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An Efficient Technique for Solution of Two - Dimensional Fractional Blach-Torrey Equations

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Abstract: ***The main aim of the current paper is to propose an efficient numerical technique for solving two-dimensional space-multi-time fractional Bloch–Torrey equations. The current research work is a generalization of [6]. The temporal direction is based on the Caputo fractional derivative with multi-order fractional derivative and the spatial directions are based on the Riemann–Liouville fractional derivative. Thus, to achieve a numerical technique, the time variable is discretized using a finite difference scheme with convergence order*** $O\left(\tau^{2-\alpha}\right)$. ***Also, the space variable is discretized using the finite element method. Furthermore, for the time-discrete and the full-discrete schemes error estimate has been presented to show the unconditional stability and convergence of the developed numerical method. Finally, four test problems have been illustrated to verify the efficiency and simplicity of the proposed technique on irregular computational domains.***

Keywords: Space fractional equation, Bloch–Torrey equations, Fractional derivative, Convergence analysis, Error estimate, Caputo derivative, Riemann–Liouville fractional

1. Introduction

The fractional PDEs play important and basic role in modeling and simulating of natural and practical phenomena [14]. In the meantime, the fractional equations based on the space fractional type are important and have more difficulty for simulating and analysis. the finite element method is applied in several fractional equations for example fractional diffusion-wave equation [13,28], time-fractional fourth-order partial differential equation [31], nonlinear time fractional fourth-order reaction–diffusion problem[32], fractional FitzHugh-Nagumo monodomain model [8], time-space fractional telegraph equation[63], space-time fractional diffusion equations [7,22,23], two-dimensional space-fractional advection–dispersion equations [29,62], space-fractional Boussinesq equation [64], space and time fractional Fokker–Planck equation[16], space-fractional parabolic equation[25] and Rayleigh–Stokes problem with fractional derivatives[12]. Also, anew Crank–Nicolson finite element method is developed in [60]for solving the time-fractional subdiffusion equation. Author of [56] developed efficient algorithms based on the Legendre-tau approximation for one- and two-dimensional fractional Rayleigh–Stokes problems for a generalized second-grade fluid. A Legendre spectral tau method is revisited in [57]to handle the multi-term time-fractional diffusion equations (MTT-FDEs) also an error estimate and rigorous convergence analysis are carried out using some critical theorems.

The space fractional equations have been solved using many numerical solutions such as finite difference technique for space-fractional sine-Gordon equations [33], Jacobi collocation approach for space-time variable-order fractional Schrödinger equations [4], an efficient operational formulation of spectral tau method for multi-term time–space fractional partial differential equations [5], general Pade approximation method [17], Fourier spectral method for fractional-in-space Cahn–Hilliard equation [45], the point interpolation method for Riesz space fractional advection–dispersion equations [55], third-order difference schemes for space fractional PDEs [50], Gaussian-sum (GauSum) solver for space fractional nonlinear Schrödinger/Gross–Pitaevskii equations [3], time-split Gauss–Seidel projection approach for the space fractional Landau–Lifshitz equations [48], finite volume method for space-fractional diffusion equations [24], finite element technique for nonlinear space fractional diffusion equations on complex domains [47], Fourier spectral method for space fractional reaction–diffusion equations [37], a Petrov–Galerkin (PG) spectral method for fractional IVP [58], discontinuous spectral element methods for time- and space-fractional advection equations [59], anew operational matrix for fractional-order differential equations [38], fourth-order finite difference schemes for time-space fractional sub-diffusion equations [36], a fast second-order accurate method for space-fractional diffusion equation [21], a fourth-order compact ADI scheme space for fractional Schrödinger equation [61], finite element method for time-space fractional Bloch–Torrey equations [6], high-order approximation to Caputo derivatives [26], a high order finite difference scheme for a two dimensional fractional Klein–Gordon equation subject to Neumann boundary conditions [43] compact finite difference schemes for the modified anomalous fractional sub-diffusion equation and fractional diffusion-wave equation [44], anew difference scheme for the time fractional diffusion equation based on the Caputo fractional derivative [1], solutions of the Dirichlet and Robin boundary value problems for the multi-term variable-distributed order diffusion equation with error estimate [2], the dual reciprocity boundary elements method for solving fractional PDEs [15], etc.

The main aim of [18]is to develop two high-order approximate formulas for solving the Riesz fractional derivative. Authors of [49] considered a fractional partial differential equation with Riesz space fractional derivatives (FPDE-RSFD) on a finite domain and they solved it by using namely, the L1/L2-approximation method, the standard/shifted Grunwald method, and the matrix transform method (MTM). A Crank–Nicolson method has been applied in [9] to a fractional diffusion equation which has the Riesz fractional derivative, to obtain a method which will be unconditionally stable and convergent. Authors of [30] proposed a second-order backward difference formula (BDF2) for solving time approximation of Riesz space-fractional diffusion equations. The main purpose of [34]is to develop a consistent and stable numerical method of second-order convergence. Authors of [19] developed a class of high-order numerical algorithms for Riesz derivatives through constructing new generating functions. The main aim of [46]is to propose two higher order numerical methods for solving fractional differential equations, the first approach is based on a direct discretisation of the fractional differential operator and the second approach is based on discretisation of the integral form of the fractional differential equation. Also see [27, 51–54].Authors of [6] considered the two-dimensional space-time fractional Bloch–Torrey equations as follows.

$$ {}_0^C D_t^{\alpha} u = K_x \frac{\partial^{2\beta} u}{\partial |x|^{2\beta}} + K_y \frac{\partial^{2\beta} u}{\partial |y|^{2\beta}} + f(x,y,t), (x,y,t) \in \Omega \quad t > 0. $$

We consider a generalization of the above model. In the current paper, we consider the following two-dimensional space/multi-time fractional Bloch–Torrey equations.

$$ P_{\alpha,\alpha_1,\alpha_2,\dots,\alpha_m}(D_t) u(x,y,t) = K_x \frac{\partial^{2\beta} u}{\partial |x|^{2\beta}} + K_y \frac{\partial^{2\beta} u}{\partial |y|^{2\beta}} + f(x,y,t), \quad (x,y,t) \in \Omega, \quad t > 0. \quad (1.1) $$

Where

$$ P_{\alpha,\alpha_1,\alpha_2,\dots,\alpha_m}(D_t) u(x,y,t) = {}_0^C D_t^{\alpha} + \sum_{j=0}^{m} d_j \, {}_0^C D_t^{\alpha j}, $$

$0 < \alpha_m < \dots < \alpha_1 < \alpha < 1, \quad 1 < 2\beta < 2$ And

$d_{j \geq 0}, \ j = 1, \dots, m, \ m \in \square$ are constants. Also, ${}_0^C D_t^{\alpha j}$ is theCaputo fractional derivative of order αj with respect to t. We consider this equation with the initial condition

$$ u(x,y,0) = \varphi(x,y), \quad (1.2) $$

and boundary condition

$$ u(x,y,t) = g(x,y,t), \quad (x,y) \in \partial\Omega. \quad (1.3) $$

The main aim of the current paper is to present an efficient numerical procedure for solving two-dimensional space-multi-time fractional Bloch–Torrey equations. The considered model is a generalized version of the studied model in [6]. For the mentioned model, a finite difference/finite element technique is proposed as its unconditional stability and convergence have been proved.

2. Some preliminaries of fractional calculus

In this paper, we will use the Caputo fractional derivative as follows.

$$ {}_0^C D_t^{a} f(t) = \frac{1}{\Gamma(n-a)} \int_a^t \frac{f^{(n)}(\tau) d\tau}{(t-\tau)^{a+1-n}}, (n-1 < a < n, \ n \in \square) \quad (2.1) $$

With Γ denoting the gamma function. Also.

$$ \frac{\partial^a u(x,t)}{\partial t^a} = \frac{1}{\Gamma(1-a)} \int_a^t \frac{\partial u(x,t)}{\partial s} \frac{ds}{(t-s)^a}, a \in (0,1) \quad (2.2) $$

The Riesz fractional derivative with respect to spatial direction is [10, 35].

$$ \frac{\partial^{2\beta} u(x,y,t)}{\partial |x|^{2\beta}} = \frac{-1}{2\cos(\beta\pi)} \left({}_{\Theta_1(y)}^{RL} D_t^{2\beta} u(x,y,t) + {}_x^{RL} D_{\Theta_2(y)}^{2\beta} u(x,y,t) \right), \quad (2.3) $$

$$ \frac{\partial^{2\beta} u(x,y,t)}{\partial |y|^{2\beta}} = \frac{-1}{2\cos(\beta\pi)} \left({}_{\Upsilon_1(x)}^{RL} D_y^{2\beta} u(x,y,t) + {}_y^{RL} D_{\Upsilon_2(x)}^{2\beta} u(x,y,t) \right), \quad (2.4) $$

In which [10, 35]

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$$ {}_{\Theta_1(y)}^{RL} D_t^{2\beta} u(x,y,t) = \frac{1}{\Gamma(2-2\beta)} \frac{d^2}{dx^2} \int_{\Theta_1(y)}^{x} (x-s)^{1-2\beta} u(s,y,t) ds, (2.5) $$

$$ {}_x^{RL} D_{\Theta_2(y)}^{2\beta} u(x,y,t) = \frac{1}{\Gamma(2-2\beta)} \frac{d^2}{dx^2} \int_{x}^{\Theta_2(y)} (s-x)^{1-2\beta} u(s,y,t) ds, (2.6) $$

$$ {}_{\Upsilon_1(x)}^{RL} D_y^{2\beta} u(x,y,t) = \frac{1}{\Gamma(2-2\beta)} \frac{d^2}{dy^2} \int_{\Upsilon_1(y)}^{y} (y-s)^{1-2\beta} u(x,s,t) ds, (2.7) $$

$$ {}_y^{RL} D_{\Upsilon_2(x)}^{2\beta} u(x,y,t) = \frac{1}{\Gamma(2-2\beta)} \frac{d^2}{dy^2} \int_{y}^{\Upsilon_2(y)} (s-y)^{1-2\beta} u(x,s,t) ds, (2.8) $$

And

$$ \Theta_1(y) = \min\{x : (x,\theta) \in \Omega, \quad \theta = y\}, $$
$$ \Theta_2(y) = \max\{x : (x,\theta) \in \Omega, \quad \theta = y\}, $$
$$ \Upsilon_1(x) = \min\{y : (\theta,y) \in \Omega, \quad \theta = x\}, $$
$$ \Upsilon_2(x) = \max\{y : (\theta,y) \in \Omega, \quad \theta = x\}, $$

Definition 2.1 ([10, 35]). Given $\theta > 0$ then the following semi-norms can be defined.

$$ |v|_{J_L^{\theta}(R^2)} = \left(\left\| {}_{\Theta_1(y)}^{RL} D_t^{2\beta} v \right\|_{L^2(R^2)}^2 + \left\| {}_{\Upsilon_1(x)}^{RL} D_y^{2\beta} v \right\|_{L^2(R^2)}^2 \right)^{\frac{1}{2}}, \quad (2.9) $$

$$ |v|_{J_R^{\theta}(R^2)} = \left(\left\| {}_x^{RL} D_{\Theta_2(y)}^{2\beta} v \right\|_{L^2(R^2)}^2 + \left\| {}_y^{RL} D_{\Upsilon_2(x)}^{2\beta} v \right\|_{L^2(R^2)}^2 \right)^{\frac{1}{2}}, \quad (2.10) $$

And norms

$$ \|v\|_{J_L^{\theta}(R^2)} = \left(\|v\|_{L^2(R^2)}^2 + |v|_{J_L^{\theta}(R^2)}^2 \right)^{\frac{1}{2}}, \quad (2.11) $$

$$ \|v\|_{J_R^{\theta}(R^2)} = \left(\|v\|_{L^2(R^2)}^2 + |v|_{J_R^{\theta}(R^2)}^2 \right)^{\frac{1}{2}}, \quad (2.12) $$

Also, $J_L^{\theta}(R^2)$ and $J_R^{\theta}(R^2)$ are the closure of $C_0^{\infty}(R^2)$ with respect to $\|v\|_{J_L^{\theta}(R^2)}$ and $\|v\|_{J_R^{\theta}(R^2)}$, respectively.

Definition 2.2 ([10, 35]). Given $\theta > 0$ then we define the following semi-norm

$$|v|_{H^\theta(R^2)} = \left\| |\xi|^\theta \hat{V}(\xi) \right\|_{L^2(R^2_\xi)}, \qquad (2.13)$$

And norm

$$\|v\|_{H^\theta(R^2)} = \left(\|v\|^2_{L^2(R^2_\xi)} + |v|^2_{H^\theta(R^2)} \right)^{\frac{1}{2}}, \qquad (2.14)$$

In which $H^\theta(R^2)$ is the closure of $C_0^\infty(R^2)$ with respect to $\|v\|_{H^\theta(R^2)}$ and $\hat{V}$ is the Fourier transform of V.

Definition 2.3 ([10, 35]). Given $\theta > 0$. If $\theta \neq n - \frac{1}{2}$ for $n \in \mathbb{N}$ we consider the following semi-norm

$$|v|_{J_s^\theta(R^2)} = \left(\left| \left({}^{RL}_{\Theta_1(y)}D_t^\theta v, {}^{RL}_{x}D^\theta_{\Theta_2(y)} v \right)_{L^2(R^2)} \right| + \left| \left({}^{RL}_{\Upsilon_1(x)}D_y^\theta v, {}^{RL}_{y}D^\theta_{\Upsilon_2(x)} v \right)_{L^2(R^2)} \right| \right)^{\frac{1}{2}}, \qquad (2.15)$$

And norm

$$\|v\|_{J_s^\theta(R^2)} = \left(\|v\|^2_{L^2(R^2)} + |v|^2_{J_R^\theta(R^2)} \right)^{\frac{1}{2}}, \qquad (2.16)$$

Also, $J_s^\theta(R^2)$ is the closure of $C_0^\infty(\mathbb{R}^2)$ with respect to $\|v\|_{J_s^\theta(R^2)}$.

Lemma 2.1 ([10, 35]). For $\theta > 0$ spaces $J_L^\theta(R^2), J_R^\theta(R^2)$ and $H^\theta(R^2)$ are equivalent.

The spaces $J_L^\theta(R^2)$) and $H^\theta(R^2)$ are equal with equivalent semi-norms and norms.

Proof. The proof follows immediately from the following lemma.

Let $\theta > 0$, be given. A function $v \in L^2(R)$ belongs to $J_L^\theta(R^2)$ if and only if

$$|\omega|^\theta \hat{v} \in L^2(R^2). \qquad (2.17)$$

Specifically,

$$|v|_{J_L^\theta(R^2)} = \left\| |\omega|^\theta \hat{v} \right\|_{L^2(R^2)} \qquad (2.18)$$

Analogous to $J_L^\theta(R^2)$ we introduce $J_R^\theta(R^2)$, the right fractional derivative space, and establish their equivalence.

Lemma 2.2 ([10, 35]). Let $v \in J^\theta_{R,0}(\Omega)$ and $0 < \varepsilon < \theta$ then

$$\|v\|_{L^2(\Omega)} \le C|v|_{J_L^\theta(\Omega)}, \; |v|_{J_L^\varepsilon(\Omega)} \le C|v|_{J_L^\theta(\Omega)} \qquad (2.19)$$

Also, if $v \in J^\theta_{R,0}(\Omega)$ and $0 < \varepsilon < \theta$ then

$$\|v\|_{L^2(\Omega)} \le C|v|_{J_R^\theta(\Omega)}, \; |v|_{J_R^\varepsilon(\Omega)} \le C|v|_{J_L^\theta(\Omega)} \qquad (2.20)$$

We can write for $v \in J^\theta_{L,0}(\Omega)$ we have $D^{-\theta}D^\theta v = v$, and for $v \in J^\theta_{R,0}(\Omega)$, we have

Proof. By the definition of $J^\theta_{L,0}(\Omega)$, there exists a sequence $\{\phi_n\}_{n=1}^\infty \subset C_0^\infty(\Omega)$ such that $\lim_{n\to\infty} \|v - \phi_n\|_{J_L^\theta(\Omega)} = 0.$

Applying the triangle inequality, we have

$$\|D^{-\theta}D^\theta v - v\|_{J_L^\theta(\Omega)} \le \|D^{-\theta}D^\theta(v - \phi_n)\|_{J_L^\theta(\Omega)} + \|D^{-\theta}D^\theta\phi_n - \phi_n\|_{J_L^\theta(\Omega)} + \|\phi_n - v\|_{J_L^\theta(\Omega)}$$

Since $\phi_n \in C_0^\infty(\Omega)$, property A.6 (in the Appendix) implies that $\|D^{-\theta}D^\theta\phi_n - \phi_n\|_{J_L^\theta(\Omega)} = 0$. By the mapping properties in lemma 2.6,

$$\|D^{-\theta}D^\theta(v - \phi_n)\|_{J_L^\theta(\Omega)} \le C\|v - \phi_n\|_{J_L^\theta(\Omega)}$$

Thus,

$$\|D^{-\theta}D^\theta v - v\|_{J_L^\theta(\Omega)} \le (C+1)\|v - \phi_n\|_{J_L^\theta(\Omega)}$$

Corollary .for $v \in J^\theta_{L,0}(\Omega)$, $0 < s < \theta$

$$\|D^{-s}D^sD^{\theta-s}v\| = D^{\theta-s}v$$

Proof. Proceeding as in the proof of Lemma 2.7,

$$\|D^{-s}D^sD^{\theta-s}v - D^{\theta-s}v\|_{L^2(\Omega)} \le \|D^{-s}D^sD^{\theta-s}(v - \phi_n)\|_{L^2(\Omega)} + \|D^{-s}D^sD^{\theta-s}\phi_n - D^{\theta-s}\phi_n\|_{L^2(\Omega)} + \|D^{\theta-s}(v - \phi_n)\|_{L^2(\Omega)} \le \|D^{-s}D^\theta(v - \phi_n)\|_{L^2(\Omega)} + \|(v - \phi_n)\|_{J_{L,0}^{\theta-s}(\Omega)} + \|D^{-s}D^sD^{\theta-s}\phi_n - D^{\theta-s}\phi_n\|_{L^2(\Omega)}$$

An elementary calculation shows that for $\phi_n \in C_0^\infty(\Omega)$.

$$D^{\theta-s}\phi_n(l) = 0.$$

Hence, by property A.6,

$$\|D^{-s}D^sD^{\theta-s}\phi_n - D^{\theta-s}\phi_n\|_{L^2(\Omega)} = 0$$

The stated result then follows from the convergence of ϕ_n to v and the mapping properties in Lemma 2.6

Analogous to fractional integral operators, fractional differential operators also satisfy a semi-group property.

Lemma 2.3 ([10, 35]). Let $v \in J_s^\theta(\Omega)$ and $\hat{u}$ be the extension of u by zero outside of Ω then

$$\left({}^{RL}_{\Theta_1(y)}D_x^\theta v, {}^{RL}_{x}D^\theta_{\Theta_2(y)} v \right)_{L^2(\Omega)} = \left({}^{RL}_{\Theta_1(y)}D_x^\theta \bar{v}, {}^{RL}_{x}D^\theta_{\Theta_2(y)} \bar{v} \right)_{L^2(\mathbb{R}^2)} = \cos(\pi\theta)\left\| {}^{RL}_{-\infty}D_x^\theta \bar{v} \right\|_{L^2(\mathbb{R}^2)} \qquad (2.21)$$

$$\left({}^{RL}_{\Upsilon_1(x)}D_y^\theta v, {}^{RL}_{y}D^\theta_{\Upsilon_2(x)} v \right)_{L^2(\Omega)} = \left({}^{RL}_{\Upsilon_1(x)}D_L^\theta \bar{v}, {}^{RL}_{y}D^\theta_{\Upsilon_2(x)} \bar{v} \right)_{L^2(\mathbb{R}^2)} = \cos(\pi\theta)\left\| {}^{RL}_{-\infty}D_y^\theta \bar{v} \right\|_{L^2(\mathbb{R}^2)} \qquad (2.22)$$

Lemma 2.4 ([10, 35]). Let $\theta\in(1,2), u,v\in J_L^{\theta}(\Omega), u|_{\partial\Omega}=0$ then

$$\left({}_{\Theta_1(y)}^{RL}D_x^{\theta}u,v\right)=\left({}_{\Theta_1(y)}^{RL}D_x^{\frac{\theta}{2}}u,\ {}_{x}^{RL}D_{\Theta_2(y)}^{\frac{\theta}{2}}v\right),$$

$$\left({}_{\Upsilon_1(x)}^{RL}D_y^{\theta}u,v\right)=\left({}_{\Upsilon_1(x)}^{RL}D_y^{\frac{\theta}{2}}u,\ {}_{y}^{RL}D_{\Upsilon_2(y)}^{\frac{\theta}{2}}v\right), \quad (2.23)$$

$$\left({}_{x}^{RL}D_{\Theta_2(y)}^{\theta}u,v\right)=\left({}_{x}^{RL}D_{\Theta_2(y)}^{\frac{\theta}{2}}v,\ {}_{\Theta_1(y)}^{RL}D_x^{\frac{\theta}{2}}v\right),$$

$$\left({}_{y}^{RL}D_{\Upsilon_2(x)}^{\theta}u,v\right)=\left({}_{y}^{RL}D_{\Upsilon_2(y)}^{\frac{\theta}{2}}v,\ {}_{\Upsilon_1(y)}^{RL}D_y^{\frac{\theta}{2}}v\right). \quad (2.24)$$

Lemma 2.5 ([42]). Suppose $0<\alpha<1$ and $v(t)\in C^2[0,t_k]$ it holds that

$$\left|\int_0^{t_k}\frac{v'(t)}{(t_k-t)^{\alpha}}dt-\frac{1}{\tau}[b_0v(t_k)-\sum_{m=1}^{k-1}(b_{k-m-1}-b_{k-m})v(t_m)-b_{k-1}v(t_0)]\right|$$

$$\le\frac{1}{1-\alpha}[\frac{1-\alpha}{12}+\frac{2^{2-\alpha}}{2-\alpha}-(1+2^{-\alpha})]_{\max\limits_{0\le t\le t_k}|v''(t)|}\tau^{2-\alpha}$$

In which

$$b_m=\frac{\tau^{1-\alpha}}{1-\alpha}[(m+1)^{1-\alpha}-m^{1-\alpha}],\qquad m\ge 0$$

3. Analytical investigation of the new numerical scheme

Now, we discrete the main problem in the time direction with a finite difference scheme [11]. Using Lemma 2.5 for $J=0,1,2,\dots,m$ we have

$${}_{\alpha}^{C}D_t^{\alpha_j}u(x,y,t_n)=\frac{1}{\Gamma(1-\alpha_j)}\int_0^{t_n}\frac{\partial u(x,y,t)}{\partial t}\frac{dt}{(t_n-t)^{\alpha_j}}$$

$$=\frac{1}{\tau\Gamma(1-\alpha_j)}[b_0^ju(x,y,t_n)-\sum_{k=1}^{n-1}(b_{n-k-1}^j-b_{n-k}^j)$$

$$u(x,y,t_k)-b_{n-1}^ju(x,y,t_0)]+O(\tau^{2-\alpha}).$$

Let

$$T_t^{\alpha_j}U^n={}_0^CD_t^{\alpha_j}U(x,y,t_n), \qquad (3.1)$$

then Eq. (1.1) may be written as follows

$$\tau\sum_{j=0}^{m}d_jT_t^{\alpha_j}U^n=\tau k_x\frac{\partial^{2\beta}U^n}{\partial|x|^{2\beta}}+\tau k_y\frac{\partial^{2\beta}U^n}{\partial|y|^{2\beta}}+\tau f(x,y,t_n).\ (3.2)$$

Then, the corresponding vibrational form for Eq. (3.2) is

$$\begin{cases}\alpha(U^n,v)=(\varsigma^n,v), & \forall v\in H_0^{\beta}(\Omega), \quad (3.3)\\ U^0=\varphi(x,y),\end{cases}$$

Where

$$\alpha(U^n,v)=\sum_{j=1}^{m}\frac{d_j}{\Gamma(1-\alpha_j)}b_0^j(U^n,v),$$

$$(\varsigma^n,v)=\sum_{j=1}^{m}\frac{d_j}{\Gamma(1-\alpha_j)}[\sum_{k=1}^{n-1}(b_{n-k-1}^j-b_{n-k}^j)(U^n,v)$$
$$+b_{n-1}^j(U^0,v)]+\tau(f^n,v),$$

$$B(U^n,v)=\tau A_x[\left({}_{\Theta_1(y)}^{RL}D_x^{\beta}U^n,\ {}_{x}^{RL}D_{\Theta_2(y)}^{\beta}v\right)$$
$$+\left({}_{x}^{RL}D_{\Theta_2(y)}^{\beta}U^n,\ {}_{\Theta_1(y)}^{RL}D_x^{\beta}v\right)]$$
$$+\tau A_y[\left({}_{\Upsilon_1(x)}^{RL}D_y^{\beta}U^n,\ {}_{y}^{RL}D_{\Upsilon_2(x)}^{\beta}v\right)$$
$$+\left({}_{y}^{RL}D_{\Upsilon_2(x)}^{\beta}U^n,\ {}_{\Upsilon_1(x)}^{RL}D_y^{\beta}v\right)].$$

And

$$A_x=\frac{k_x}{2\cos(\beta\pi)},\qquad A_x=\frac{k_y}{2\cos(\beta\pi)},$$

$$b_m^j=\frac{\tau^{1-\alpha_j}}{1-\alpha_j}[(m+1)^{1-\alpha_j}-m^{1-\alpha_j}]. \qquad (3.4)$$

Theorem 3.1. The bilinear form $\alpha(U^n,v)$ is continuous and V –elliptic

Proof: Since for arbitrary $0\le\alpha\le 1,\ \Gamma(1-a)\ge 1$ thus we have

$$\left|\sum_{j=0}^{m}\left\{\frac{d_jb_0^j}{\Gamma(1-\alpha_j)}(U^n,v)\right\}\right|\le\sum_{j=0}^{m}\left\{\frac{d_jb_0^j}{\Gamma(1-\alpha_j)}\|U^n\|_{L^2(\Omega)}\right\}\|v\|_{L^2(\Omega)}$$
$$\le 2d^*\sum_{j=1}^{m}\|U^n\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}$$
$$\le 2md^*\|U^n\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}$$
$$\le C\|U^n\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}$$
$$\le C\|U^n\|_{H^{\beta}(\Omega)}\|v\|_{H^{\beta}(\Omega)}$$

And [6]

$$\left|B(U^n,v)\right|\le\tau\left|A_x\right|[\left|\left({}_{\Theta_1(y)}^{RL}D_x^{\beta}U^n,\ {}_{x}^{RL}D_{\Theta_2(y)}^{\beta}v\right)\right|+\left|\left({}_{x}^{RL}D_{\Theta_2(y)}^{\beta}U^n,\ {}_{\Theta_1(y)}^{RL}D_x^{\beta}v\right)\right|]$$
$$+\tau\left|A_y\right|[\left|\left({}_{\Upsilon_1(x)}^{RL}D_y^{\beta}U^n,\ {}_{y}^{RL}D_{\Upsilon_2(x)}^{\beta}v\right)\right|+\left|\left({}_{y}^{RL}D_{\Upsilon_2(x)}^{\beta}U^n,\ {}_{\Upsilon_1(x)}^{RL}D_y^{\beta}v\right)\right|]$$
$$\le\tau\left|A_x\right|[\left\|{}_{\Theta_1(y)}^{RL}D_x^{\beta}U^n\right\|_{L^2(\Omega)}\left\|{}_{y}^{RL}D_{\Upsilon_2(x)}^{\beta}v\right\|_{L^2(\Omega)}+\left\|{}_{x}^{RL}D_{\Theta_2(y)}^{\beta}U^n\right\|_{L^2(\Omega)}\left\|{}_{\Theta_1(y)}^{RL}D_x^{\beta}v\right\|_{L^2(\Omega)}]$$
$$+\tau\left|A_y\right|[\left\|{}_{\Upsilon_1(x)}^{RL}D_y^{\beta}U^n\right\|_{L^2(\Omega)}\left\|{}_{y}^{RL}D_{\Upsilon_2(x)}^{\beta}v\right\|_{L^2(\Omega)}+\left\|{}_{y}^{RL}D_{\Upsilon_2(x)}^{\beta}U^n\right\|_{L^2(\Omega)}\left\|{}_{\Upsilon_1(x)}^{RL}D_y^{\beta}v\right\|_{L^2(\Omega)}]$$
$$\le C\|U^n\|_{H^{\beta}(\Omega)}\|v\|_{H^{\beta}(\Omega)} \qquad (3.5)$$

As a result, we have

$$\left|B\left(U^n,v\right)\right| = \left|\sum_{j=0}^{m}\left\{\frac{d_j b_0^j}{\Gamma\left(1-\alpha_j\right)}\left(U^n,v\right)\right\}+B\left(U^n,v\right)\right|$$
$$\leq C\left\|U^n\right\|_{H^\beta(\Omega)}\|v\|_{H^\beta(\Omega)}+C\left\|U^n\right\|_{H^\beta(\Omega)}\|v\|_{H^\beta(\Omega)}$$
$$\leq \max\{C,C\}\left\|U^n\right\|_{H^\beta(\Omega)}\|v\|_{H^\beta(\Omega)}.$$

Then, the bilinear form $\alpha\left(U^n,v\right)$ is bounded. In other words

$$\sum_{j=1}^{m}\frac{d_j b_0^j}{\Gamma\left(1-\alpha_j\right)}\left(U^n,U^n\right)=\sum_{j=1}^{m}\frac{d_j b_0^j}{\Gamma\left(1-\alpha_j\right)}\left\|U^n\right\|^2_{L^2(\Omega)}$$
$$\geq \frac{d_{\min}}{\Gamma\left(1-\alpha_{\min}\right)}\left\|U^n\right\|^2_{L^2(\Omega)}=C\left\|U^n\right\|^2_{L^2(\Omega)}. \qquad (3.6)$$

Also from [6]we obtain

$$\left|B\left(U^n,U^n\right)\right|=\tau\left|2A_x\right|\left|\left({}_{\Theta_1(y)}^{RL}D_x^\beta U^n,\ {}_x^{RL}D_{\Theta_2(y)}^\beta U^n\right)\right|$$
$$+\tau\left|2A_y\right|\left|\left({}_{\Upsilon_1(x)}^{RL}D_y^\beta U^n,\ {}_y^{RL}D_{\Upsilon_2(x)}^\beta U^n\right)\right|$$
$$\geq \tau\min\left\{\left|2A_x\right|,\left|2A_y\right|\right\}\left\{\left|\left({}_{\Theta_1(y)}^{RL}D_x^\beta U^n,\ {}_x^{RL}D_{\Theta_2(y)}^\beta U^n\right)\right|\right.$$
$$\left.+\left|\left({}_{\Upsilon_1(x)}^{RL}D_y^\beta U^n,\ {}_y^{RL}D_{\Upsilon_2(x)}^\beta U^n\right)\right|\right\}\geq \tau k\left\|U^n\right\|^2_{H^\beta(\Omega)}$$

As a result, we can write

$$\alpha\left(U^n,U^n\right)=\sum_{j=1}^{m}\frac{d_j b_0^j}{\Gamma\left(1-\alpha_j\right)}\left(U^n,U^n\right)+B\left(U^n,U^n\right)$$
$$\geq C\left\|U^n\right\|^2_{L^2(\Omega)}+\tau k\left\|U^n\right\|^2_{L^2(\Omega)}$$

So, the bilinear form $\alpha\left(U^n,v\right)$ is v-elliptic.

Theorem 3.2. The semi-discrete (3.3)is unconditionally stable

Proof. Let $\bar{U}^n$ be approximate solution of (3.3)

$$\begin{cases}\alpha\left(\bar{U}^n,v\right)=\tau\left(\sum_{j=1}^{m}d_j\,\mathrm{I}_t^{\alpha}\bar{U}^n,v\right)+B\left(\bar{U}^n,v\right)=\tau\left(f^n,v\right), & \forall v\in H_0^\beta(\Omega) \qquad (3.7)\\ U^0=g(x,y). & \end{cases}$$

Then we define

$$\rho^n=U^n-\bar{U}^n. \qquad (3.8)$$

Subtracting Eq. (3.7) from Eq. (3.3) yields

$$\alpha\left(\rho^n,v\right)=\tau\left(\sum_{j=1}^{m}d_j\,\mathrm{I}_t^{\alpha}\rho^n,v\right)+B\left(\rho^n,v\right)=0,\ \forall v\in H_0^\beta(\Omega) \qquad (3.9)$$

Or

$$\tau\sum_{n=1}^{M}\left(\sum_{j=1}^{m}d_j\,\mathrm{I}_t^{\alpha}\rho^n,v\right)+\sum_{n=1}^{M}B\left(\rho^n,v\right)=0,\ \forall v\in H_0^\beta(\Omega) \qquad (3.10)$$

So, the left hand side of Eq. (3.10)can be rewritten as

$$\tau\sum_{n=1}^{M}\left(\sum_{j=1}^{m}d_j\,\mathrm{I}_t^{\alpha}\rho^n,\rho^n\right)$$
$$=\sum_{n=1}^{M}\sum_{j=1}^{m}\frac{d_j}{\Gamma\left(1-\alpha_j\right)}[b_0^j\left(\rho^n,\rho^n\right)-\sum_{k=1}^{n-1}\left(b_{n-k-1}^j-b_{n-k}^j\right)\left(\rho^n,\rho^n\right)-b$$
$$\geq\sum_{n=1}^{M}\sum_{j=1}^{m}\frac{d_j}{\Gamma\left(1-\alpha_j\right)}[b_0^j\left\|\rho^n\right\|^2_{L^2(\Omega)}-\sum_{k=1}^{n-1}\left(b_{n-k-1}^j-b_{n-k}^j\right)\left\|\rho^k\right\|_{L^2(\Omega)}\left\|\rho^n\right.$$
$$=\sum_{n=1}^{M}\sum_{j=1}^{m}\frac{d_j}{\Gamma\left(1-\alpha_j\right)}[b_0^j\left\|\rho^n\right\|^2_{L^2(\Omega)}-\sum_{k=1}^{n-1}\left(b_{n-k-1}^j-b_{n-k}^j\right)\left\|\rho^k\right\|_{L^2(\Omega)}-b$$
$$\geq\sum_{j=1}^{m}d_j\left\{\frac{\tau t_M^{-\alpha_j}}{2\Gamma\left(1-\alpha_j\right)}\sum_{n=1}^{M}\left\|\rho^n\right\|^2_{L^2(\Omega)}-\frac{\tau t_M^{1-\alpha_j}}{2\left(1-\alpha_j\right)\Gamma\left(1-\alpha_j\right)}\left\|\rho^0\right\|^2_{L^2(\Omega)}\right.$$

Clearly

$$\sum_{n=1}^{M}B\left(\rho^n,\rho^n\right)\geq 0.$$

Then also we write

$$\sum_{n=1}^{m}\frac{\tau d_j t_M^{-\alpha_j}}{2\Gamma\left(1-\alpha_j\right)}\left\|\rho^M\right\|^2_{L^2(\Omega)}\leq\sum_{n=1}^{m}\frac{\tau d_j t_M^{-\alpha_j}}{2\Gamma\left(1-\alpha_j\right)}\sum_{n=1}^{M}\left\|\rho^n\right\|^2_{L^2(\Omega)}$$
$$\leq\left\{\sum_{n=1}^{m}\frac{d_j t_M^{1-\alpha_j}}{2\left(1-\alpha_j\right)\Gamma\left(1-\alpha_j\right)}\right\}\left\|\rho^0\right\|^2_{L^2(\Omega)}.$$

By changing the notations, we have

$$\sum_{j=1}^{m}\frac{\tau d_j t_n^{-\alpha_j}}{2\Gamma\left(1-\alpha_j\right)}\left\|\rho^n\right\|_{L^2(\Omega)}^2 \le \sum_{j=1}^{m}\frac{\tau d_j t_n^{-\alpha_j}}{2\Gamma\left(1-\alpha_j\right)}\sum_{k=1}^{n}\left\|\rho^k\right\|_{L^2(\Omega)}^2$$

$$\le\left\{\sum_{j=1}^{m}\frac{d_j t_n^{1-\alpha_j}}{2\left(1-\alpha_j\right)\Gamma\left(1-\alpha_j\right)}\right\}\left\|\rho^0\right\|_{L^2(\Omega)}^2$$

Also we write

$$\underbrace{\sum_{j=1}^{m}\frac{\tau d_j t_n^{-\alpha_j}}{2\Gamma\left(1-\alpha_j\right)}\left\|\rho^n\right\|_{L^2(\Omega)}^2}_{c_2} \le \underbrace{\left\{\sum_{j=1}^{m}\frac{d_j t_n^{1-\alpha_j}}{2\left(1-\alpha_j\right)\Gamma\left(1-\alpha_j\right)}\right\}}_{c_1}\left\|\rho^0\right\|_{L^2(\Omega)}^2 \text{ Or}$$

$$\left\|\rho^n\right\|_{L^2(\Omega)} \le \sqrt{\frac{c_1}{c_2}}\left\|\rho^0\right\|_{L^2(\Omega)},$$

Thus

$$\left\|\rho^n\right\|_{L^2(\Omega)} \le c\left\|\rho^0\right\|_{L^2(\Omega)},$$

in which the last inequality completes the proof

Theorem 3.3. Let $U^n\left(x,y,t\right)$ be solution of the semi-discrete scheme. Then the convergence order of the numerical technique is $\mathrm{O}\left(\tau^{2-\alpha}\right)$.

Proof. We consider

$$\begin{cases}\left(\tau\sum_{j=0}^{m}d_j \mathrm{I}_t^{\alpha j}u^n,v\right)+B\left(u^n,v\right)=\tau\left(f^n,v\right)+\tau\left(R_t^{\alpha},v\right), & \forall v\in H_0^{\beta}\left(\Omega\right) \\ u^0=g\left(x,y\right)\end{cases} \quad (3.11)$$

and

$$\begin{cases}\left(\tau\sum_{j=0}^{m}d_j \mathrm{I}_t^{\alpha j}U^n,v\right)+B\left(U^n,v\right)=\tau\left(f^n,v\right), & \forall v\in H_0^{\beta}\left(\Omega\right) \\ U^0=g\left(x,y\right)\end{cases} \quad (3.12)$$

We set $\eta^n=u^n-U^n$ then by subtracting

$$\left(\tau\sum_{j=0}^{m}d_j \mathrm{I}_t^{\alpha j}\eta^n,v\right)+B\left(\eta^n,v\right)=\tau\left(R_t^n,v\right), \quad \forall v\in H_0^{\beta}\left(\Omega\right) \quad (3.13)$$

Similar to Theorem 3.2, we have

$$\tau\sum_{n=1}^{M}\left(\sum_{j=0}^{m}d_j \mathrm{I}_t^{\alpha j}\eta^n,\eta^n\right)\ge\sum_{j=1}^{m}\left\{\frac{\tau d_j t_M^{-\alpha j}}{2\Gamma\left(1-\alpha j\right)}\sum_{n=1}^{M}\left\|\eta^n\right\|_{L^2(\Omega)}^2\right\}, \quad (3.14)$$

And we write

$$\sum_{n=1}^{M}B\left(\eta^n,\eta^n\right)\ge\tau k\left\|\eta^n\right\|_{H^{\beta}(\Omega)}^2$$

Then, by combining the above inequalities, we can get

$$\sum_{j=1}^{m}\left\{\frac{\tau d_j t_n^{-\alpha j}}{2\Gamma\left(1-\alpha j\right)}\sum_{k=1}^{n}\left\|\eta^k\right\|_{L^2(\Omega)}^2\right\}+\sum_{k=1}^{n}B\left(\eta^k,\eta^k\right)\le\tau\sum_{k=1}^{n}\left(R_t^k,\eta^k\right).$$

Thus the previous relation is equal to

$$\tau k\left\|\eta^{n}\right\|_{H^{\beta}(\Omega)}^{2}+\frac{d_{j}\tau T^{-\alpha}}{2\Gamma(1-\alpha)}\sum_{k=1}^{n}\left\|\eta^{k}\right\|_{L^{2}(\Omega)}^{2}$$

$$\tau\sum_{j=1}^{m}\left\{\frac{\tau d_{j}t_{n}^{-\alpha j}}{2\Gamma(1-\alpha)}\sum_{k=1}^{n}\left\|\eta^{k}\right\|_{L^{2}(\Omega)}^{2}\right\}+\sum_{k=1}^{n}B\left(\eta^{k},\eta^{k}\right)\leq\tau\sum_{k=1}^{n}B\left(R_{t}^{k},\eta^{k}\right)$$

$$\leq\tau\sum_{k=1}^{n}\left\|R_{t}^{k}\right\|_{L^{2}(\Omega)}\left\|\eta^{k}\right\|_{L^{2}(\Omega)}$$

Now, we can write

$$\tau k\left\|\eta^{n}\right\|_{H^{\beta}(\Omega)}^{2}\leq\tau^{2}\left\|R_{t}^{k}\right\|_{L^{2}(\Omega)}^{2}+\sum_{k=1}^{n}\tau\left\|R_{t}^{k}\right\|_{L^{2}(\Omega)}\left\|\eta^{k}\right\|_{L^{2}(\Omega)}$$

$$\leq\tau^{2}\left\|R_{t}^{k}\right\|_{L^{2}(\Omega)}^{2}\sum_{k=1}^{n}\left(1+\tau\left\|R_{t}^{k}\right\|_{L^{2}(\Omega)}\right)$$

$$\leq\tau T\left\|R_{t}^{k}\right\|_{L^{2}(\Omega)}^{2}+\tau^{2}T\left\|R_{t}^{k}\right\|_{L^{2}(\Omega)}^{3}$$

$$\leq 2\tau T\left\|R_{t}^{k}\right\|_{L^{2}(\Omega)}^{2}\leq c\tau\left(\tau^{2-\alpha}\right)^{2},$$

Then $\left\|\eta^{n}\right\|_{H^{\beta}(\Omega)}\leq c\tau^{2-\alpha}$.

We divide the computational domain Ω into a number of triangular sub-domains. Let T_h be some triangulation as $\Omega_h=\{e_h:e_h\in T_h\}$,

And h represents the maximum diameter of the triangulation. Let v_k be the space of linear piecewise continuous polynomials. Let the finite element subspace be

$$X_h^k=\{v_h\in c(\bar{\Omega}),\ v_h|_{\partial\Omega}=0, v_h|_e=v_k(x,y), v_h|_{(x,y)\neq e}=0\}.$$

Also, consider P_h^m as the interpolation operator. Thus, for any $w\in H^r(\Omega)$ we can get the following estimation

$$\left\|w-P_h^m\right\|_{H^r(\Omega)}\leq Ch^{m-r}\left\|w\right\|_{H^m(\Omega)},\qquad 0\leq r\leq m. \qquad (3.15)$$

In the next, we define the projection operator $\Pi_h:H_0^\beta(\Omega)\to X_h^k$ that

$$B(\Pi_h u,v)=B(u,v),\qquad \forall v\in X_h^k.$$

Lemma 3.1 ([6]) Let $w\in H^\mu(\Omega)\cap H_0^\beta(\Omega)$ and $\beta\leq\mu\leq k+1$

$$\left\|w-\Pi_h w\right\|_{H^\mu(\Omega)}\leq Ch^{\mu-\beta}H^\mu\left\|w\right\|_{H^\mu(\Omega)}$$

Lemma 3.2 ([41]). Let a_n and f_n be real valued functions defined for $n\in\mathbb{N}$ and suppose that $f_n\geq 0$ for every $n\in\mathbb{N}$ if

$$a_n\leq x_0+\sum_{s=0}^{n-1}f_S a_s,\qquad n\in\mathbb{N},$$

in which x_0 is a non-negative constant then

$$a_n\leq x_0\sum_{s=0}^{n-1}[1+f_s],\qquad n\in\mathbb{N},$$

Now, we are ready to propose the convergence theorem.

Theorem 3.4 Assume $u^n=u(x,y,t_n)$ is the exact solution of problem (1.1) and $U_h{}^n=u(x,y,t_n)$ is its approximate solution respect to the finite element method. Then the numerical solution satisfies the following inequality

$$\left\|u^n-U_h{}^n\right\|\leq C^*\left(\tau^{2-\alpha}+h^2\right).$$

Proof: We have

$$a\left(u_h^n,v_h\right)=\left(\tau\sum_{j=0}^{m}d_j\mathrm{I}_t^{\alpha j}u_h^n,v_h\right)+B\left(u_h^n,v_h\right)=\tau\left(f^n,v_h\right)+\tau\left(R_t^n,v_h\right),\quad v_h\in H_0^\beta(\Omega)$$

And

$$a\left(U_h^n,v_h\right)=\left(\tau\sum_{j=0}^{m}d_j\mathrm{I}_t^{\alpha j}U_h^n,v_h\right)+B\left(U_h^n,v_h\right)=\tau\left(f^n,v_h\right),\quad v_h\in H_0^\beta(\Omega)$$

Subtracting the above relations, we have

$$a\left(u_h^n-U_h^n,v_h\right)=\left(\tau\sum_{j=0}^{m}d_j\mathrm{I}_t^{\alpha j}\left(u_h^n-U_h^n\right),v_h\right)+B\left(u_h^n-U_h^n,v_h\right)=\tau\left(f^n,v_h\right),\quad v_h\in H_0^\beta(\Omega)$$

In other hand we can write

$$\left(\tau\sum_{j=0}^{m}d_j\mathrm{I}_t^{\alpha j}\left(\Pi_h u_h^n-U_h^n\right),v_h\right)+\left(\tau\sum_{j=0}^{m}d_j\mathrm{I}_t^{\alpha j}\left(u_h^n-\Pi_h u_h^n\right),v_h\right)+B\left(\Pi_h u_h^n-U_h^n,v_h\right)=\tau\left(R_t^n,v_h\right),\qquad\forall v_h\in H_0^\beta(\Omega)$$

that using the following notations

$$\Phi_h^n=\Pi_h u_h^n-U_h^n,\quad \psi_h^n=u_h^n-\Pi_h u_h^n, \qquad (3.16)$$

Yields

$$\left(\tau\sum_{j=0}^{m} d_j \mathrm{I}_t^{\alpha j}\Phi_h^n, v_h\right) + B\left(\Phi_h^n, v_h\right) = \tau\left(R_t^n, v_h\right) - \left(\tau\sum_{j=0}^{m} d_j \mathrm{I}_t^{\alpha j}\psi_h^n, v_h\right), \qquad \forall v_h \in H_0^{\beta}(\Omega).$$

We set $v_h = \Phi_h^n$ then

$$\left(\tau\sum_{j=0}^{m} d_j \mathrm{I}_t^{\alpha j}\Phi_h^n, \Phi_h^n\right) + B\left(\Phi_h^n, \Phi_h^n\right) = \tau\left(R_t^n, \Phi_h^n\right) - \left(\tau\sum_{j=0}^{m} d_j \mathrm{I}_t^{\alpha j}\psi_h^n, \Phi_h^n\right), \qquad \forall v_h \in H_0^{\beta}(\Omega).$$

thus using Lemma 3.2, we can write

$$\tau\kappa\left\|\Phi_h^n\right\|_{H^{\beta}(\Omega)}^2 + \frac{dm\tau T^{-\alpha}}{2\Gamma(1-a)}\sum_{k=1}^{n}\left\|\Phi_h^n\right\|_{L^2(\Omega)}^2$$

$$\leq \tau\sum_{j=1}^{m}\left\{\frac{t_n^{-\alpha j}}{2\Gamma(1-\alpha j)}\sum_{k=1}^{n}\left\|\Phi_h^k\right\|_{L^2(\Omega)}^2\right\} + \tau\sum_{k=1}^{n}\left(\Phi_h^k, \Phi_h^k\right)$$

$$\tau\sum_{k=1}^{n}\left(R_t^k, \Phi_t^k\right) + \tau\sum_{k=1}^{n}\left(\sum_{j=0}^{m} d_j \mathrm{I}_t^{\alpha j}\psi_h^k, \Phi_h^k\right)$$

$$\leq \tau\sum_{k=1}^{n}\left\|R_t^k\right\|_{L^2(\Omega)}\left\|\Phi_h^k\right\|_{L^2(\Omega)} + \tau\sum_{k=1}^{n}\left\|\sum_{j=0}^{m} d_j \mathrm{I}_t^{\alpha j}\psi_h^k\right\|_{L^2(\Omega)}\left\|\Phi_h^k\right\|_{L^2(\Omega)}$$

$$\leq \tau\sum_{k=1}^{n}\left\{\left\|R_t^k\right\|_{L^2(\Omega)} + \left\|\sum_{j=0}^{m} d_j \mathrm{I}_t^{\alpha j}\psi_h^k\right\|_{L^2(\Omega)}\right\}\left\|\Phi_h^k\right\|_{L^2(\Omega)}$$

$$\leq \tau C\sum_{k=1}^{n}\left\{\tau^{2-\alpha} + \left(\tau^{2-\alpha} + h^2\right)\right\}\left\|\Phi_h^k\right\|_{L^2(\Omega)}$$

$$\leq \tau^2\left(\tau^{2-\alpha} + h^2\right)^2 + \tau c\left(\tau^{2-\alpha} + h^2\right)\sum_{k=1}^{n}\left\|\Phi_h^k\right\|_{L^2(\Omega)}$$

$$\leq \tau^2\left(\tau^{2-\alpha} + h^2\right)^2\sum_{k=1}^{n}\left[1 + \tau c\left(\tau^{2-\alpha} + h^2\right)\right]$$

$$\leq \tau T\left(\tau^{2-\alpha} + h^2\right)^2 + \tau^2 TC\left(\tau^{2-\alpha} + h^2\right)^3$$

$$\leq \tau T\left(\tau^{2-\alpha} + h^2\right)^2 + \tau TC\left(\tau^{2-\alpha} + h^2\right)^2$$

$$\leq C_1\tau T\left(\tau^{2-\alpha} + h^2\right).$$

Thus

$$\left\|\Phi_h^n\right\|_{H^{\beta}(\Omega)} \leq C_1^*\left(\tau^{2-\alpha} + h^2\right). \tag{3.17}$$

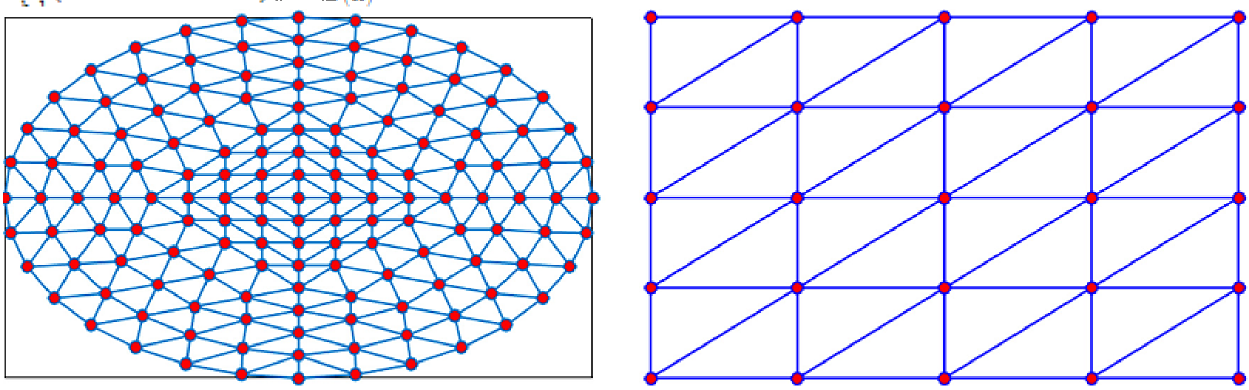

Figure 1: The employed rectangular and non-rectangular domains.

The use of triangle inequality, yields

$$\left\|u_h^n - U_h^n\right\|_{H^{\beta}(\Omega)} \leq \left\|\Phi_h^n\right\| + \left\|\psi_h^n\right\| \leq C_1^*\left(\tau^{2-\alpha} + h^2\right) + C_u^* h^2 \leq C^*\left(\tau^{2-\alpha} + h^2\right) \tag{3.18}$$

Which completes the proof.

Usually, for calculating the appeared integrals in variational weak forms, the Gauss quadrature formula should be employed. But, in the fractional calculus, we can not use the Gauss quadrature formula directly. For the convex domains there are some papers such as two-dimensional Riesz space fractional diffusion equations with nonlinear source term on irregular convex domains [47] and a novel unstructured mesh finite element method for solving the time space fractional wave equation on a two-dimensional irregular convex domain [20]. In the current paper, we employ the Gauss quadrature formula in [47]for convex domains.

4. Numerical investigation of the developed scheme

We present some test problem to check the stability and convergence of proposed numerical technique by performing the mentioned method for different values of h and τ for simulating the approximation solutions; we applied the Matlab 16 software on a machine with 8Gbyte of memory. Also, for comparing the theoretical convergence orders with the computational convergence orders the following relation is employed

$$C_1-order=\frac{\log\left(\frac{E_1}{E_2}\right)}{\log\left(\frac{h_1}{h_2}\right)},$$

In which E_1 and E_2 are the obtained error respect to the space-sizes h_1 and h_2 respectively the employed rectangular and non-rectangular domains in the current paper are demonstrated in Fig.1

4.1. Test problem 1

We consider the following equation [6]

$$\begin{cases}\sum_{j=0}^{2}{}_0^cD_t^{\alpha j}u=\frac{\partial^{2\beta}u}{\partial|x|^{2\beta}}+\frac{\partial^{2\beta}u}{\partial|y|^{2\beta}}+f(x,y,t), & (x,y,t)\in\Omega\times(0,0.5], \qquad (4.1)\\ u(x,y,0)=0, & (x,y)\in\Omega,\\ u(x,y,t)_{\partial\Omega}=0, & t\in(0,0.5],\end{cases}$$

in which $0<\alpha<1,\frac{1}{2}<\beta<1$ and

$$f(x,y,t)=10(x-x^2)(y-y^2)\left[\frac{t^{1-\alpha_0}}{\Gamma(2-\alpha_0)}+\frac{t^{1-\alpha_1}}{\Gamma(2-\alpha_1)}+\frac{t^{1-\alpha_2}}{\Gamma(2-\alpha_2)}\right]$$
$$-\frac{5t(y^2-y)}{\cos(\beta\pi)}\left[\frac{x^{1-2\beta}+(1-x)^{1-2\beta}}{\Gamma(2-2\beta)}-\frac{2x^{2-2\beta}+2(1-x)^{2-2\beta}}{\Gamma(3-2\beta)}\right]$$
$$-\frac{10t(x^2-x)}{\cos(\beta\pi)}\left[\frac{y^{1-2\beta}+(1-y)^{1-2\beta}}{\Gamma(2-2\beta)}-\frac{2y^{2-2\beta}+2(1-y)^{2-2\beta}}{\Gamma(3-2\beta)}\right].$$

Table 1: Errors and computational orders obtained with $\tau=\frac{1}{1000}$, $\alpha_0=0,25,\alpha_1=0,45\,and\,\alpha_2=0,65$ for Test problem 1

h	$2\beta=1.25$		$2\beta=1.75$	
	L_∞	C_1-order	L_∞	C_1-order
1/4	7.4810×10^{-2}	–	8.1151×10^{-2}	–
1/8	2.3421×10^{-2}	1.6754	2.6390×10^{-2}	1.6206
1/16	7.0231×10^{-3}	1.7376	8.0203×10^{-3}	1.7183
1/32	1.9890×10^{-3}	1.8201	2.1698×10^{-3}	1.8861
1/64	4.9539×10^{-4}	2.0054	5.5731×10^{-4}	1.9610
1/128	1.2345×10^{-4}	2.0045	1.4023×10^{-4}	1.9907

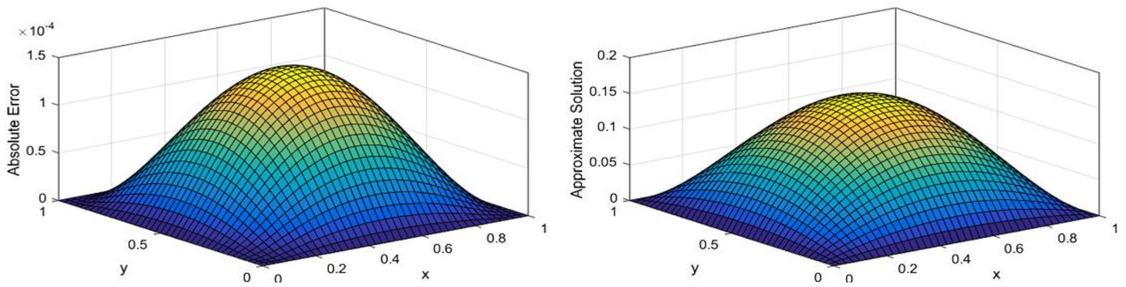

Figure 2: Approximate solution and absolute error with $a_0=0.15,a_1=0.55,a_2=0.95,\beta=0.6,h=1/40$ And $\tau=1/100$ on rectangular domain for Test problem 1

Then the exact solution is

$$u(x,y,t)=10(x-x^2)(y-y^2).$$

We solve this problem using the proposed technique on rectangular domain. Table 1 shows errors and computational orders obtained with $\tau=1/100$, $a_0=0.25,a_1=0.45\ and\ a_2=0.65$

for Test problem 1. Table 1 presents the theoretical order is closed to the computational order in the spatial direction.

Fig. 2 demonstrates approximate solution and absolute error with $a_0 = 0.15, a_1 = 0.55, a_2 = 0.95,\ 2\beta = 1.6, h = 1/40$ and $\tau = 1/100$ on rectangular domain for Test problem 1.

4.2. Test problem 2

For the second example, we want to solve [47]

$$\begin{cases} \sum_{j=0}^{1} {}_0^c D_t^{\alpha j} = \frac{\partial^\alpha u}{\partial |x|^\alpha} + \frac{\partial^\beta u}{\partial |y|^\beta} + f(x,y,t), & (x,y,t) \in \Omega \times (0,1], \\ u(x,y,0) = 0, & (x,y) \in \Omega, \\ u(x,y,t)_{\partial\Omega} = 0, & t \in (0,1], \end{cases} \quad (4.2)$$

in which $0 < \alpha_j < 1,\ 1 < \alpha,\ \ \beta < 2$ *and*

$$\begin{aligned} f(x,y,t) = {} & 100t^3 g_1\left(x + \sqrt{1-y^2}, -\sqrt{1-y^2}\right) \\ & + 100t^3 g_1\left(x + \sqrt{1-y^2} - x, \sqrt{1-y^2}\right) \\ & + 100t^3 g_2\left(-\sqrt{1-x^2}, y + \sqrt{1-x^2}\right) \\ & + 100t^3 g_2\left(\sqrt{1-x^2}, \sqrt{1-x^2} - y\right) \\ & + \frac{100\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\alpha}\left(x^2+y^2-1\right)^2, \end{aligned}$$

and also we have

$$g_1(x,y) = \frac{8y^2 x^{2-2\alpha}}{\Gamma(3-2\alpha)} + \frac{24yx^{3-2\alpha}}{\Gamma(4-2\alpha)} + \frac{24x^{4-2\alpha}}{\Gamma(5-2\alpha)},$$

$$g_2(x,y) = \frac{8x^2 y^{2-2\alpha}}{\Gamma(3-2\beta)} + \frac{24xy^{3-2\alpha}}{\Gamma(4-2\beta)} + \frac{24y^{3-2\alpha}}{\Gamma(5-2\beta)}.$$

Table 2: Errors and computational orders obtained with $\tau = \frac{1}{1000}, a_0 = 0.45, a_1 = 0.85$ for Test problem 2.

h	$2\beta = 1.15$		$2\beta = 1.65$	
	L_∞	C_1-order	L_∞	C_1-order
1/4	9.1817×10^{-2}	–	1.4872×10^{-1}	–
1/8	3.2231×10^{-2}	1.5103	5.3220×10^{-2}	1.4826
1/16	1.0431×10^{-2}	1.6276	1.5989×10^{-2}	1.7352
1/32	3.1108×10^{-3}	1.7455	4.5323×10^{-3}	1.8184
1/64	8.0115×10^{-4}	1.9571	1.1370×10^{-3}	1.9950
1/128	1.9908×10^{-4}	2.0087	2.8343×10^{-4}	2.0042

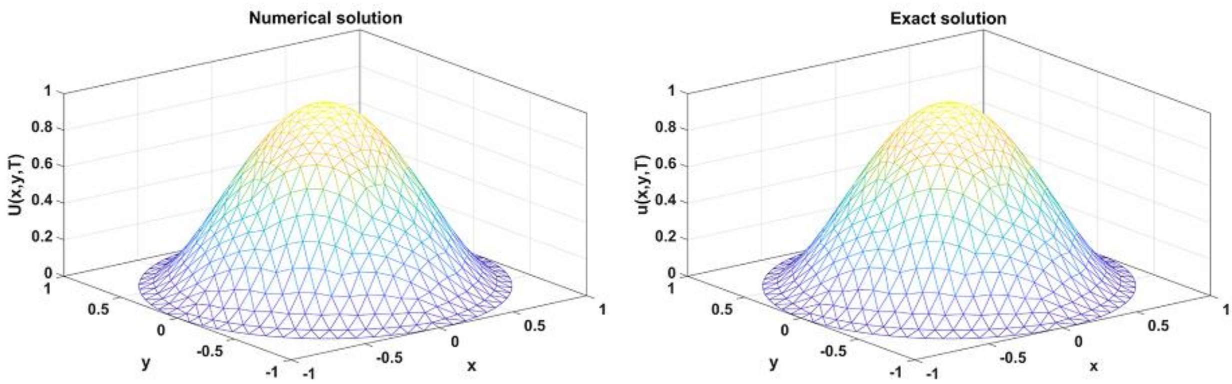

Figure 3: Approximate and exact solutions with $a_0 = 0.45, a_1 = 0.85, a = 1.3, \beta = 1.6, h = 1/40$ and $\tau = 1/1000$ on circular domain for Test problem 2.

Then the exact solution is
$u(x,y,t) = 100t^3(x^2 + y^2 - 1)^2$

We solve this problem using the proposed technique on rectangular and circular domains.

Table 2
Presents errors and computational orders obtained with $\tau = 1/1000, a_0 = 0.45, a_1 = 0.85, a = 1.15$ and $\beta = 1.7$ for Test problem 2. Fig. 3 illustrates the approximate and exact solutions with $a_0 = 0.45, a_1 = 0.85, a = 1.3, \beta = 1.6, h = 1/40$ and $\tau = 1/1000$ on circular domain for Test problem2.

4.3. Test problem 3

We consider the following example [39, 40]

$${}_0^c D_t^\alpha u(x,y,t) = \frac{1}{2}\left(\frac{\partial^\beta u(x,y,t)}{\partial |x|^\beta} + \frac{\partial^\beta u(x,y,t)}{\partial |y|^\beta}\right) + f(x,y,t), \quad (x,y) \in \Omega, 0 < t \le 1,$$

$$u(x,y,0)=0, \quad (x,y)\in \bar{\Omega},$$
$$u(x,y,t)=0, \quad (x,y)\in \partial\Omega, \qquad 0\le t\le 1,$$

in which $0<\alpha<1,\ \ 1<\beta<2$ and

$$\begin{aligned} f(x,y,t)=\{[&\frac{9}{\Gamma(9-\beta)}\left(x^{8-\beta}+(1-x)^{8-\beta}\right)-\frac{4\Gamma(8)}{\Gamma(8-\beta)}\left(x^{7-\beta}+(1-x)^{7-\beta}\right)\\ &+\frac{6\Gamma(7)}{\Gamma(7-\beta)}\left(x^{6-\beta}+(1-x)^{6-\beta}\right)-\frac{4\Gamma(6)}{\Gamma(6-\beta)}\left(x^{5-\beta}+(1-x)^{5-\beta}\right)\\ &++\frac{\Gamma(5)}{\Gamma(5-\beta)}\left(x^{4-\beta}+(1-x)^{4-\beta}\right)]y^4(1-y)^4\\ &+[+\frac{9}{\Gamma(9-\beta)}\left(y^{8-\beta}+(1-y)^{8-\beta}\right)-+\frac{4\Gamma(8)}{\Gamma(8-\beta)}\left(y^{7-\beta}+(1-y)^{7-\beta}\right)\\ &+\frac{6\Gamma(7)}{\Gamma(7-\beta)}\left(y^{6-\beta}+(1-y)^{6-\beta}\right)-\frac{4\Gamma(6)}{\Gamma(6-\beta)}\left(y^{5-\beta}+(1-y)^{5-\beta}\right)\\ &+\frac{\Gamma(5)}{\Gamma(5-\beta)}\left(y^{4-\beta}+(1-y)^{4-\beta}\right)]x^4(1-x)^4\}\frac{t^3}{4\cos\left(\frac{\beta\pi}{3}\right)}+\frac{\Gamma(4)}{\Gamma(4-\alpha)}t^{3-\alpha}x^4(1-x)^4y^4(1-y)^4. \end{aligned}$$

Table 3
Comparison between errors obtained with $h=1/50$, $\alpha=0.8$, $\beta=1.5$ for Test problem 3.

τ	Method of [39]	Method of [40] scheme (4.10)–(4.12)	Method of [40] scheme (4.21)–(4.23)	Present method
1/2	2.0511×10^{-6}	2.6102×10^{-6}	2.6213×10^{-6}	2.3185×10^{-6}
1/4	1.1254×10^{-6}	6.3747×10^{-7}	6.6560×10^{-7}	1.0181×10^{-6}
1/8	5.9281×10^{-7}	1.4901×10^{-7}	1.6263×10^{-7}	5.3319×10^{-7}
1/16	3.0833×10^{-7}	2.6903×10^{-8}	4.0640×10^{-8}	2.8918×10^{-7}

Then, the exact solution is

$$u(x,y,t)=t^3x^4(1-x)^4y^4(1-y)^4. \qquad (4.3)$$

Also, for this problem, we assume

$$S_\infty(h,\tau)=\max_{0\le k\le N}\left\|U^k-u^k\right\|_\infty. \qquad (4.4)$$

We solve the current example using the developed method. Also, we compare the obtained numerical results of the finite element method with methods of [39] and [40]. Table 3 illustrates a comparison between errors obtained with $h=1/50, a=0.8, \beta=1.5$ for Test problem 3

4.4. Test problem 4

We consider the following example [18,30]

$${}_0^C D_t^\alpha u(x,y,t)=\left(\frac{\partial^\beta u(x,y,t)}{\partial|x|^\beta}+\frac{\partial^\beta u(x,y,t)}{\partial|y|^\beta}\right)+ f(x,y,t),\quad (x,y)\in\Omega, 0<t\le 1,$$

$$u(x,y,0)=0, \quad (x,y)\in \bar{\Omega},$$
$$u(x,y,t)=0, \quad (x,y)\in \partial\Omega, \qquad 0\le t\le 1,$$

in which $0<\alpha<1,\ \ 1<\beta<2$ and

$$\begin{aligned} f(x,y,t)=\{[&\frac{\Gamma(11)}{\Gamma(11-\beta)}\left(x^{10-\beta}+(1-x)^{10-\beta}\right)-\frac{10\Gamma(12)}{\Gamma(12-\beta)}\left(x^{11-\beta}+(1-x)^{11-\beta}\right)\\ &+\frac{45\Gamma(13)}{\Gamma(13-\beta)}\left(x^{12-\beta}+(1-x)^{12-\beta}\right)-\frac{10\Gamma(14)}{\Gamma(14-\beta)}\left(x^{13-\beta}+(1-x)^{13-\beta}\right)\\ &+\frac{210\Gamma(15)}{\Gamma(15-\beta)}\left(x^{14-\beta}+(1-x)^{14-\beta}\right)-\frac{252\Gamma(15)}{\Gamma(16-\beta)}\left(x^{15-\beta}+(1-x)^{15-\beta}\right)]\\ &+\frac{210\Gamma(17)}{\Gamma(17-\beta)}\left(x^{16-\beta}+(1-x)^{16-\beta}\right)-\frac{120\Gamma(15)}{\Gamma(18-\beta)}\left(x^{17-\beta}+(1-x)^{17-\beta}\right)]\\ &+\frac{45\Gamma(19)}{\Gamma(19-\beta)}\left(x^{18-\beta}+(1-x)^{18-\beta}\right)-\frac{10\Gamma(20)}{\Gamma(20-\beta)}\left(x^{19-\beta}+(1-x)^{19-\beta}\right)\\ &+\frac{\Gamma(21)}{\Gamma(21-\beta)}\left(x^{20-\beta}+(1-x)^{20-\beta}\right)\}y^{10}(1-y)^{10}\\ &+\{[\frac{\Gamma(11)}{\Gamma(11-\beta)}\left(y^{10-\beta}+(1-y)^{10-\beta}\right)-\frac{10\Gamma(12)}{\Gamma(12-\beta)}\left(y^{11-\beta}+(1-y)^{11-\beta}\right)\\ &+\frac{45\Gamma(13)}{\Gamma(7-\beta)}\left(y^{12-\beta}+(1-y)^{12-\beta}\right)-\frac{10\Gamma(12)}{\Gamma(12-\beta)}\left(y^{11-\beta}+(1-y)^{11-\beta}\right)\\ &+\frac{210\Gamma(15)}{\Gamma(15-\beta)}\left(y^{14-\beta}+(1-y)^{14-\beta}\right)-\frac{252\Gamma(15)}{\Gamma(16-\beta)}\left(y^{15-\beta}+(1-y)^{15-\beta}\right)]. \end{aligned}$$

Table 4: Errors and computational orders obtained with $\tau=1/1000, a=0.45$ and $\beta=1.25$ for Test problem 4.

h	Method of [18]		Present method	
	L_∞	C_1-order	L_∞	C_1-order
1/8	4.3856×10^{-4}	–	1.8570×10^{-3}	–
1/16	1.2475×10^{-5}	5.1357	5.2125×10^{-4}	1.8329
1/32	2.4013×10^{-7}	5.6990	1.3142×10^{-4}	1.9878
1/64	3.8207×10^{-9}	5.9738	3.2927×10^{-5}	1.9968

Table 5: Errors and computational orders obtained with $\tau = 1/1000, a = 0.15$ and $\beta = 1.7$ for Test problem4

h	Method of [30]		Present method	
	L_∞	C_1-order	L_∞	C_1-order
1/8	1.4713×10^{-3}	–	1.8524×10^{-3}	–
1/16	4.3519×10^{-4}	1.7574	5.2013×10^{-4}	1.8324
1/32	1.1102×10^{-4}	1.9708	1.314×10^{-4}	1.9877
1/64	2.7748×10^{-5}	2.0003	3.2857×10^{-5}	1.9968

$$+\frac{210\Gamma(17)}{\Gamma(17-\beta)}\left(y^{16-\beta}+(1-y)^{16-\beta}\right)-\frac{120\Gamma(15)}{\Gamma(18-\beta)}\left(y^{17-\beta}+(1-y)^{17-\beta}\right)]$$

$$+\frac{45\Gamma(19)}{\Gamma(19-\beta)}\left(y^{18-\beta}+(1-y)^{18-\beta}\right)-\frac{10\Gamma(20)}{\Gamma(20-\beta)}\left(y^{19-\beta}+(1-y)^{19-\beta}\right)]$$

$$+\frac{\Gamma(21)}{\Gamma(21-\beta)}\left(y^{20-\beta}+(1-y)^{20-\beta}\right)]x^{10}(1-x)^{10}$$

$$+200\left(t^2-2t\right)x^{10}(1-x)^{10}y^{10}(1-y)^{10}.$$

Then, the exact solution is

$$u(x,y,t)=200\left(t^2-2t\right)x^{10}(1-x)^{10}y^{10}(1-y)^{10}.$$

Table 4 shows errors and computational orders obtained with $\tau = 1/1000, a = 0.45$ and $\beta = 1.25$ for Test problem 4. Also in Table 4, we compare our numerical results with the proposed method in [18]. Table 5 presents errors and computational orders obtained with $\tau = 1/1000, a = 0.15$ and $\beta = 1.7$ for Test problem 4. Furthermore, the numerical results of the current paper have been compared with the method of [30].

5. Conclusion

In this investigation, we proposed a new numerical technique by employing the finite element method and finite difference approach for the fractional Bloch–Torrey equations. The space derivative is based on the Riemann–Liouville fractional derivative and the time derivative is based on the Caputo fractional derivative. At first, we obtained a time-discrete scheme for the temporal direction using a difference scheme with convergence order $o\left(\tau^{2-\alpha}\right)$ in the next, the Galerkin finite element technique is applied on spatial direction to obtain a full-discrete scheme. We showed the appeared bilinear form in the Galerkin finite element method is continuous and V -elliptic. The unconditional stability and convergence of the proposed numerical plan are studied using the energy method. We solved the considered model on some non-rectangular domains. The numerical results demonstrate the developed technique is useful and effective for solving fractional Bloch–Torrey equations and other fractional partial differential equation equations on irregular domains.

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