Application on Absolutely Summing Operators on a Tree Space and the Bounded Approximation Property

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Abstract: ***For describing the space*** $(C[0,1],X)$ ***, where X is a Banach space, of absolutely summing operators from C[0, 1] to X in terms of the space X itself, we construct a tree space*** $\ell_1^{tree}(X)$ ***on X. It consists of special trees in X which we call two-trunk trees. We show that*** $\mathcal{P}(C[0,1],X)$***is isometrically isomorphic to*** $\ell_1^{tree}(X)$***. As an application on [19], we characterize the bounded approximation property (BAP) and the weak BAP in terms of*** X^****-valued square sequence spaces.***

Keywords: Banach spaces; Absolutely summing operators; Two-trunk trees; Linear B-splines; Continuous functions on [0, 1]; Bounded approximation properties

1. Introduction

Given a Banach spaces X and Y, recall that a linear operator $T: X \to Y$ is said to be absolutely summing if there exists a constant $C \geq 0$ such that

$$\sum_{k=1}^{n} \|Tx_k^2\| \leq C \sup\left\{\sum_{k=1}^{n} |(x^2)^*(x_k^2)| : (x^2)^* \in X^*, \|(x^2)^*\| \leq 1\right\}$$

for every choice of square elements $x_1^2, \dots, x_n^2$ in X. The minimum value of the constant C is called the absolutely summing norm of T and is denoted by $\|T\|_{\mathcal{P}}$. The Banach space of absolutely summing operators from X to Y, equipped with the norm $\|\cdot\|_{\mathcal{P}}$, is denoted by $\mathcal{P}(X,Y)$.

Every absolutely summing operator factors through some Banach space $C(K)$ of continuous functions on a compact Hausdorff space. There is a vast study on absolutely summing operators from $C(K)$-spaces to Banach spaces; see, e.g., [5] and [6] for results. Asvald Lima, Vegard Lima and Eve Oja [19] are interested in the classical case $K = [0,1]$. We describe the space $\mathcal{P}(C[0,1],X)$ in terms of the space X itself. This aim is motivated by [8-15] on the classical bounded approximation property (BAP) and its weak counterpart . We construct a tree space on X and we show that $\mathcal{P}(C[0,1],X)$ is isometrically isomorphic to this tree space. We follow the original proved of [19] and show an application, X^*-valued square sequence spaces which be applied (see [19]).

The tree space will be called the ℓ_1-tree space on X and denoted $\ell_1^{\text{tree}}(X)$. It consists of special trees in X which will be called two-trunk trees. The representation theorem for $\mathcal{P}(C[0,1],X)$from section 3 is applied in section 4 and 5 to study tree $\ell_1^{\text{tree}}(X^*)$ as a dual Banach space, to derive a representation theorem for the Banach space $C_X[0,1]$ of continuous X-valued functions on $[0,1]$, and to characterize the weak BAP and the BAP of X in terms of X^*-valued square sequence spaces. We show that the Banach space $\ell_1(X)$ of absolutely summable X-valued square sequences nicely embeds in $\ell_1^{\text{tree}}(X)$.

The representation of $\mathcal{P}(C[0,1],X)$ relies on linear B-splines. Since the linear splines are known to be efficient for both computational and implementation purposes. The representation of absolutely summing operators might also be useful in Numerical Analysis.

We consider Banach spaces over the real field $\mathbb{R}$. We denote by $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operators from X to Y, and we write $\mathcal{L}(X)$ for $\mathcal{L}(X,X)$. Besides the operator ideal $\mathcal{P}$of absolutely summing operators, we also need the ideals $\mathcal{I}$ and $\mathcal{N}$of integral operators and of nuclear operators. Integral and nuclear norms of operators are denoted by $\|\cdot\|_{\mathcal{I}}$ and $\|\cdot\|_{\mathcal{N}}$, respectively (see [19]). For ,$\mathcal{I}$, and $\mathcal{N}$, see Diestel, Jarchow, and Tonge [5], Pietsch [17], and Ryan [18]. And see Diestel and Uhl [6] and Ryan [18] for the classical approximation properties and tensor products.

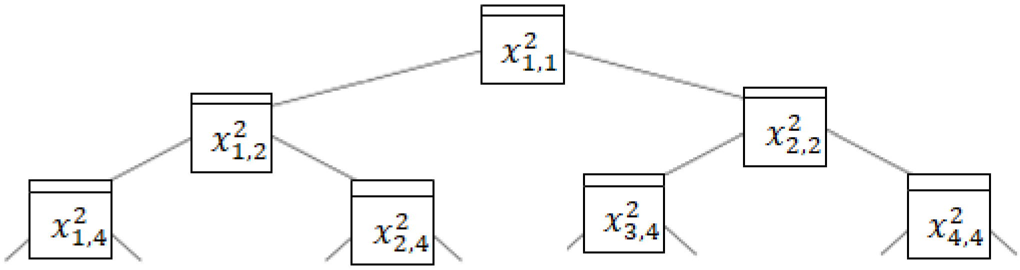
2. Two trees and the $\ell_1 - tree$ space

Let X be a Banach space. A (standard) tree in X is a system $\left(\left(x_{k,2^n}^2\right)_{k=1}^{2^n}\right)_{n=0}^{\infty}$ of square elements of X with the property that

$$x_{k,2^n}^2 = \frac{1}{2} x_{2k-1,2^{n+1}}^2 + \frac{1}{2} x_{2k,2^{n+1}}^2$$

for all $n = 0,1,\dots$ and $k = 1,2,\dots,2^n$. Hence, a tree looks as follows:

where, for each connecting line, its connecting coefficient is 1/2.



The first square element $x^2_{1,1}$ is called the trunk of the tree. The square elements $x^2_{1,2^n}, x^2_{2,2^n}, \ldots, x^2_{2^n,2^n}$ are said to be on the same (n-th) level, or to form the n-th (year) growth.

The study of trees in Banach spaces was initiated by [7]. By now, there is a vast literature on various variants of trees and related tree spaces. We need to introduce the following version of tree which will be called a two-trunk tree .

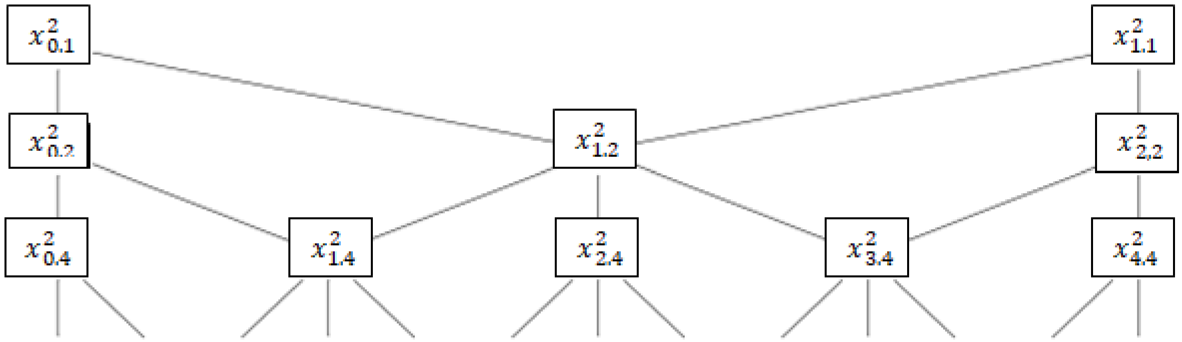
Definition 2.1 Let X be a Banach space. We say that a system $\left(\left(x^2_{k,2^n}\right)_{k=1}^{2^n}\right)_{n=0}^{\infty}$ of square elements of X is a two-trunk tree in X if for all $n = 0,1,\ldots$ and $k = 1,2,\ldots,2^n - 1$

$$x^2_{k,2^n} = \frac{1}{2}x^2_{k-1,2^{n+1}} + x^2_{2k,2^{n+1}} + \frac{1}{2}x^2_{2k+1,2^{n+1}},$$

$$x^2_{0,2^n} = x^2_{0,2^{n+1}} + \frac{1}{2}x^2_{1,2^{n+1}},$$

$$x^2_{2^n,2^n} = \frac{1}{2}x^2_{2^{n+1}-1,2^{n+1}} + x^2_{2^{n+1},2^{n+1}}.$$

A two-trunk tree looks as follows:



where connecting coefficients are 1 for the vertical lines and 1/2 for the sloping lines.

Compared to a standard tree which has 2^n square elements on its n-th level, a two-trunk tree has $2^n + 1$ square elements on its n-th level.

The basic example, which is also a prototype of our concept of a two-trunk tree, is the two- trunk tree in $C[0,1]$ consisting of linear B –splines (see [19]).

Example 2.2 For $n = 0,1,\ldots$, let S_n denote the space of all linear splines on the interval $[0,1]$ with the knots $\{k/2^n : k = 0,1,\ldots,2^n\}$. The spline space S_n, equipped with the maximum norm from $C[0,1]$, is a (2^n+1)-dimensional subspace of the space $C[0,1]$. The space S_n has a natural square basis $\left(g^2_{k,2^n}\right)_{k=0}^{2^n}$ where $g^2_{k,2^n} \in S_n$, $k = 0,\ldots,2^n$, are defined by the conditions

$$g^2_{k,2^n}\left(\frac{k}{2^n}\right) = 1 \quad \text{and} \quad g^2_{k,2^n}\left(\frac{j}{2^n}\right) = 0 \quad \text{if } j \neq k$$

If $g^2 \in S_n$, then clearly

$$g^2 = \sum_{k=0}^{2^n} g^2\left(\frac{k}{2^n}\right) g^2_{k,2^n}.$$

The square functions $g^2_{k,2^n}, n = 0,1,\ldots, k = 0,1,\ldots,2^n$, are called linear B-splines ("B " comes from "basis"). Sometimes they are also called "the second order cardinal B-spline functions" (see, e.g., [4]). (The square functions $g^2_{k,2^n}$ are generated by the scaling function $\varphi: \mathbb{R} \to \mathbb{R}, \varphi(\epsilon - 1) = \epsilon$ for $\epsilon < 1$ $\varphi(1-\epsilon) = \epsilon$ for $0 \le \epsilon \le 1$, and $\varphi(1-\epsilon) = 0$ for $\epsilon \le 0$; so that $g^2_{k,2^n}(1-\epsilon) = \varphi(2^n(1-\epsilon) - k)$, $\epsilon \le 0$. But we shall not need this description of the square functions $g^2_{k,2^n}$.)

Since

$$g^2_{k,2^n} = \sum_{j=0}^{2^{n+1}} g^2_{k,2^n}\left(\frac{j}{2^{n+1}}\right) g^2_{j,2^{n+1}}$$

and, by the definition of the $g^2_{k,2^n}$,

$$g^2_{k,2^n}\left(\frac{j}{2^{n+1}}\right) = \begin{cases} 0, & \text{for } j \notin \{2k-1, 2k, 2k+1\}, \\ 1, & \text{for } j = 2k, \\ \frac{1}{2}, & \text{for } j \in \{2k-1, 2k+1\}, \end{cases}$$

it is immediate that $\left(\left(g^2_{k,2^n}\right)_{k=0}^{2^n}\right)_{n=0}^{\infty}$ is a two-trunk tree in $C[0,1]$. Its trunks are $g^2_{0,1} = g^2_{0,1}(1-\epsilon) = \epsilon$ and $g^2_{1,1} = g^2_{1,1}(1-\epsilon) = 1-\epsilon, 0 \le \epsilon \le 1$.

Definition 2.3 Let X be a Banach space. The ℓ_1-tree space $\ell_1^{\text{tree}}(X)$ consists of all square two-trunk trees $x^2 = \left(x^2_{k,2^n}\right) = \left(\left(x^2_{k,2^n}\right)_{k=0}^{2^n}\right)_{n=0}^{\infty}$ in X such that

$$\|x^2\| := \sup_n \sum_{k=0}^{2^n} \left\|x^2_{k,2^n}\right\| < \infty.$$

Thus, the space $\ell_1^{\text{tree}}(X)$ is defined as a subspace of the space $\ell_\infty\left(\ell_1^{2^n+1}(X)\right)$. It shown that $\ell_1^{\text{tree}}(X)$ is isometrically isomorphic to $\mathcal{P}(C[0,1], X)$. Hence, in particular, $\ell_1^{\text{tree}}(X)$ is a Banach space (see [19]).

Before proceeding to the description of $\mathcal{P}(C[0,1],X)$, let us reformulate the notion of a square two-trunk tree in terms of connecting matrices.

Looking at Definition 2.1 , for $n=0,1,\ldots$, denote by M_n the matrix whose k-th row is formed by the square coefficients of $x^2_{k,2^n}$ in $(x^2_{0,2^{n+1}}, x^2_{1,2^{n+1}},\ldots,x^2_{2^{n+1},2^{n+1}})$. The matrix M_n is of order $(2^n+1)\times(2^{n+1}+1)$, and we have

$$M_0=\begin{pmatrix}1 & 1/2 & 0\\ 0 & 1/2 & 1\end{pmatrix}$$

$$M_1=\begin{pmatrix}1 & 1/2 & 0 & 0 & 0\\ 0 & 1/2 & 1 & 1/2 & 0\\ 0 & 0 & 0 & 1/2 & 1\end{pmatrix}$$

$$M_2=\begin{pmatrix}1 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1/2 & 1 & 1/2 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1/2 & 1 & 1/2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1/2 & 1 & 1/2 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1\end{pmatrix}$$

etc. By Example 2.2 , we can write that

$$M_n=\left(g^2_{k,2^n}\left(\frac{j}{2^{n+1}}\right)\right)_{k=0,1,\ldots,2^n;\,j=0,1,\ldots,2^{n+1}}.$$

Considering $M_n=(m^n_{kj})$ as the matrix operator from $X^{2^{n+1}+1}$ to X^{2^n+1} defined in a usual way, i.e.,

$$M_n(x^2_0,x^2_1,\ldots,x^2_{2^n+1})=\left(\sum_{j=0}^{2^{n+1}} m^n_{kj}\,x^2{}_j\right)_{k=0}^{2^n},$$

we have the following reformulations (see [19]).

Proposition 2.4 Let X be a Banach space. A system $\left(\left(x^2_{k,2^n}\right)_{k=0}^{2^n}\right)_{n=0}^{\infty}$ of square elements of X is a square two-trunk tree in X if and only if for all $n=0,1,\ldots$

$$\left(x^2_{k,2^n}\right)_{k=0}^{2^n}=M_n\left(\left(x^2_{j,2^{n+1}}\right)_{k=0}^{2^{n+1}}\right),$$

and

$$\ell_1^{\text{tree}}(X)=\left\{(z^2_n)_{n=0}^{\infty}\in\ell_\infty\left(\ell_1^{2^{n+1}}(X)\right): z^2_n=M_n z^2_{n+1}\right\}.$$

It is easily computed that for $M_n:\ell_1^{2^{n+1}+1}(X)\to\ell_1^{2^n+1}(X)$ one has $\|M_n\|=1$. Therefore, in Proposition 2.4 , $\|z^2_n\|\le\|z^2_{n+1}\|$,$n=0,1,\ldots$, and

$$\|(z^2_n)_{n=0}^{\infty}\|=\lim_n\|z^2_n\|.$$

3. A representation theorem for the absolutely summing operators on *C*[0, 1]

Let X and Y be Banach spaces. Let $T_n\in\mathcal{L}(Y,X)$ be such that the $Ty^2:=\lim_n T_n y^2$ exists for every $y^2\in Y$. It is a well-known result from Banach's thesis [1,p.157] that then (T_n) is bounded in $\mathcal{L}(Y,X)$, $T\in\mathcal{L}(Y,X)$, and $\|T\|\le\sup_n\|T_n\|$. The following version of Banach's result is now immediate from the definition of absolutely summing operators (see [19]).

Lemma 3.1 Let X and Y be Banach spaces, and let $T_n\in\mathcal{P}(Y,X)$. If the sequence (T_n) is bounded in $\mathcal{P}(Y,X)$, and for every $y^2\in Y$ the limits $Ty^2:=\lim_n T_n y^2$ exists, then $T_n\in\mathcal{P}(Y,X)$ and $\|T_n\|_{\mathcal{P}}\le\sup_n\|T_n\|_{\mathcal{P}}$.

Theorem 3.2. Let X be a Banach space. Then $\mathcal{P}(C[0,1],X)$ is isometrically isomorphic to $\ell_1^{\text{tree}}(X)$, by the mapping

$$T\mapsto\left(\left(Tg^2_{k,2^n}\right)_{k=0}^{2^n}\right)_{n=0}^{\infty},\qquad T\in\mathcal{P}(C[0,1],X),$$

where $g^2_{k,2^n}$, $k=0,1,\ldots,2^n$, are the linear B-splines on $[0,1]$ with knots $0/2^n,1/2^n,\ldots,2^n/2^n$. The inverse mapping

$$\left(\left(x^2_{k,2^n}\right)_{k=0}^{2^n}\right)_{n=0}^{\infty}\mapsto T$$

is given by

$$Tf=\lim_n\sum_{k=0}^{2^n} f\left(\frac{k}{2^n}\right)x^2_{k,2^n},\quad f\in C[0,1].$$

Proof : **(a)** Let us start with some preparation. For $n=0,1,\ldots$, define projections P_n from $C[0,1]$ onto its subspace of linear splines S_n (see Example 2.2) by

$$P_n f=\sum_{k=0}^{2^n} f\left(\frac{k}{2^n}\right)g^2_{k,2^n},\quad f\in C[0,1].$$

Since $P_n f\in S_n$ and $(P_n f)(k/2^n)=f(k/2^n)$, $k=0,1,\ldots,2^n$, meaning that $P_n f$ is the piecewise linear interpolant of f ($P_n f$ agrees with f at the knots and interpolates linearly in between), $P_{n+\epsilon}P_n=P_n$ for $\epsilon\ge 0$,

$$\|P_n f\|=\max\left\{\left|f\left(\frac{k}{2^n}\right)\right|: k=0,1,\ldots,2^n\right\},\qquad f\in C[0,1],$$

and therefore $\|P_n\|=1$. Since $P_n 1=1$, we have

$$\sum_{k=0}^{2^n} g^2_{k,2^n}=1.$$

Using the uniform continuity of f on $[0,1]$, it follows that $P_n f\to f$ in $C[0,1]$.

(b) Let $T\in\mathcal{P}(C[0,1],X)$ be arbitrary. Since $\left(\left(g^2_{k,2^n}\right)_{k=0}^{2^n}\right)_{n=0}^{\infty}$ is a square two-trunk tree in $C[0,1]$ and T is a linear operator, $z^2_T:=\left(\left(Tg^2_{k,2^n}\right)_{k=0}^{2^n}\right)_{n=0}^{\infty}$ is a square two-trunk tree in X.

Recall that $[0,1]$ can be identified with a weak* compact norming subset of $B_{(C[0,1])^*}$ (by the correspondence $\mapsto\delta_t$, where $\delta_t(f)=f(t)$, $f\in C[0,1]$). Therefore, by the Pietsch domination theorem (see [5]), there exists a regular Borel probability measure μ on $[0,1]$ such that for each $g^2_{k,2^n}$

$$\|Tg^2_{k,2^n}\|\le\|T\|_{\mathcal{P}}\int_0^1\left|g^2_{k,2^n}(t)\right|d\mu(t).$$

Hence for each n

$$\sum_{k=0}^{2^n}\|Tg^2_{k,2^n}\|\le\|T\|_{\mathcal{P}}\int_0^1\sum_{k=0}^{2^n} g^2_{k,2^n}(t)\,d\mu(t)=\|T\|_{\mathcal{P}}\qquad(1)$$

showing that $z^2_T\in\ell_\infty\left(\ell_1^{2^n+1}(X)\right)$, hence $z^2_T\in\ell_1^{\text{tree}}(X)$ as desired, and $\|z^2_T\|\le\|T\|_{\mathcal{P}}$. Moreover, the mapping $T\mapsto z^2_T$ is clearly linear. To show that it is also an isometric mapping, it suffices to show that $\|T\|_{\mathcal{P}}\le\|z^2_T\|$.

Since $P_n f\to f$ for every $f\in C[0,1]$, also $TP_n f\to Tf$ for every $f\in C[0,1]$. The sequence (TP_n) is bounded in $\mathcal{P}(C[0,1],X)$ because $\|TP_n\|_{\mathcal{P}}\le\|T\|_{\mathcal{P}}\|P_n\|=\|T\|_{\mathcal{P}}$ for all n. Hence, by Lemma 3.1,

$$\|T\|_{\mathcal{P}}\le\sup_n\|TP_n\|_{\mathcal{P}}.\qquad(2)$$

In fact,

$$\|T\|_{\mathcal{P}} = \sup_n \|TP_n\|_{\mathcal{P}} = \lim_n \|TP_n\|_{\mathcal{P}}$$

because, as we saw, $\|TP_n\|_{\mathcal{P}} \le \|T\|_{\mathcal{P}}$, and $\|TP_n\|_{\mathcal{P}} = \|TP_{n+1}P_n\|_{\mathcal{P}} \le \|TP_{n+1}\|_{\mathcal{P}}$.

We shall now estimate $\|TP_n\|_{\mathcal{P}}$ from above. Let $I_n: S_n \to \ell_\infty^{2^n+1}$ be defined by $I_n g^2 = \left(g^2(k/2^n)\right)_{k=0}^{2^n}$. Then

$$\|I_n g^2\| = \max\left\{\left|g^2\left(\frac{k}{2^n}\right)\right| : k = 0,1,\ldots,2^n\right\} = \|g^2\|,$$

and I_n is a linear isometry, whose inverse mapping I_n^{-1} is given by

$$I_n^{-1}\alpha = \sum_{k=0}^{2^n} e_k(\alpha)\, g_{k,2^n}^2\,, \quad \alpha \in \ell_\infty^{2^n+1},$$

where $e_0 = (1,0,\ldots,0),\ldots,e_{2^n} = (0,\ldots,0,1)$ are the unit vectors in $\left(\ell_\infty^{2^n+1}\right)^* = \ell_1^{2^n+1}$.

The operator TP_n being of finite-rank, we can look at its nuclear norm $\|TP_n\|_{\mathcal{N}}$. We have

$$\|TP_n\|_{\mathcal{P}} \le \|TP_n\|_{\mathcal{N}} = \|TI_n^{-1}I_n\, P_n\|_{\mathcal{N}} \le \|TI_n^{-1}\|_{\mathcal{N}}$$

$$= \left\|\sum_{k=0}^{2^n} e_k \otimes g_{k,2^n}^2\right\|_{\mathcal{N}} \le \sum_{k=0}^{2^n} \|e_k\|\, \|Tg_{k,2^n}^2\|$$

$$= \sum_{k=0}^{2^n} \|Tg_{k,2^n}^2\|. \qquad (3)$$

In fact,

$$\|TP_n\|_{\mathcal{P}} = \sum_{k=0}^{2^n} \|Tg_{k,2^n}^2\| \qquad (4)$$

because replacing T with TP_n in (1) yields that

$$\sum_{k=0}^{2^n} \|Tg_{k,2^n}^2\| = \sum_{k=0}^{2^n} \|TP_n g_{k,2^n}^2\| \le \|TP_n\|_{\mathcal{P}}.$$

From (2) and (3), it is clear that

$$\|T\|_{\mathcal{P}} \ge \|z_T^2\|.$$

This proves that the mapping $T \mapsto z_T^2$ is an isometric isomorphism of $\mathcal{P}(C[0,],X)$ into $\ell_1^{\text{tree}}(X)$.

(c) Let now $z^2 : (z_n^2)_{n=0}^{\infty} := ((x_{k,2^n}^2)_{k=0}^{2^n})_{n=0}^{\infty} \in \ell_1^{\text{tree}}(X)$ be arbitrary. Set

$$T_n f = \sum_{k=0}^{2^n} f\left(\frac{k}{2^n}\right) x_{k,2^n}^2, \quad f \in C[0,1].$$

Then $T_n \in \mathcal{F}(C[0,1],X)$ and $T_n g_{k,2^n}^2 = x_{k,2^n}^2$ for $k = 0,1,\ldots,2^n$, because $g_{k,2^n}^2(j/2^n) = \delta_{kj}$. Hence, $T_nP_nf = T_nf$, $f \in C[0,1]$, and, since $T_n \in \mathcal{P}(C[0,1],X)$, using (4), we have

$$\|T_n\|_{\mathcal{P}} = \|T_nP_n\|_{\mathcal{P}} = \sum_{k=0}^{2^n} \|T_n g_{k,2^n}^2\| = \sum_{k=0}^{2^n} \|x_{k,2^n}^2\| \le \|z^2\|.$$

Next we show that the sequence (T_n) converges pointwise in $\mathcal{L}(C[0,1],X)$. Since the square functions $g_{k,2^n}^2$, $n = 0,1,\ldots,k = 0,1,\ldots,2^n$, span a dense subset of $C[0,1]$ and the sequence (T_n) is bounded in $\mathcal{L}(C[0,1],X)$, it suffices to prove that the limit $lim_n\, T_n g_{j,2^{n+\epsilon}}^2$ exists for every $g_{j,2^{n+\epsilon}}^2$, $n+\epsilon = 0,1,\ldots,j = 0,1,\ldots,2^{n+\epsilon}$. We know already that $T_{n+\epsilon} g_{j,2^{n-\epsilon}}^2 = x_{j,2^{n+\epsilon}}^2$. Recalling about matrices M_n and Proposition 2.4, we have

$$\left(T_{n+\epsilon+1} g_{j,2^{n+\epsilon}}^2\right)_{j=0}^{2^{n+\epsilon}}$$

$$= \left(\sum_{k=0}^{2^{n-\epsilon+1}} g_{j,2^{n+\epsilon}}^2\left(\frac{k}{2^{n+\epsilon+1}}\right) x_{k,2^{n+\epsilon+1}}^2\right)_{j=0}^{2^{n+\epsilon}}$$

$$= M_{n+\epsilon} z_{n+\epsilon+1}^2 = z_{n+\epsilon}^2,$$

in particular, $T_{n+\epsilon+1} g_{j,2^{n+\epsilon}}^2 = x_{j,2^{n+\epsilon}}^2$ for each $j = 0,1,\ldots,2^{n+\epsilon}$. Consequently, $T_{n+\epsilon+2} g_{k,2^{n+\epsilon+1}}^2 = x_{k,2^{n+\epsilon+1}}^2$ for each $k = 0,1,\ldots,2^{n+\epsilon+1}$, and therefore

$$T_{n+\epsilon+2} g_{j,2^{n+\epsilon}}^2 = T_{n+\epsilon+2}\left(\sum_{k=0}^{2^{n+\epsilon+1}} g_{j,2^{n+\epsilon}}^2\left(\frac{k}{2^{n+\epsilon+1}}\right) g_{k,2^{n+\epsilon+1}}^2\right)$$

$$= \sum_{k=0}^{2^{n+\epsilon+1}} g_{j,2^{n+\epsilon}}^2\left(\frac{k}{2^{n+\epsilon+1}}\right) x_{k,2^{n+\epsilon+1}}^2$$

$$= T_{n+\epsilon+1} g_{j,2^{n+\epsilon}}^2 = x_{j,2^{n+\epsilon}}^2.$$

Continuing similarly, we get that for each $\epsilon \ge 0$

$$T_n g_{j,2^{n+\epsilon}}^2 = x_{j,2^{n+\epsilon}}^2, \qquad j = 0,1,\ldots,2^{n+\epsilon}.$$

Hence $lim_n\, T_n g_{j,2^{n+\epsilon}}^2 = x_{j,2^{n+\epsilon}}^2$ for each $n+\epsilon = 0,1,\ldots$ and $j = 0,1,\ldots,2^{n+\epsilon}$.

It follows that (T_n) converges pointwise. Thus the operator T is well defined, $T \in \mathcal{P}(C[0,1],X)$ by Lemma 3.1, and $T \mapsto z^2$ because $Tg_{j,2^{n+\epsilon}}^2 = x_{j,2^{n+\epsilon}}^2$.

Remark 3.1 Let X be a Banach space, let Σ be the σ-field of Borel sets in $[0,1]$, and let G be a countably additive vector measure of bounded variation with values in X. Then we can define

$T \in \mathcal{P}(C[0,1],X)$ by $Tf = \int_0^1 f\, dG$ for $f \in C[0,1]$ (see [19]). The formula

$$Tf = \lim_n \sum_{k=0}^{2^n} f\left(\frac{k}{2^n}\right) x_{k,2^n}^2$$

in Theorem 3.2 can be rewritten as follows:

$$\int_0^1 f\, dG = \lim_n \sum_{k=0}^{2^n} f\left(\frac{k}{2^n}\right) \int_0^1 g_{k,2^n}^2\, dG.$$

In the special case when $X = \mathbb{R}$ and $G = m$, the Lebesgue measure on $[0,1]$, this formula looks as follows:

$$\int_0^1 f\, d(n+\epsilon) = \lim_n \frac{1}{2^{n+1}}\left(f(0) + 2f\left(\frac{1}{2^n}\right) + 2f\left(\frac{2}{2^n}\right) + \cdots + +2f\left(\frac{2^n-1}{2^n}\right) + f(1)\right),$$

which is the Trapezoidal Rule for numerical integration.

Let us denote $\ell_1^{\text{tree}} = \ell_1^{\text{tree}}(\mathbb{R})$. It is essentially well known that $\mathcal{P}(X,\mathbb{R}) = X^*$ and $\|f\|_{\mathcal{P}} = \|f\|$, for all $f \in X^*$. This easily follows from the fact that if $f \in X^*$, then

$$\sum_{k=1}^{n}|f(x_k^2)| = \|f\| \sum_{k=1}^{n}\left|\frac{f}{\|f\|}(x_k^2)\right| \leq \|f\| \sup\left\{\sum_{k=1}^{n}|(x^2)^*(x_k^2)| : (x^2)^* \in X^*, \|(x^2)^*\| \leq 1\right\}$$

for every choice of square elements $x_1^2, \ldots, x_n^2$ in X. The following is now immediate from Theorem3.2 (see[19]).

Corollary 3.3 Through the canonical isometric isomorphism $\mu \mapsto (\mu(g_{k,2^n}^2)) \in \ell_1^{\text{tree}}$, $\mu \in C[0,1]^*$, one has the identification

$$C[0,1]^* = \ell_1^{\text{tree}}.$$

Corollary 3.3 enables us to obtain a new equivalent formulation of the Radon–Nikodým property of Banach spaces; see [6,pp.217-218] for a summary containing more than twenty equivalent formulations of the Radon–Nikodým property. Below, $\widehat{\otimes} = \widehat{\otimes}_\pi$ stands for the (completed) projective tensor product (see [19]).

Corollary 3.4 A Banach space X has the Radon–Nikodým property if and only if

$$\ell_1^{\text{tree}} \widehat{\otimes} X = \ell_1^{\text{tree}}(X),$$

where the canonical isometric isomorphism is given by the mapping $(a_{k,2^n}) \otimes x^2 \mapsto (a_{k,2^n}x^2)$, $(a_{k,2^n}) \in \ell_1^{\text{tree}}$, $x^2 \in X$.

Proof: By the above, $\ell_1^{\text{tree}} \otimes X = C[0,1]^* \widehat{\otimes} X = \mathcal{N}(C[0,1], X)$ (the latter equality is well known and it holds because $C[0,1]^*$ has the approximation property (see, e.g., [18,p.76])) and, under this identification, the elementary tensor $(a_{k,2^n}) \otimes x^2$ corresponds to the operator $\mu \otimes x^2$, where $a_{k,2^n} = \mu(g_{k,2^n}^2)$, $\mu \in C[0,1]^*$. It is known (see, [6,pp.174-175]) that X has the Radon–Nikodým property if and only if $\mathcal{N}(C[0,1], X) = \mathcal{P}(C[0,1], X)$ with the equality of nuclear and absolutely summing norms. By Theorem 3.2, $\mathcal{P}(C[0,1], X) = \ell_1^{\text{tree}}(X)$and the operator $\mu \otimes x^2$ corresponds to the square two-trunk tree $((\mu \otimes$ *x2gk,2n2=μgk,2n2 x2=ak,2n x2∈ℓ1*tree*X.*

Remark 3.2. It is well known (see, e.g., [18, p. 19]) that for *every* Banach space X, one has

$$\ell_1 \widehat{\otimes} X = \ell_1(X)$$

where the canonical isometric isomorphism is given by the same mapping, as in Corollary 3.4, namely, $(a_n) \otimes x^2 \mapsto (a_n x^2), (a_n) \in \ell_1, x^2 \in X$.

4. Preduals of $\ell_1^{\text{tree}}(X^*)$ and a representation theorem for $C_X[0, 1]$

An easy application of Theorem 3.2 shows that $\ell_1^{\text{tree}}(X^*)$ is a dual Banach space. Indeed, let X be a Banach space. By the well-known results of Grothendieck (see, e.g., [5,p.99] and [18,p.67]),

$$\mathcal{P}(C[0,1], X^*) = \mathcal{J}(C[0,1], X^*) = \left(C[0,1] \check{\otimes} X\right)^*$$

as Banach spaces, where $\mathcal{J}$ denotes the ideal of integral operators and $\check{\otimes} = \widehat{\otimes}_\varepsilon$ stands for the (completed) injective tensor product. Hence, by Theorem 3.2, $\ell_1^{\text{tree}}(X^*)$ is isometrically isomorphic to $\left([0,1] \check{\otimes} X\right)^*$, so that $[0,1] \check{\otimes} X$ is a predual of $\ell_1^{\text{tree}}(X^*)$.

The fact, that $\ell_1^{\text{tree}}(X^*)$ is a dual Banach space, can also be seen in a straight forward manner, without having recourse to Theorem 3.2. Indeed, $\ell_1^{\text{tree}}(X^*)$ is a closed subspace of $\ell_\infty\left(\ell_1^{2^n+1}(X^*)\right) = \ell_\infty\left(\left(\ell_\infty^{2^n+1}(X)\right)^*\right) = \left(\ell_1\left(\ell_\infty^{2^n+1}(X)\right)\right)^*$. We shall show that $\ell_1^{\text{tree}}(X^*) = Z^\perp$, the annihilator of some closed subspace Z of $\ell_1\left(\ell_\infty^{2^n+1}(X)\right)$, to be specified below. In particular, $\ell_1^{\text{tree}}(X^*)$ is a weakly* closed subspace of $\left(\ell_1\left(\ell_\infty^{2^n+1}(X)\right)\right)^*$ and a dual Banach space (see[19]).

Definition 4.1. Let X be a Banach space. Let us define Z to be the closed subspace of $\ell_1\left(\ell_\infty^{2^n+1}(X)\right)$ spanned by the sequences of the form

$$(0, \ldots, 0, z_n^2, -M_n^* z_n^2, 0, 0, \ldots), \quad n = 0, 1, \ldots,$$

where $z_n^2 \in \ell_\infty^{2^n+1}(X)$ and $M_n^*: X^{2^n+1} \to X^{2^{n+1}+1}$ denotes the matrix operator defined by the transpose of the matrix M_n (see [19]).

Proposition 4.2. Let X be a Banach space. Then $\ell_1^{\text{tree}}(X^*) = Z^\perp$ in $\left(\ell_1\left(\ell_\infty^{2^n+1}(X)\right)\right)^*$.

Proof: Let $(z^2)^* = ((z_n^2)^*)_{n=0}^\infty \in \ell_1^{\text{tree}}(X^*)$, where $(z_n^2)^* = \left(x_{k,2^n}^2\right)_{n=0}^{2^n} \in \ell_1^{2^n+1}(X^*)$. Let $z^2 = (0, \ldots, 0, z_n^2, -M_n^* z_n^2, 0, 0, \ldots)$, for some fixed n, with $z_n^2 = \left(x_{k,2^n}^2\right)_{n=0}^{2^n} \in \ell_\infty^{2^n+1}(X)$.

Then

$$\begin{aligned}(z^2)^*(z^2) &= (z_n^2)^*(z_n^2) - (z_{n+1}^2)^*(M_n^* z_n^2) \\ &= (z_n^2)^*(z_n^2) - (M_n^* z_{n+1}^2)(z_n^2) \\ &= (z_n^2)^*(z_n^2) - (z_n^2)^*(z_n^2) = 0\end{aligned}$$

(see Proposition 2.4). Hence, $\ell_1^{\text{tree}}(X^*) \subset Z^\perp$.

On the other hand, if $(z^2)^* = ((z_n^2)^*)_{n=0}^\infty \in \ell_\infty\left(\ell_1^{2^n+1}(X^*)\right) \backslash \ell_1^{\text{tree}}(X^*)$, then $(z_n^2)^* \neq M_n^* z_{n+1}^2$ in $\ell_1^{2^n+1}(X^*) = \left(\ell_\infty^{2^n+1}(X)\right)^*$ for some fixed n. Hence, there is $z_n^2 \in \ell_\infty^{2^n+1}(X)$ such that $(z_n^2)^*(z_n^2) \neq (M_n^* z_{n+1}^2)(z_n^2)$. Using the above notation, we get by the above that $(z^2)^*(z^2) \neq 0$. Hence, $(z^2)^* \neq Z^\perp$.

Since $Z^\perp$ can be canonically identified with $\left(\ell_1\left(\ell_\infty^{2^n+1}(X)\right)/Z*\right.$, we immediately have the following result (see [19]).

Corollary 4.3. Let X be a Banach space. Then $\ell_1^{\text{tree}}(X^*)$ is isometrically isomorphic to $\left(\ell_1\left(\ell_\infty^{2^n+1}(X)\right)/Z\right)^*$. Thus, $\ell_1^{\text{tree}}(X^*)$ has two preduals: $C[0,1] \check{\otimes} X$ and $\ell_1\left(\ell_\infty^{2^n+1}(X)\right)/Z$. Theorem 4.4 below will provide a description of $C[0,1] \check{\otimes} X$, which will show, in particular, that these preduals are isometrically isomorphic. For completeness, let us recall that two preduals of a dual Banach space need not be isomorphic, in general. For instance, $\ell_1 = c_0^*$ and also $\ell_1 = (C(\omega^\omega))^*$, where $C(\omega^\omega)$

denotes the Banach space of continuous functions on the compact Hausdorff space ω^ω of ordinal numbers $\leq \omega^\omega$, but c_0 is not isomorphic to $C(\omega^\omega)$ (see [2]).

For a Banach space X, let us denote by $\Phi: \mathcal{P}(C[0,1], X^*) \to \ell_\infty(\ell_1^{2^n+1}(X^*))$ the into isometry given by Theorem 3.2. Recall that ran $\Phi = \ell_1^{\text{tree}}(X^*)$. Identifying $\mathcal{P}(C[0,1], X^*)$ with $(C[0,1] \check{\otimes} X)^*$ and $\ell_\infty(\ell_1^{2^n+1}(X^*))$ with $\left(\ell_1(\ell_\infty^{2^n+1}(X))\right)^*$, we have

$$\Phi: (C[0,1] \check{\otimes} X)^* \to \left(\ell_1(\ell_\infty^{2^n+1}(X))\right)^*.$$

Theorem 4.4 Let X be a Banach space. Define $\Psi: \ell_1(\ell_\infty^{2^n+1}(X)) \to C[0,1] \check{\otimes} X$ by

$$\Psi\left(\left(x_{k,2^n}^2\right)_{k=0}^{2^n}\right)_{n=0}^{\infty} = \sum_{n=0}^{\infty}\sum_{k=0}^{2^n} g_{k,2^n}^2 \otimes x_{k,2^n}^2,$$

where $g_{k,2^n}^2$, $k = 0,1,\dots,2^n$, are the linear B-splines on $[0,1]$ with knots $0/2^n, 1/2^n, \dots, 2^n/2^n$. Then Ψ is a quotient mapping, ker $\Psi = Z$, and $\Phi = \Psi^*$.

Proof: First of all, Ψ is correctly defined, because the defining series converges absolutely in $[0,1] \check{\otimes} X$. Indeed, let $\varepsilon = \|\cdot\|_\varepsilon$ denote the injective tensor norm. Denoting $z_n^2 = \left(x_{k,2^n}^2\right)_{k=0}^{2^n} \in \ell_\infty^{2^n+1}(X)$ and recalling that $C[0,1] \otimes_\varepsilon X$ can be identified with a subspace of $\mathscr{L}(X, C[0,1])$,
we have

$$\begin{aligned}
\left\|\sum_{k=0}^{2^n} g_{k,2^n}^2 \otimes x_{k,2^n}^2\right\|_\varepsilon
&= \sup_{\|(x^2)^*\|\leq 1}\left\|\sum_{k=0}^{2^n}(x^2)^*(x_{k,2^n}^2)g_{k,2^n}^2\right\|_{C[0,1]}\\
&= \sup_{\|(x^2)^*\|\leq 1}\max_{0\leq k\leq 2^n}\left|(x^2)^*(x_{k,2^n}^2)\right|\\
&= \max_{0\leq k\leq 2^n}\|x_{k,2^n}^2\| = \|z_n^2\|,
\end{aligned}$$

where the second equality holds because a linear spline attains its extremal value at some of its knots. Since $\|(z_n^2)\| = \sum_n \|z_n^2\| < \infty$, Ψ is correctly defined and linear. We also clearly have that $\|\Psi\| \leq 1$.

Let now $T \in \mathcal{P}(C[0,1], X^*) = (C[0,1] \check{\otimes} X)^*$ and $(z_n^2) = \left(\left(x_{k,2^n}^2\right)_{k=0}^{2^n}\right) \in \ell_1\left(\ell_\infty^{2^n+1}(X)\right)$. Then

$$\begin{aligned}
(\Phi T)(z_n^2) &= \left(\left(T g_{k,2^n}^2\right)_{k=0}^{2^n}\right)(z_n^2) = \sum_{n=0}^{\infty}\sum_{k=0}^{2^n}\left(T g_{k,2^n}^2\right)\left(x_{k,2^n}^2\right)\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{2^n}\langle T g_{k,2^n}^2 \otimes x_{k,2^n}^2\rangle = \langle T, \Psi(z_n^2)\rangle\\
&= (\Psi^* T)(z_n^2),
\end{aligned}$$

meaning that $\Phi = \Psi^*$. Since Ψ^* is an into isometry, it is well known (see, e.g., [17,B.3.9]) that Ψ is a quotient mapping. Using Proposition 4.2, we also have

$$\ker \Psi = (\operatorname{ran} \Psi^*)_\perp = \left(\ell_1^{\text{tree}}(X^*)\right)_\perp = (Z^\perp)_\perp = Z.$$

It is well known (see, e.g., [179]) that the Banach space $C_X[0,1]$ can be identified with $C[0,1] \check{\otimes} X$. Since Ψ is a quotient mapping with $\ker \Psi = Z$, the following representation result is immediate from Theorem4.4 (see [19]).

Theorem 4.5. $C[0,1] \check{\otimes} X$ and $C_X[0,1]]$ are both isometrically isomorphic to the quotient space $\ell_1\left(\ell_\infty^{2^n+1}(X)\right)/Z$.

5. Bounded approximation properties and embedding $\boldsymbol{\ell_1(X)}$ in $\boldsymbol{(\ell_1^{tree}(X)}$

Let X be a Banach space. We denote by $\mathcal{F}(X)$ the subspace of $\mathcal{L}(X)$ of finite-rank operators. Let I_X denote the identity operator on X.

Recall that a Banach space X is said to have the approximation property (AP) if there exists a net $(S_\alpha) \subset \mathcal{F}(X)$ such that $S_\alpha \to I_{x^2}$ uniformly on compact subsets of X. If (S_α) can be chosen with $\sup_\alpha \|S_\alpha\| \leq 1+\epsilon$ for some $\epsilon > 0$, then X is said to have the $(1+\epsilon)$-bounded approximation property ($(1+\epsilon)$-BAP). Recently, the weak bounded approximation property was introduced in [10]: X has the weak $(1+\epsilon)$-bounded approximation property (weak $(1+\epsilon)$-BAP) if for every Banach space Y and for every weakly compact operator $T: X \to Y$ there exists a net $(S_\alpha) \subset \mathcal{F}(X)$ such that $S_\alpha \to I_{x^2}$ uniformly on compact subsets of X and $\sup_\alpha \|TS_\alpha\| \leq (1+\epsilon)\|T\|$.

By [12] (see [15] for a simpler proof), the weak (1+∈)-BAP and the (1+∈)-BAP are equivalent for a Banach space X whenever X^* or X^{**} has the Radon–Nikodým property. It remains open whether the weak $(1+\epsilon)$-BAP is strictly weaker than the $(1+\epsilon)$-BAP. If they were equivalent, then, by [10], the answer to the long-standing famous open problem [3.8,in3], whether the AP of a dual Banach space implies the 1-BAP, would be "yes". For a recent survey on bounded approximation properties, see [16].

Recall that $\mathcal{I}$ denotes the ideal of integral operators. In [9,Theorem 1.3 and corollary 3.4] we show that X has the $(1+\epsilon)$-BAP if and only if for every $T \in \mathcal{I}(C[0,1], X^*)$ there exists a net $(S_\alpha) \subset \mathcal{F}(X)$ such that $S_\alpha \to I_{x^2}$ pointwise and lim $\sup_\alpha \|S_\alpha^* T\|_{\mathcal{I}} \leq (1+\epsilon)\|T\|_{\mathcal{I}}$. It is well known that $\mathcal{I}(C[0,1], X^*) = \mathcal{P}(C[0,1], X^*)$ with equality of norms (see [5,p.99]). By Theorem 3.2, if $T \in \mathcal{P}(C[0,1], X^*)$ is canonically identified with $\left(\left((x_{k,2^n}^2)^*\right)_{k=0}^{2^n}\right)_{n=0}^{\infty} \in \ell_1^{\text{tree}}(X^*)$ and $S \in \mathcal{F}(X)$, then $S_\alpha^* T$ is canonically identified with $\left(\left(S^*(x_{k,2^n}^2)^*\right)_{k=0}^{2^n}\right)_{n=0}^{\infty}$. Hence, the BAP of X can be characterized in terms of the X^*-valued square sequence space $\ell_1^{\text{tree}}(X^*)$ as follows (see [19]).

Theorem 5.1. Let X be a Banach space and let $0 \leq \epsilon < \infty$. The following statements are equivalent.
(a) X has the $(1+\epsilon)$-BAP.
(b) For every $(x_{k,2^n}^2)^* = \left(\left((x_{k,2^n}^2)^*\right)_{k=0}^{2^n}\right)_{n=0}^{\infty} \in \ell_1^{\text{tree}}(X^*)$ there exists a net $(S_\alpha) \subset \mathcal{F}(X)$ such that $S_\alpha \to I_{x^2}$ pointwise and lim $\sup_\alpha \left\|S_\alpha^*(x_{k,2^n}^2)^*\right\| \leq (1+\epsilon)\left\|(x_{k,2^n}^2)^*\right\|$.

Recall that $\mathcal{N}$ denotes the ideal of nuclear operators. In [8] we proved that X has the weak (1+∈)-BAP if and only if for every $T \in \mathcal{N}(X, \ell_1)$ there exists a net $(S_\alpha) \subset \mathcal{F}(X)$ such that $S_\alpha \to I_{x^2}$ pointwise and $\lim\sup_\alpha \|TS_\alpha\|_{\mathcal{N}} \leq (1+\in)\|T\|_{\mathcal{N}}$. It is well known that $\mathcal{N}(X, \ell_1)$ is isometrically isomorphic to $\ell_1(X^*)$ since both spaces can be identified with the projective tensor product $X^* \widehat{\otimes} \ell_1$ (see, e.g., [18,pp.19-20,76]). It can be easily verified that a linear isometry between $\mathcal{N}(X, \ell_1)$ and $\ell_1(X^*)$ is explicitly given by the mapping $T \mapsto (T^* e_n), T \in \mathcal{N}(X, \ell_1)$, where (e_n) is the unit vector basis of c_0. And the inverse mapping $(x_n^2)^* \mapsto T$, $(x_n^2)^* \in \ell_1(X^*)$, is given by $Tx^2 = \sum_{n=1}^{\infty}(x_n^2)^*(x^2)e_n^*$, $x^2 \in X$, where (e_n^*) is the unit vector basis of ℓ_1. Therefore, if $T \in \mathcal{N}(X, \ell_1)$ is canonically identified with $(x_n^2)^* \in \ell_1(X^*)$ and $S \in \mathcal{F}(X)$, then TS is canonically identified with $(S^*(x_n^2)^*)$. Hence, the weak BAP of X can be characterized as follows (see [19]).

Theorem 5.2. Let X be a Banach space and let $0 \leq \epsilon < \infty$. The following statements are equivalent.
(a) X has the weak $(1+\epsilon)$-BAP.
(b) For every $(x_n^2)^* = ((x_n^2)^*)_{n=1}^{\infty} \in \ell_1(X^*)$ there exists a net $(S_\alpha) \subset \mathcal{F}(X)$ such that $S_\alpha \to I_{(x^2)^*}$ point-wise and $\lim \sup_\alpha \|(S_\alpha^*(x_n^2)^*)\| \leq (1+\epsilon)\|(x_n^2)^*\|$.

Since the $(1+\epsilon)$-BAP of X implies the weak $(1+\epsilon)$-BAP of X, condition (b) of Theorem 5.1 implies condition (b) of Theorem 5.2. However, to see the latter implication, there is no need to have recourse to the approximation properties (i.e. conditions (a) of Theorems 5.1 and 5.2: our final result Theorem 5.3 shows that $\ell_1(X^*)$ embeds in $\ell_1^{\text{tree}}(X^*)$ in the way that makes the implication hold.

It is an easy to show that ℓ_1 embeds isometrically in $C[0,1]^*$. Since $\ell_1^{\text{tree}} = C[0,1]^*$ (see Corollary 3.3), $\ell_1 = \ell_1(\mathbb{R})$ embeds in $\ell_1^{\text{tree}} = \ell_1^{\text{tree}}(\mathbb{R})$. It is not clear a priori that $\ell_1(X)$ embeds in $\ell_1^{\text{tree}}(X)$ for an arbitrary Banach space X. Our final purpose is to show that this is indeed the case, and moreover, $\ell_1(X)$ embeds in $\ell_1^{\text{tree}}(X)$ in such a way that the embedding respects all bounded linear operators on X. To make this precise, we need some notation.

Let $S \in \mathcal{L}(X)$. Define

$$\bar{S}(x_n^2) = (Sx_n^2), \quad (x_n^2) = (x_n^2)_{n=1}^{\infty} \in \ell_1(X),$$
$$\tilde{S}(x_{k,2^n}^2) = (Sx_{k,2^n}^2), \quad (x_{k,2^n}^2) = \left(\left(x_{k,2^n}^2\right)_{k=0}^{2^n}\right)_{n=1}^{\infty} \in \ell_1^{\text{tree}}(X).$$

It is straightforward to verify that $\bar{S} \in \mathcal{L}(\ell_1(X))$, $\tilde{S} \in \mathcal{L}(\ell_1^{\text{tree}}(X))$, $\|S\| = \|\bar{S}\| = \|\tilde{S}\|$, and the mappings $S \mapsto \bar{S}$ and $S \mapsto \tilde{S}$ are linear isometries from $\mathcal{L}(X)$ into $\mathcal{L}(\ell_1(X))$and into $\mathcal{L}(\ell_1^{\text{tree}}(X))$, respectively. Thus, one can naturally embed $\mathcal{L}(X)$ both in $\mathcal{L}(\ell_1(X))$ and $\mathcal{L}(\ell_1^{\text{tree}}(X))$. The following result shows that there exists a linear isometry from $\ell_1(X)$ into $\ell_1^{\text{tree}}(X)$ which identifies S with $\tilde{S}$for all $S \in \mathcal{L}(X)$ (see[19]).

Theorem 5.3. Let X be a Banach space. Then there exists a linear isometry φ from $\ell_1(X)$ into $\ell_1^{\text{tree}}(X)$ such that $\varphi S = \tilde{S}\varphi$ for all $S \in \mathcal{L}(X)$.

Proof: We shall construct φas the pointwise limit of a sequence $\varphi_m \in \mathcal{L}(\ell_1(X), \ell_1^{\text{tree}}(X))$.
Let $x^2 = (x_n^2)_{n=1}^{\infty} \in \ell_1(X)$. We define $\varphi_1 x^2 = ((z^2)_n^1)_{n=1}^{\infty} \in \ell_1^{\text{tree}}(X)$ as follows. We first put $(z^2)_1^1 = (x_1^2, x_2^2, x_3^2)$, and define $(z^2)_0^1 = M_0(z^2)_1^1$.

Then, departing again from $(z^2)_1^1$, we put $(z^2)_2^1 = (x_1^2, 0, x_2^2, 0, x_3^2)$, $(z^2)_3^1 = (x_1^2, 0,0,0, x_2^2, 0,0,0, x_3^2)$, ….

Thus, $(z^2)_{n+1}^1$ is obtained from $(z^2)_n^1$, $n = 1,2,\ldots$, by inserting zeros between the components of the previous vector $(z^2)_n^1$. To define $\varphi_2 x^2 = ((z^2)_n^2)_{n=1}^{\infty} \in \ell_1^{\text{tree}}(X)$, we first put

$$(z^2)_2^2 = (z^2)_2^1 + (0, x_4^2, 0, x_5^2, 0) = (x_1^2, x_4^2, x_2^2, x_5^2, x_3^2).$$

Then, departing from $(z^2)_2^2$, we define $z_1^4 = M_1 z_2^4, z_0^4 = M_0 z_1^4$, and we define $z_3^4, z_4^4, \ldots$ as above, by inserting zeros between the components of $z_2^4, z_3^4, \ldots$. In general, to define $\varphi_{(n+\epsilon)} x^2 = ((z^2)_n^{n+\epsilon})_{n=1}^{\infty} \in \ell_1^{\text{tree}}(X)$, we first put

$$(z^2)_{n+\epsilon}^{n+\epsilon} = (z^2)_{n+\epsilon}^{n+\epsilon-1} + \left(0, x_{2^{n+\epsilon-1}+2}^2, 0, x_{2^{n+\epsilon-1}+3}^2, \ldots, 0, x_{2^{n+\epsilon}+1}^2, 0\right).$$

Then $(z^2)_{n+\epsilon-1}^{n+\epsilon}, (z^2)_{n+\epsilon-2}^{n+\epsilon}, \ldots, (z^2)_0^{n+\epsilon}$ are defined by

$$(z^2)_n^{n+\epsilon} = M_n(z^2)_{n+1}^{n+\epsilon}, \qquad \epsilon = 0,1,2,\ldots$$

and $(z^2)_{n+\epsilon+1}^{n+\epsilon}, (z^2)_{n+\epsilon+2}^{n+\epsilon}, \ldots$ are defined by inserting zeros between the components of the previous vector. This latter procedure can be formalized by introducing the $(2^{n+1}+1) \times (2^n+1)$ matrices A_n with

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \cdots$$

so that $M_n A_n = I_{2^n+1}$, the identity matrix of order 2^n+1. Then the recurring rule is $(z^2)_{n+1}^{n+\epsilon} = A_n(z^2)_n^{n+\epsilon}$ for $= 0,1,2,\ldots$.

This construction guarantees that $(z^2)_n^{n+\epsilon} = M_n(z^2)_{n+1}^{n+\epsilon}$ for $n+\epsilon = 1,2,\ldots$, $n = 0,1,\ldots$, so that $\varphi_{(n+\epsilon)}x^2 = ((z^2)_n^{n+\epsilon})_{n=0}^{\infty} \in \ell_1^{\text{tree}}(X)$ (see Proposition 2.4). We also clearly have that $\varphi_{(n+\epsilon)}$is linear and $\varphi_{(n+\epsilon)}\bar{S} = \tilde{S}\varphi_{(n+\epsilon)}$, $n+\epsilon = 1,2,\ldots, S \in \mathscr{L}(X)$.

To estimate the norm of $\varphi_{(n+\epsilon)}x^2$, set $s = \|x^2\| = \|x_1^2\| + \|x_2^2\| + \cdots$, and $s_n^2 = \|x_1^2\| + \cdots + \|x_n^2\|$If $n = n+\epsilon, n+\epsilon+1, \ldots$, then, by construction,

$$\|(z^2)_n^{n+\epsilon}\| = \|(z^2)_{n+\epsilon}^{n+\epsilon}\| = s_{2^{n+\epsilon}+1}^2.$$

Hence (see Proposition 2.4),

$$\|\varphi_{(n+\epsilon)}x^2\| = \lim_m \|(z^2)_n^{n+\epsilon}\| = s_{2^{n+\epsilon}+1}^2 \leq s^2 = \|x^2\|,$$

so that $\varphi_{(n+\epsilon)} \in \mathscr{L}\left(\ell_1(X), \ell_1^{\text{tree}}(X)\right)$, $n+\epsilon = 1,2,\ldots$, and

$$\|\varphi_{(n+\epsilon)}x^2\| \mapsto \|x^2\|, \quad x^2 \in \ell_1(X).$$

Similarly,

$$\begin{aligned}\|\varphi_{(n+\epsilon)}x^2 - \varphi_{(n+\epsilon-1)}x^2\| &= \|(z^2)_{n+\epsilon}^{n+\epsilon} - (z^2)_{n+\epsilon}^{n+\epsilon-1}\| \\ &= \left\|\left(0, x^2_{2^{n+\epsilon-1}+2}, \dots, 0, x^2_{2^{n+\epsilon}+1}, 0\right)\right\| \\ &= s^2_{2^{n+\epsilon}+1} - s^2_{2^{n+\epsilon-1}+1},\end{aligned}$$

so that for $\epsilon > 0$

$$\begin{aligned}&\|\varphi_{(n+\epsilon)}x^2 - \varphi_n x^2\| \\ &\le s^2_{2^{n+\epsilon}+1} - s^2_{2^{n+\epsilon-1}+1} + s^2{}_{2^{n+\epsilon-1}+1} - \cdots \\ &+ s^2_{2^{n+1}+1} - s^2_{2^n+1} \\ &= s^2_{2^{n+\epsilon}+1} - s^2_{2^n+1} \xrightarrow[n+\epsilon,n]{} 0.\end{aligned}$$

Hence, the limit

$$\varphi x^2 = \lim_{(n+\epsilon)} \varphi_{n+\epsilon}\, x^2$$

exists for all $x^2 \in \ell_1(X)$, and $\varphi \in \mathscr{L}\left(\ell_1(X), \ell_1^{\text{tree}}(X)\right)$. Moreover,

$$\|\varphi x^2\| = \lim_{(n+\epsilon)} \|\varphi_{(n+\epsilon)}x^2\| = \|x^2\|, \quad x^2 \in \ell_1(X),$$

meaning that φ is isometric, and for all $S \in \mathscr{L}(X)$,

$$\varphi \bar{S} x^2 = \lim_{(n+\epsilon)} \varphi_{(n+\epsilon)}\, \bar{S} x^2 = \lim_{(n+\epsilon)} \varphi_{(n+\epsilon)}\, \tilde{S} x^2 = \tilde{S}\varphi x^2, \quad x^2 \in \ell_1(X).$$

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