principles. In particular, the Collatz–Wielandt characterization for nonnegative irreducible matrices and Rayleigh quotient techniques are employed, with the eccentricity vector serving as a positive test vector. This strategy allows global distance information to enter spectral estimates naturally without modifying the underlying matrix structure.

Within this framework, computable lower and upper bounds are obtained for the spectral radius of both the adjacency and signless Laplacian matrices in terms of eccentricity distributions and degree eccentricity interactions. Equality conditions are fully characterized, and the bounds are exact for broad families of regular and self-centered graphs. Structural limitations are also identified for graphs with highly nonuniform eccentricity distributions, clarifying the settings in which eccentricity-based bounds become inherently coarse.

Overall, the results establish vertex eccentricity as a distance sensitive spectral control parameter that complements degree-based bounds. By integrating global distance information into classical spectral frameworks, the analysis provides a clearer understanding of how local and global graph structure jointly influence spectral radii.

2. Related Work

The study of spectral radii of graph matrices has a long- and well-established history in spectral graph theory. Foundational treatments of graph spectra and their structural implications are presented in standard monographs by Cvetković, Doob, and Sachs [3], Chung [2], and Brouwer and

Haemers [1]. These works formalized the role of eigenvalues of adjacency and Laplacian type matrices in capturing fundamental graph properties and laid the groundwork for subsequent extremal investigations.

Early research on bounding the adjacency spectral radius focused primarily on degree-based parameters. Hong [4] derived bounds relating the largest adjacency eigenvalue to vertex degrees, initiating a systematic study of extremal degree-based inequalities. This line of research was further developed by Nikiforov [5], who established sharp bounds using degree sequences and extremal configurations. Additional refinements were introduced by Das and Kumar [6] and by Das and Bapat [7], extending degree-based techniques to weighted graphs and related spectral settings. These results remain central to spectral extremal theory and are particularly effective for regular or nearly regular graphs.

Alongside degree-based approaches, distance-based graph invariants have been extensively studied in structural and chemical graph theory. Wiener's pioneering work [8] introduced distance dependent indices to relate molecular structure to physical properties, initiating a broad research direction centered on global distance measures. Subsequent studies by Dobrynin, Entringer, and Gutman [9] systematically developed the theory of distance dependent invariants and highlighted the structural information captured by global distance parameters. Estrada [10] further emphasized the role of global structure in network analysis, reinforcing the importance of distance sensitive descriptors.

Despite their structural relevance, distance parameters such as eccentricity, radius, and diameter have appeared only sporadically in classical spectral radius bounds. More recent spectral investigations involving eccentricity have largely focused on matrices constructed explicitly from eccentricity values. Mahato and collaborators [11] studied the spectral radius and energy of eccentricity matrices, while Qiu, Li, and Zhang [12] examined eccentricity energy and eccentricity spectral radius for graphs under diameter constraints. Extensions to directed graphs were considered by Yang and Wang [13]. These contributions demonstrate that eccentricity-based matrices possess rich spectral behaviour, but they do not directly address how eccentricity influences the spectral radius of the standard adjacency or signless Laplacian matrices.

Research on the signless Laplacian spectral radius has also expanded in recent years. Chen, Cioabă, and Lin [14] investigated extremal properties of the signless Laplacian spectral radius under forbidden odd cycle conditions, while Chen [15] studied signless Laplacian bounds for book free graphs. More recently, Malathy and Desikan [16] derived bounds for the adjacency and signless Laplacian spectral radii of generalized core satellite graphs. These studies reflect ongoing interest in refining spectral bounds under increasingly specialized structural constraints.

The approach adopted here differs fundamentally from the above directions. Rather than introducing new matrix constructions or relying solely on degree-based parameters, the classical adjacency and signless Laplacian matrices are retained, and vertex eccentricity is incorporated as a control parameter within established spectral frameworks. By applying the Collatz–Wielandt characterization and Rayleigh quotient techniques with eccentricity-based test vectors, distance sensitive bounds for the spectral radius are obtained while maintaining direct comparability with classical results.

This perspective also complements recent work on vertex eccentricity labeled energy [17], where eccentricity is incorporated into energy based spectral descriptors. In contrast, the analysis presented here shows that eccentricity alone, without redefining matrix structure, can effectively influence the spectral radius of standard graph matrices, thereby linking distance based structural theory with classical spectral analysis.

3. Preliminaries

Let $G = (V(G), E(G))$ be a connected simple undirected graph with vertex set
$V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$. Throughout this paper, graphs are assumed to be connected unless stated otherwise.

3.1 Distance and Eccentricity

The distance $d(u,v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path connecting them in G. The eccentricity of a vertex $v \in V(G)$ is defined as

$$e(v) = \max_{u \in V(G)} d(u,v).$$

The radius and diameter of G are defined respectively by

$$r(G) = \min_{v \in V(G)} e(v), D(G) = \max_{v \in V(G)} e(v).$$

A graph is said to be self-centered if all vertices have the same eccentricity.

3.2 Graph Matrices and Spectral Radius

The adjacency matrix of G, denoted by $A(G) = (a_{ij})$, is defined by

$$a_{ij} = \begin{cases} 1, & \text{if } v_i v_j \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$D(G) = \operatorname{diag}(d(v_1), d(v_2), \dots, d(v_n))$$

denote the diagonal matrix of vertex degrees. The signless Laplacian matrix of G is defined as

$$Q(G) = D(G) + A(G).$$

Since both $A(G)$ and $Q(G)$ are real symmetric matrices, all of their eigenvalues are real. The spectral radius of a matrix M, denoted by $\rho(M)$, is defined as the largest eigenvalue of M. For connected graphs, both $A(G)$ and $Q(G)$ are irreducible nonnegative matrices, and therefore their spectral radii coincide with their Perron eigenvalues.

3.3 Rayleigh Quotient

Let M be a real symmetric matrix. The Rayleigh quotient associated with M and a nonzero vector $x \in \mathbb{R}^n$ is given by

$$R_M(x) = \frac{x^\top M x}{x^\top x}.$$

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The spectral radius of M satisfies

$$\rho(M) = \max_{x \neq 0} R_M(x).$$

This characterization is used to derive lower bounds for the spectral radius by selecting appropriate test vectors.

3.4 Collatz–Wielandt Characterization

Let M be a nonnegative irreducible matrix and let $x \in \mathbb{R}^n$ be a positive vector. The spectral radius $\rho(M)$ satisfies

$$\min_{1 \le i \le n} \frac{(Mx)_i}{x_i} \le \rho(M) \le \max_{1 \le i \le n} \frac{(Mx)_i}{x_i}.$$

Equality holds if and only if x is a Perron eigenvector of M.

This result is used to establish both lower and upper bounds for the spectral radii of $A(G)$ and $Q(G)$ by choosing the eccentricity vector as a test vector.

3.5 Eccentricity Vector

The eccentricity vector of G is defined by

$$x = (e(v_1), e(v_2), \dots, e(v_n))^{\top}.$$

Since G is connected, $e(v) \ge 1$ for all $v \in V(G)$, and hence x is a positive vector. This vector plays a central role in the spectral bounds derived in subsequent sections.

4. Methodology and Approach

Let $G = (V(G), E(G))$ be a simple connected graph of order n. The methodology adopted here relies on classical variational principles for real symmetric matrices, combined with vertex eccentricity as a global structural parameter.

The approach retains the standard graph matrices, namely the adjacency matrix $A(G)$ and the signless Laplacian matrix $Q(G)$, and incorporates eccentricity information through the choice of appropriate test vectors rather than by modifying the matrix structure itself. In this way, global distance characteristics influence spectral estimates while preserving full compatibility with classical spectral theory.

4.1 Variational Framework

Since $A(G)$ and $Q(G)$ are real symmetric matrices, their spectral radii admit Rayleigh quotient characterizations. For any nonzero vector $x \in \mathbb{R}^n$,

$$\rho(M) \ge \frac{x^{\top} M x}{x^{\top} x},$$

where $M \in \{A(G), Q(G)\}$. Equality holds if and only if x is an eigenvector corresponding to the spectral radius.

Lower bounds are obtained by selecting test vectors derived from vertex eccentricities. Specifically, the eccentricity vector

$$x = (e(v_1), e(v_2), \dots, e(v_n))^{\top}$$

is used to introduce global distance information into the Rayleigh quotient. Since G is connected, this vector is positive and therefore admissible for spectral estimation.

4.2 Collatz–Wielandt Bounds

Upper bounds are derived using Perron–Frobenius theory for nonnegative irreducible matrices. The Collatz–Wielandt characterization states that

$$\min_i \frac{(Mx)_i}{x_i} \le \rho(M) \le \max_i \frac{(Mx)_i}{x_i},$$

for any positive vector x. Substituting the eccentricity vector into this inequality yields bounds expressed in terms of extremal eccentricity contributions across adjacent vertices.

This formulation connects the spectral radius to structural configurations in which adjacency and global remoteness interact, emphasizing vertex pairs that are both adjacent and distant from the remainder of the graph.

4.3 Diameter-Based Arguments

Lower bounds depending on the diameter are obtained using induced subgraphs and eigenvalue interlacing. In particular, any diametral path of G induces a path subgraph whose spectral radius provides a theoretical lower bound for $\rho(A(G))$. This argument shows that increased graph length directly influences the spectral radius.

4.4 Extension to the Signless Laplacian

The same framework extends naturally to the signless Laplacian matrix $Q(G) = D(G) + A(G)$. Since $Q(G)$ combines degree and adjacency information, eccentricity-based test vectors lead to bounds involving both local connectivity and global distance parameters.

This unified methodology produces parallel results for $A(G)$ and $Q(G)$, allowing a consistent comparison between adjacency based and degree augmented spectral behavior.

4.5 Validation via Standard Graph Families

To assess the effectiveness and limitations of the derived bounds, the results are evaluated on classical graph families, including complete graphs, stars, paths, cycles, and complete bipartite graphs. These examples illustrate cases in which eccentricity-based bounds are sharp, asymptotically tight, or necessarily conservative, depending on the underlying distance structure.

5. Results and Discussion

This section presents eccentricity-based bounds for the spectral radius of the adjacency matrix and the signless Laplacian matrix of a connected graph. The results are derived using classical variational principles and show how vertex eccentricity functions as a global distance sensitive control parameter for spectral radii. Equality conditions and structural limitations are also identified.

Throughout this section, $G = (V(G), E(G))$ denotes a connected simple graph of order n.

5.1 Eccentricity Based Bounds for the Adjacency Matrix

Theorem 5.1 (Collatz–Wielandt Eccentricity Bounds)
Let G be a connected graph. Then

$$\min_{v\in V(G)} \frac{\sum_{u\sim v} e(u)}{e(v)} \le \rho(A(G)) \le \max_{v\in V(G)} \frac{\sum_{u\sim v} e(u)}{e(v)}.$$

Proof.
Since G is connected, the adjacency matrix $A(G)$ is nonnegative and irreducible. Let $x \in \mathbb{R}^n$ be the eccentricity vector defined by $x_v = e(v)$. By the Collatz–Wielandt characterization for nonnegative irreducible matrices,

$$\min_i \frac{(Ax)_i}{x_i} \le \rho(A(G)) \le \max_i \frac{(Ax)_i}{x_i}.$$

For each vertex v,

$$(Ax)_v = \sum_{u\sim v} e(u).$$

Dividing by $e(v)$ yields the stated inequalities.

Theorem 5.2 (Rayleigh Quotient Lower Bound)
For any connected graph G,

$$\rho(A(G)) \ge \frac{2\sum_{uv\in E(G)} e(u)e(v)}{\sum_{v\in V(G)} e(v)^2}.$$

Proof.
The adjacency matrix is real and symmetric. By the Rayleigh quotient,

$$\rho(A) = \max_{y\neq 0} \frac{y^\top A y}{y^\top y}.$$

Choosing $y = x$, where $x_v = e(v)$, gives

$$\rho(A(G)) \ge \frac{x^\top A x}{x^\top x}.$$

Since

$$x^\top A x = 2\sum_{uv\in E(G)} e(u)e(v),$$

the result follows.

Theorem 5.3 (Degree Eccentricity Envelope)
Let

$$e_{\min} = \min_{v\in V(G)} e(v), e_{\max} = \max_{v\in V(G)} e(v).$$

Then

$$e_{\min} \min_{v\in V(G)} \frac{d(v)}{e(v)} \le \rho(A(G)) \le e_{\max} \max_{v\in V(G)} \frac{d(v)}{e(v)}.$$

Proof.
For each vertex v,

$$d(v)e_{\min} \le \sum_{u\sim v} e(u) \le d(v)\, e_{\max}.$$

Substituting these bounds into Theorem 5.1 yields the result.

Theorem 5.4 (Equality Characterization)
Equality holds in both bounds of Theorem 5.1 if and only if

$$A(G)x = \rho(A(G))x,$$

where $x_v = e(v)$.

Proof.
By the equality condition of the Collatz–Wielandt theorem, equality occurs if and only if the chosen positive vector x is a Perron eigenvector of $A(G)$.

Theorem 5.5 (Self-centered Regular Graphs)
If G is k-regular and self-centered, then

$$\rho(A(G)) = k,$$

and Theorem 5.1 yields exact bounds in this case.

Proof.
In a self-centered graph, all vertices have the same eccentricity, so x is a scalar multiple of the all-ones vector. Since G is k-regular,

$$A(G)\mathbf{1} = k\mathbf{1}.$$

Hence $\rho(A(G)) = k$.

5.2 Eccentricity Based Bounds for the Signless Laplacian Matrix

Theorem 5.6 (Collatz–Wielandt Bounds for $Q(G)$)

Let G be connected. Then

$$\min_{v\in V(G)} \frac{d(v)e(v) + \sum_{u\sim v} e(u)}{e(v)} \le \rho(Q(G)) \le \max_{v\in V(G)} \frac{d(v)e(v) + \sum_{u\sim v} e(u)}{e(v)}.$$

Proof.
The signless Laplacian matrix $Q(G)$ is nonnegative and irreducible. Applying the Collatz–Wielandt inequalities with the eccentricity vector x yields the stated bounds.

Theorem 5.7 (Rayleigh Lower Bound for $Q(G)$)
For any connected graph G,

$$\rho(Q(G)) \ge \frac{\sum_{v\in V(G)} d(v)e(v)^2 + 2\sum_{uv\in E(G)} e(u)e(v)}{\sum_{v\in V(G)} e(v)^2}.$$

Proof.
By the Rayleigh quotient,

$$\rho(Q) = \max_{y\neq 0} \frac{y^\top Q y}{y^\top y}.$$

Substituting $y = x$ and expanding $Q = D + A$ gives the result.

Theorem 5.8 (Exactness for Regular Self-Centered Graphs)
If G is k-regular and self-centered, then

$$\rho(Q(G)) = 2k,$$

and the bounds in Theorem 5.6 are exact.

Proof.
For a k-regular graph,

$$Q(G)\mathbf{1} = 2k\mathbf{1}.$$

Since x is proportional to $\mathbf{1}$, equality follows from Theorem 5.6.

5.3 Corollaries and Structural Consequences

Corollary 5.1.
For any connected graph, $e_{\min} = r(G), e_{\max} = D(G)$.

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Corollary 5.2.
The quantity $\min_{v \in V(G)} \frac{\sum_{u \sim v} e(u)}{e(v)}$ provides a computable lower bound for $\rho(A(G))$.

Corollary 5.3.
The quantity $\min_{v \in V(G)} \frac{d(v)e(v) + \sum_{u \sim v} e(u)}{e(v)}$ provides a computable lower bound for $\rho(Q(G))$.

Proposition 5.4 (Cycles). For the cycle graph C_n,

$$\rho(A(C_n)) = 2, \rho(Q(C_n)) = 4,$$

and the eccentricity-based bounds are exact.

Proposition 5.5 (Complete Graphs).
For the complete graph K_n,

$$\rho(A(K_n)) = n - 1, \rho(Q(K_n)) = 2(n - 1),$$

and the eccentricity-based bounds are exact.

Proposition 5.6 (Complete Bipartite Graphs).
For $K_{t,t}$,

$$\rho(A(K_{t,t})) = t, \rho(Q(K_{t,t})) = 2t,$$

and the eccentricity-based bounds are exact.

Remark 5.7 (Structural Limitation).
For graphs with highly nonuniform eccentricity distributions, such as star graphs, ratio-based bounds may be relatively loose. In such cases, Rayleigh type bounds provide sharper estimates, reflecting an inherent limitation of ratio-based eccentricity bounds.

5.3 Discussion

The results presented above establish vertex eccentricity as a rigorous and effective distance-sensitive parameter for bounding the spectral radius of classical graph matrices. The bounds are exact for broad families of graphs and offer useful insights for graphs with extended structures, while their limitations are explicitly identified.

6. Conclusion and Future Scope

This study examined the influence of vertex eccentricity on the spectral radius of classical graph matrices. By using eccentricity as a distance sensitive test vector within established variational frameworks, bounds were obtained for the spectral radii of the adjacency and signless Laplacian matrices of connected graphs. The approach preserves the standard matrix structures while incorporating global distance information, thereby linking distance-based graph parameters with classical spectral theory.

The derived bounds are sharp for several well-known graph families, including complete graphs, cycles, and regular self-centered graphs. For graphs with nonuniform eccentricity distributions, the results illustrate how global remoteness interacts with local connectivity to constrain spectral growth. The analysis also identifies the structural settings in which eccentricity-based bounds are exact and those in which Rayleigh type estimates provide more effective control.
The methodology complements existing degree based spectral bounds and recent studies on eccentricity related matrix spectra. Unlike approaches that modify the underlying matrix structure, the framework presented here shows that eccentricity can be incorporated directly into classical spectral analysis through standard tools such as the Rayleigh quotient and Collatz–Wielandt inequalities.

Several directions remain open for further investigation. The results may be extended to other matrix families, including the normalized Laplacian and Seidel type matrices. Eccentricity based spectral bounds for directed graphs, weighted graphs, and graphs with additional structural constraints such as forbidden subgraphs also warrant deeper investigation. In addition, combining eccentricity with other global invariants may lead to refined hybrid bounds capable of capturing more subtle structural features of large-scale networks.

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Author Profile

Sunilgar L. Gusai is a faculty member in the Faculty of Computer Applications at Marwadi University, Rajkot, India. His research interests include spectral graph theory, graph invariants, and the study of eigenvalue-based properties of graphs. He also works on the application of graph-theoretic methods in computer science, network analysis, and interdisciplinary areas such as mathematical modelling in chemistry and data-driven systems.