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On Differentiability and Integrability of Rings

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Abstract: ***In this paper we study properties of the differential ideals of a ring R and of the iterated skew polynomial rings over R defined with respect to a finite set of commuting derivations of R. The concept of the integration of R associated to a given derivation of R is also introduced and some fundamental properties are studied. This new concept generalizes basic features of the indefinite integrals.***

Keywords: Derivations and integrations associated to derivations

1. Introduction

All the rings considered in this paper are with identity and they have characteristic zero. A derivation on a ring is a function which generalizes certain features of the traditional derivative operator. On the other hand the term integration is connected to the computation of an integral.

In the present work properties of the differential ideals of a ring R and of the iterated skew polynomial rings over R is defined with respect to a finite set of commuting derivations of R. The concept of the integration of R associated to a given derivation of R is also introduced and some fundamental properties of it are studied. This new concept generalizes basic features of the indefinite integrals.

2. Derivations and Differential Simplicity of Rings

We start by recalling the following definitions:

2.1 Definition: Let R be a ring. Then a map $d:R \to R$ is called a derivation of R, if and only if, $d(x + y) = d(x) + d(y)$ and $d(x,y) = xd(y) + d(x)y$, for all x, y in R. Observe that $d(1) = d(1.1) = 2d(1)$, therefore $d(1) = 0$. The set of all derivations of R is denoted by Deri(R) Given a non commutative ring R and an element s in R it is easy to check that the map $d: R \to R$ defined by $d(r) = sr - rs$ is a derivation of R, called the inner derivation of R induced by s. For distinguishing between the two cases, a derivation of R which is not inner is called an outer derivation.

2.2 Definition: Let R be a ring and let d be a derivation of R**.** Then an ideal I of R is said to be a d-ideal, if $d(I) \subseteq I$. If the only d-ideals of R are 0 and R, then R is called as d-simple ring and d is called a simple derivation of R. Non commutative d-simple rings exist in abundance; for example every simple ring is d-simple for any derivation d of R. On the other hand, there is not known any general criterion under which one can decide whether or not a commutative ring possesses simple derivations. Typical examples of such rings are the polynomial rings in finitely many variables over a field 1 and the regular local rings of finitely generated type over a field 2. More examples can be found in 1, whereas in 3 geometrical examples are presented of smooth varieties (algebraic sets) over a field with coordinate rings possessing simple derivations.

It is well known that if a commutative ring R is d-simple then R is an integral domain and also that if R has no non zero prime d-ideals, then R is a d-simple ring (4; Corollary 1.5).

Definition 2.2 can be generalized for a finite set D of derivations of R as follows:

2.3 Definition: Let D be a finite set of derivations of R. Then an ideal I of R is called a D-ideal if $d(I) \subseteq I$ for all d in D and R is called a D – simple ring, if it has no proper nonzero. D-ideals (differential simplicity of R). Obviously, if R is a d-simple ring for some d in D, then R is also a D-simple ring, but the converse is not true; e.g. this happens with the coordinate ring of the real sphere (5, Lemma 3.1).

2.4 Definition: Let R be a ring and let d be a derivation of R. Define on the set S of all polynomials in one variable x over R addition in the usual way and multiplication by the rule; $xr = rx + d(r)$, for all r in R. It is well known then that S becomes a non commutative ring denoted by R [x, d] and called a skew polynomial ring (of derivation type) over R (e.g. sec 6, p.35).

Such rings, which are also known as Ore extensions, have been firstly introduced by O. Ore 7 to be used as counter examples. Note that skew polynomial rings can also be defined over R with respect to an endomorphism f of R and in a more general context with respect to f and an f-derivation d of R 6, which is a generalization of the concept of the ordinary derivation. We continue with the following useful Lemma:

2.5 Lemma: Let R be a ring, let d be a derivation of R and let S = R[x, d] be the corresponding skew polynomial ring over R. Let also d* be another derivation of R. Then d* extends to a derivation of S by $d^*(x) = 0$, if, and only if, d* commutes with d.

Proof 8: Obviously d* extends to a derivation of S, if, and only if, $d^*(x)$ can be defined in a way compatible to multiplication in S. In other words, if $d^*(x) = h$, then for all r in R we must have $d^*(xr) = d^*(rx) + d^*[d(r)] \Leftrightarrow xd^*(r)+hr =$

rh +d*(r)x+d*[d(r)] ⇔ d*(r)x+d[d*(r)]+hr 4.rh+d*(r)x+d*[d(r)],

Therefore h=0 ⇔ d[d*(r)]=d*[d(r)], which completes the proof.

Let now D ={$d_1, d_2,\dots, d_n$} be a finite set of derivations of R commuting to each other, i.e. we have that d_i o d_j = d_j o d_i, i, j = 1, 2, …, n. Consider the set S_n of all polynomials in n variables $x_1, x_2, \dots x_n$ and defined addition in S_n in the usual way and multiplication by the rules $x_ir = rx_i+d_i(r)$, $x_ix_j = x_jx_i$, for all r in R and all I, j = 1,2,…,n.

Set $S_1 = R[x_1, d_1]$ and, using Lemma 2.5, consider the skew polynomial rings $S_2 = S_1(x_2, d_2)$, $S_{k+1} = S_k [x_k, d_k],\dots,S_n=S_{n-1}[x_n, d_n]$.

The ring S_n = R $[x_1, d_1]$ $[x_2, d_2]$ …, $[x_n, d_n]$, introduced by Voskoglou 8, is called an iterated skew polynomial ring (ISPR) of derivation type over R and for brevity will be denoted by S_n = R[X, D].

Voskoglou 9 has also introduced ISPRs over R with respect to a finite set {$f_1, f_2,\dots,f_n$} of monomorphisms of R and a corresponding set {$d_1, d_2, \dots. d_n$} of f_i – derivations of R, such that d_i o d_j = d_j o d_i o f_j = f_j o d_i and f_i o f_j = f_j o f_i.

Other types of ISPRs, in which multiplication I defined only by the rule $x_ir = rx_i + d_i(r)$, for all r in R and all i = 1, 2,…,n, but the variables need not to commute, have been introduced by Kishmoto 10 and by others. To distinguish between the two cases we denote the ISPRs of the second case by S_n*.

Note that in S_n* the derivations of D need not commute to each other. We prove the following result about this:

2.6 Proposition: Let R be a ring and let D be a finite set of derivations of R. Then, if the variables of an ISPR over R defined with respect to D commute, the derivations of D commute too.

Proof: Given r in R and two variables x_i and x_j of the ISPR over R we have that

$$\begin{aligned} x_ix_jr &= x_i\left[rx_j+d_j(r)\right]=(x_ir)x_j+x_id_j(r) \\ &= \left[rx_i+d_i(r)\right]x_j+d_j(r)x_i+d_id_j(r) \\ &= rx_ix_j+d_i(r)x_j+d_j(r)x_i+(d_iod_j)(r) \end{aligned}$$

In the same way we find that

$$x_jx_ir = rx_jx_i+d_j(r)x_i+d_i(r)x_j+(d_jod_i)(r)$$

Assuming that $x_ix_j = x_jx_i$ the result follows by equating the right members of the last two equations.

The ISPRs have found recently two important applications resulting to the renewal of the researcher’s interest about them. The former concerns the ascertainment that many Quantum Groups (i.e. Hopf algebras having in addition a structure analogous to that of a Lee group 11), which are used as a basic tool in Theoretical Physics, can be expressed and studied in the form of an ISPR. The latter concerns the utilization of ISPRs in Cryptography for analyzing the structure of certain codes 12.

Voskoglou has also proved the following result 8.

2.7 Theorem: Let R be a ring, let D = {d_1, , …., d_n} be a finite set of derivations of R commuting to each other and let S_n = R[X, D] be the corresponding ISPR over R. Assume further that d_i is an outer derivation of S_{i-1}, where S_O = R. Then S_n is a simple ring, if, and only if, R is a D – simple ring.

As an example, consider the polynomial ring R = $k[y_1, y_2,\dots y_n]$ over a field k and the set $D=\left\{\frac{\partial}{\partial y_1},\frac{\partial}{\partial y_2},\dots,\frac{\partial}{\partial y_n}\right\}$ of partial derivatives of R. Then it is straightforward to check that R is a D – simple ring (13; Example 1), therefore by the previous theorem the ISPR R [X, D] is a simple ring.

Theorem 2.6 for n = 1 is due to D. Jordan 14.

The following definition generalizes the notion of a prime ideal of a ring:

2.8 Definition: Let R be a ring and let D be a finite set of derivations of R. Then a D-ideal I of R is said to be a D – prime ideal, if given any two D-ideals A and B of R such that AB ⊆ I, it is either A ⊆ I or B ⊆ I. In particular, R is called a D – prime ring, if (0) is a D – prime ideal of R. The next result 15 establishes a relationship among the prime ideals of S_n and the D – prime ideals of R.

2.9 Theorem: Let R be a ring, let D be a finite set of derivations of R commuting to each other and let S_n = R[X,D] be the corresponding ISPR over R. Then: Y If P is a prime ideal of S_n P ∩ R is a D – prime ideal of R. Y If I is a prime ideal of R, IS_n is a prime ideal of S_n.

3. Main Results

Let R be a commutative ring, let d be a derivation and let I be an ideal of R, Then it is straightforward to check that $d^{-1}(I)$ = {r ∈ R: d(r) ∈I} is a sub-ring of R. We shall prove the following result:

3.1 Theorem: Let P be a prime d-ideal of R, then the ring $d^{-1}(P)$ is integrally closed in R.

Proof: It suffice to show that, if r is an element of R integral over $d^{-1}(P)$, then r is in $d^{-1}(P)$. In fact, since r is integral over $d^{-1}(P)$, there exists a monic polynomial $f(x) = x^n + a_{n-1}x^{n-1}+\dots+a_1x+a_0$ of minimal degree n with coefficients in $d^{-1}(P)$, such that $f(r) = r^n+a_{n-1}r^{n-1}+\dots+a_1r+ao = 0$. Differentiating this equation with respect to d one gets that

$$[n\,r^{n-1}+(n-1)a_{n-1}r^{n-2}+\cdots+a_1]d(r)$$
$$+d(a_{n-1})r^{n-1}+\cdots+d(a_1)r=0 \text{ or}$$
$$r_0d(r)=-[d(a_{n-1})r^{n-1}+\cdots+d(a_1)r]$$

with

$$r_0=nr^{n-1}+(n-1)a_{n-1}r^{n-2}+\cdots+a_1$$

But, since $a_{n-1},\dots,a_1$ are in $d^{-1}(P)$, we get that $d(a_{n-1}),\dots, d(a_1)$ are in P. Therefore $r_o d(r)$ is in P, which implies that either r_o is in P or d(r) is in P. But, if r_o is in P, $d(r_o)$ is also in P, therefore r_o is in $d^{-1}(P)$. Thus equation (1) contradicts to the minimality of n in f(x). Consequently d(r) is in P, which shows that r is in $d^{-1}(p)$ and this completes the proof of the theorem.

Let now s = a+d(b) be an element of I + d(I), with a, b in the ideal I of R. Then d(rb) = rd(b)+d(r)b, therefore rs = ra + rd(b) = ra + [d(rb)-d(r)b]=[ra-d(r)b] +d (rb) is in I+d(I), for all r in R. Consequently I + d(I) is an ideal of R.

Assume now that R is a local ring, i.e. a Noetherian ring with a unique maximal ideal M. If M is not a d-ideal of R, then M + d(M) is an ideal of R containing properly M, therefore M +d(M) = R. On the other hand, it becomes clear that the ideal $M^k+d(M^k)\subseteq M$, for all integers k, k ≥ 2. In particular, for k = 2 we shall prove the following result:

3.2 Theorem

Let R be a local ring with maximal ideal M and let d be a derivation of R such that M is not a d-ideal of M. Then $M^2+d\left(M^2\right)=M$.

Proof: Since R is a Noetherian ring, M is a finitely generated ideal of R. Therefore, we can write M = $(m_1, m_2,\dots,m_k)$ for some positive integer k.

Since M is not a d-ideal of R, there exists at least one generator m_s of M such that $d(m_s)$ is not in M. We can write then M = $(m_1 + m_s, m_2+m_s,\dots,m_k + m_s)$. Therefore, without loss of generality we may assume that $d(m_i)$ is not in M, for all i = 1, 2,…,k. Consequently $d(m_i)$ is a unit of R, because otherwise we should have that $(d(m_i))$ is a proper ideal of R, which implies that $(d(m_i)) \subseteq M$, or $d(m_i) \in M$, a contradiction. In other words, there exists r_i in R such that $r_i d(m_i) = 1$.

Then $d\left(m_i^2\right)=2m_i d\left(m_i\right)=2m_i\left(r_i^{-1}\right)$ is in $M^2+d\left(M^2\right)$, therefore $m_i=\frac{r_i}{2}\left[2m_i\left(r_i^{-1}\right)\right]$ is also in $M^2+d\left(M^2\right)$, which completes the proof.

We now introduce the following concept:

3.3 Definition: Let R be a ring and let d be in DerR. Then the integration of R associated to d is a map i: R → R such that d[i(x)] = x, for all x in R.
Next we shall prove:

3.4 Theorem: Let d be an injective derivation of a ring R and let i be the integration of R associated to d. Then i is a derivation of R, if, and only if,

$$xy=-\left[i\left(x\right)d\left(y\right)+d\left(x\right)i\left(y\right)\right] \text{ For all x, y in R.}$$

Proof: For all x, y in R we have by definition 2.2 that $d\left[i\left(x+y\right)\right]=x+y$. We also have that

$$d\left[i\left(x\right)+i\left(y\right)\right]=d\left[i\left(x\right)\right]+d\left[i\left(y\right)\right]=x+y.$$

Therefore, since d is an injective map, we obtain that $i\left(x+y\right)=i\left(x\right)+i\left(y\right)$

On the other hand, we have that
$d[i(xy)] = xy$ and
$d[xi(y) + i(x)y] = d[xi(y)] + d[i(x)y]$
$= x[d[i(y)]] + d(x)i(y) + i(x)d(y) + d[i(x)]y$
$= 2xy + d(x)i(y) + i(x)d(y)$

On comparing the last two equations we obtain that

$$d\left[i\left(xy\right)\right]=d\left[xi\left(y\right)+i\left(x\right)y\right],$$

if, and only if,

$$xy=2xy+d\left(x\right)i\left(y\right)+i\left(x\right)d\left(y\right)\cdot$$

This, combined to the fact that d is an injective map, it finally shows that

$$\left[i\left(xy\right)\right]=xi\left(y\right)+i\left(x\right)y,$$

if, and only if,

$$xy=-\left[i\left(x\right)d\left(y\right)+d\left(x\right)i\left(y\right)\right]$$

which, together with equation (2) completes the proof of the theorem.

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