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Algebraic Structures on Multi Groups II

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Abstract: ***Cosets in the sense of multi-groups were discussed, also factor groups in the multi-group perspective.***

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1. Introduction

If G is any group and A is any subgroup of G we know that:

gA is a *Left Coset* of A in G where

$gA=\{ga:a\in A\}$ where $g\in G$. Also

$Ag=\{ag:a\in A\}$ is said to be right Coset of A in G, any two left (right) Cosets either disjoint or identical and we mention some basic results in this subject.

Lemma 1.1: Let G be a group and A is a sub-group of G then we has:

(i) $xA=A$ iff $x\in A$

(ii) $xA=yA$ iff $y^{-1}x\in A$

Same results can be written with respect to right Cosets.

Now if A is normal subgroup of G, we can define an operation on Left Cosets of A in G as follows:

$xAyA=xyA$ and in this case, this operation is well defined and the set of all *Left Cosets* of A in G form a group under this operation called the Factor group or Quotient group and is denoted be G/A. In the next section we want to extend these results in the case of multi-groups.

2. Cosets in the Sense of Multi-Groups

In this section we introduce the concept of multi Cosets i.e. Cosets in multi-group perspective Recall the definition 3.5 in [16] that is if B is a multi-group over the group Γ and $\underset{B}{C}(e)=\zeta$, $B_\upsilon=\{g\in\Gamma:C_B(g)\geq\upsilon,\upsilon\in Z^+\}$

Where Z^+ is the set of all positive integers.

$B_{\zeta(e)}=\{g\in\Gamma:C_B(g)=\zeta=C_B(e)\}$

$B_0=\{g\in\Gamma:C_B(g)\geq 0\}$

In a previous note [16] we have proved that B_υ $B_{\zeta(e)}$ and B_0 are sub-groups of Γ and in the following theorem we will prove that they are normal sub-groups of Γ.

Theorem 2.1: let Γ be a group and let $B\in AMG(\Gamma)$ then $B_0, B_{\zeta(e)}$ and B_υ where $\upsilon\in Z^+$ are normal sub-groups of the group Γ.

Proof (1):To prove that B_0 is normal in Γ:

Since $B_0=\{g\in\Gamma:C_B(g)\geq 0\}$

We had shown that $B_0, B_{\zeta(e)}$ and B_υ (see [16]) are sub-groups and so it remains to show that they are normal in Γ. Let $g\in\Gamma$ taken arbitrarily and $b\in B_0$, then $C_B(b)\geq 0$ and Recall theorem 3.15 in [16] that $\underset{B}{C}\left(gbg^{-1}\right)=\underset{B}{C}(b)\geq 0$ and hence we get $gbg^{-1}\in B_0$, then B_0 is a normal sub-groups of Γ.

(2) to prove that $B_{\zeta(e)}$ is normal in Γ, let $b\in B_{\zeta(e)}$ then $C_B(b)=C_B(e)$ and for $g\in\Gamma$ and again by theorem 3.15 in [16]

$\underset{B}{C}\left(gbg^{-1}\right)=\underset{B}{C}(b)=\underset{B}{C}(e)$ then $gbg^{-1}\in\underset{\zeta(e)}{B}$, then $\underset{\zeta(e)}{B}$ is normal in Γ.

(3) now we want to show that $\underset{\upsilon}{B}$ is normal in Γ, $\upsilon\in Z^+$, take $g\in\Gamma$, $b\in\underset{\upsilon}{B}$, since $b\in\underset{\upsilon}{B}$ then $\underset{B}{C}(b)\geq\upsilon$, also by theorem 3.15 in [16] $\underset{B}{C}\left(gbg^{-1}\right)=\underset{B}{C}(b)\geq\upsilon$ thus $gbg^{-1}\in\underset{\upsilon}{B}$ and so $\underset{\upsilon}{B}$ is normal in Γ which completes the proof of the theorem. Now we introduce the concept of singleton set in multi-set perspective.

Definition 2.2: Let Γ be any set (need not be a group) and $G=\{\upsilon/g\}$ that is $G=\{g\}_\upsilon$ and we write it as $\{g\}_\upsilon=\underset{g}{\upsilon}$.

Definition 2.3: Let Γ be a group with identity e, $B\in MG(\Gamma)$ and $\underset{B}{C}(e)=\underset{B}{\zeta}(e)=\zeta$ then the multi-set $\underset{g}{\zeta}\circ B$ is called a left *mCoset* of B in Γ and we write it

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as gB similarly $B \circ \underset{g}{\zeta}$ is called the right $mCoset$ of B in Γ, and can be written as $B \circ \underset{g}{\zeta} = Bg$.

Lemma 2.4:Let Γ be a group, $B \in MG(\Gamma)$, then

(1) $\underset{gB}{C}(a) = \underset{B}{C}\left(g^{-1}a\right)$ and

(2) $\underset{Bg}{C}(a) = \underset{B}{C}\left(ag^{-1}\right)$

Proof (1): Let $\underset{B}{C}(e) = \zeta$

$$\underset{gB}{C}(a) = \underset{\underset{g}{\zeta} \circ B}{C}(a)$$

$$= \max_{\substack{g,b \in \Gamma \\ gb = a}} \left\{ \min \left\{ \underset{\{g\}_{\zeta}}{C}(g), \underset{B}{C}(b) \right\} \right\}$$

$$= \max_{b = g^{-1}a} \left\{ \min \left\{ \underset{B}{\zeta}(e), \underset{B}{C}(b) \right\} \right\}$$

since any element in B not exceeds in multiplicity the multiplicity of e then $\underset{gB}{C}(a) = \max_{gb=a} \left\{ \underset{B}{C}(b) \right\}$

$$\underset{gB}{C}(a) = \underset{B}{C}\left(g^{-1}a\right)$$

Proof (2): $\underset{Bg}{C}(a) = \underset{B \circ \underset{g}{\zeta}}{C}(a)$

$$= \max_{\substack{bg = a \\ b,g \in \Gamma}} \left\{ \min \left\{ \underset{B}{C}(b), \underset{\{g\}}{C}_{\zeta}(g) \right\} \right\}$$

$$= \max_{b = ag^{-1}} \left\{ \min \left\{ \underset{B}{C}(b), \underset{B}{\zeta}(g) \right\} \right\}$$

$$= \max_{b = ag^{-1}} \left\{ \underset{B}{C}(b) \right\}$$

$$= \underset{B}{C}\left(ag^{-1}\right)$$

Hence our result follows.

Theorem 2.5: In $AMG(\Gamma)$, every left $mCoset$ is the same as right $mCoset$ corresponding to the same element, same sub-group of Γ i.e. $\forall B \in AMG(\Gamma)$, $\forall g \in \Gamma$ then: $gB = Bg$.

Proof: Let $B \in AMG(\Gamma)$, then $\underset{B}{C}(gh) = \underset{B}{C}(hg)$ $\forall g, h \in \Gamma$. see [16]

$$\underset{gB}{C}(a) = \underset{B}{C}\left(g^{-1}a\right) = \underset{B}{C}\left(ag^{-1}\right) = \underset{Bg}{C}(a) \forall a \in \Gamma$$

then:

$\underset{gB}{C}(a) = \underset{Bg}{C}(a) \forall a \in \Gamma$, and hence our result follows.

Theorem 2.6: Let B be an element of $MG(\Gamma)$ then $\forall g, h \in \Gamma$ $Bg = Bh$ iff $\underset{\zeta(e)}{B} g = \underset{\zeta(e)}{B} h$.

Proof (1)**"Necessity"**:

Assume that $Bg = Bh$ then $B \circ \underset{g}{\zeta} = B \circ \underset{h}{\zeta}$

$\underset{B}{C}\left(ag^{-1}\right) = \underset{B}{C}\left(ah^{-1}\right) \forall a \in \Gamma$ since a is taken arbitrary then if we put $a = h$

$\underset{B}{C}\left(hg^{-1}\right) = \underset{B}{C}\left(hh^{-1}\right) = \underset{B}{C}(e) = \zeta(e) = \zeta$

$\underset{B}{C}\left(hg^{-1}\right) = \underset{B}{C}(e) = \zeta(e) = \zeta$ which means that

$hg^{-1} \in \underset{\zeta(e)}{B}$ i.e. $\underset{\zeta(e)}{B}(hg^{-1}) = \underset{\zeta(e)}{B}$ (by Lemma 1.1) part (i)

$\underset{\zeta(e)}{B} h = \underset{\zeta(e)}{B} g$ (by Lemma 1.1) part (ii)

Proof (2)**"Sufficiency"**: Assume that $\underset{\zeta(e)}{B} g = \underset{\zeta(e)}{B} h$ which implies immediately to $gh^{-1} \in \underset{\zeta(e)}{B}$, $hg^{-1} \in \underset{\zeta(e)}{B}$

i.e. $\underset{B}{C}\left(gh^{-1}\right) = \underset{B}{C}(e)$, now let $a \in \Gamma$ taken arbitrary then:

$$\underset{B}{C}\left(ah^{-1}\right) = \underset{B}{C}\left(ag^{-1}gh^{-1}\right)$$

$$= \underset{B}{C}\left((ag^{-1})(gh^{-1})\right)$$

$$\geq \min\left\{\underset{B}{C}(ag^{-1}), \underset{B}{C}(gh^{-1})\right\}$$

$$\geq \min\left\{\underset{B}{C}(ag^{-1}), \underset{B}{C}(\mathrm{e})\right\}$$

$$\underset{B}{C}(ah^{-1}) \geq \underset{B}{C}(\mathrm{a}\, g^{-1}) = \underset{Bg}{C}(a)$$

$\underset{B}{C}(ah^{-1}) \geq \underset{Bg}{C}(a)$ ..... (1)

Now $\underset{B}{C}(ag^{-1}) = \underset{B}{C}(\mathrm{a}\, h^{-1}hg^{-1})$

$$= \underset{B}{C}\left((\mathrm{a}\, h^{-1})(hg^{-1})\right)$$

$$\underset{B}{C}(ag^{-1}) \geq \min\left\{\underset{B}{C}(ah^{-1}), \underset{B}{C}(hg^{-1})\right\}$$

$$\geq \min\left\{\underset{B}{C}(ah^{-1}), \underset{B}{C}(\mathrm{e})\right\}$$

$\geq \underset{B}{C}(ah^{-1}) = \underset{Bh}{C}(\mathrm{a})$ ..... (2)

$(1) \Rightarrow \underset{B}{C}(ah^{-1}) \geq \underset{Bg}{C}(\mathrm{a})$ ..... (1)

$\underset{B}{C}(ag^{-1}) \geq \underset{Bh}{C}(\mathrm{a})$ ..... (2)

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$(1) \Rightarrow \underset{Bh}{C}(a) \geq \underset{Bg}{C}(\mathrm{a})$ ..... (1)'

$(2) \Rightarrow \underset{Bg}{C}(a) \geq \underset{Bh}{C}(\mathrm{a})$ ..... (2)'

which means that $Bg = Bh$ and hence the theorem is proved.

Remark 2.7: Theorem 2.6 is also true in the case of left $mCosets$ i.e. if $B \in MG(\Gamma)$ then $\forall g, h \in \Gamma$ $gB = hB$ iff $g \underset{\zeta(e)}{B} = h \underset{\zeta(e)}{B}$ the proof can be carried by a similar arguments as in theorem 2.6.

Theorem 2.8: In the space of all multi-groups over a group Γ if two right $mCosets$ Bg, Bh (where B is any member of that space) are identically equal, then g, h have the same multicity in B.

Proof: Let $B \in AMG(\Gamma)$ where Γ is a group and $Bg = Bh$, $g, h \in \Gamma$, then by theorem 2.6 $\underset{\zeta(e)}{B} g = \underset{\zeta(e)}{B} h$, $gh^{-1} \in \underset{\zeta(e)}{B}$ and $hg^{-1} \in \underset{\zeta(e)}{B}$, from the properties of abelianmulti-groups see [16], we can write

$$\underset{B}{C}(\mathrm{g}) = \underset{B}{C}(\mathrm{hg}\,h^{-1})$$
$$\geq \min\left\{\underset{B}{C}(h), \underset{B}{C}(\mathrm{g}\,h^{-1})\right\}$$
$$\geq \min\left\{\underset{B}{C}(h), \underset{B}{C}(\mathrm{e})\right\}$$
$$\underset{B}{C}(\mathrm{g}) \geq \underset{B}{C}(\mathrm{h}) \quad \text{..... (1)}$$

also $\underset{B}{C}(h) = \underset{B}{C}(\mathrm{gh}\,g^{-1})$

$$\underset{B}{C}(h) \geq \min\left\{\underset{B}{C}(\mathrm{g}), \underset{B}{C}(\mathrm{h}\,g^{-1})\right\}$$
$$= \min\left\{\underset{B}{C}(\mathrm{g}), \underset{B}{C}(\mathrm{e})\right\}$$
$$\underset{B}{C}(h) \geq \underset{B}{C}(\mathrm{g}) \quad \text{..... (2)}$$

From (1), (2) we deduce that

$$\underset{B}{C}(\mathrm{g}) = \underset{B}{C}(\mathrm{h})$$

i.e. g, h have the same multiplicity in B, Which completes the proof of the theorem.

3. Quotient Group in the Multi-Group Perspective

In this section we want to extend the concept of quotient group in the multi-group. For this we restrict our salvesin the space of all abelian mulit-groups over a given group Γ i.e. $AMG(\Gamma)$.

Theorem 3.1: Assume that B is any member of $AMG(\Gamma)$, and $\Gamma / \mathrm{B} = \{Bg : g \in \Gamma\}$, then $Bg \circ Bh = B(gh) \forall g, h \in \Gamma$, moreover this operation is well defined.

Proof:Assume that $g, h \in \Gamma$ then

$$Bg \circ Bh = B \circ \underset{g}{\zeta} \circ B \circ \underset{h}{\zeta},\ \underset{g}{\zeta} = \{g\}_{\underset{B}{C}(\mathrm{e})},\ \underset{h}{\zeta} = \{h\}_{\underset{B}{C}(\mathrm{e})}$$

$$Bg \circ Bh = (\underset{g}{\zeta} \circ B) \circ (B \circ \underset{h}{\zeta})$$
$$= \underset{g}{\zeta} \circ (B \circ B) \circ \underset{h}{\zeta}$$

But $B \circ B = B$ see [16]

$$Bg \circ Bh = (\underset{g}{\zeta} \circ B) \circ \underset{h}{\zeta}$$
$$= (\mathrm{B} \circ \underset{g}{\zeta}) \circ \underset{h}{\zeta}$$
$$= \mathrm{B} \circ (\underset{g}{\zeta} \circ \underset{h}{\zeta})$$
$$= \mathrm{B} \circ \underset{gh}{\zeta}$$
$$= Bgh$$

Now to show thatthis operation is well defined, let $Bg = Bg_1$, $Bh = Bh_1$, so we can write:

$\underset{B}{C}(\mathrm{a}g^{-1}) = \underset{B}{C}(\mathrm{a}g_1^{-1})$ and $\underset{B}{C}(\mathrm{a}\,h^{-1}) = \underset{B}{C}(\mathrm{a}\,h_1^{-1})$, now we have

$\underset{B}{C}\left(\mathrm{a}(\mathrm{g}\,h)^{-1}\right) = \underset{B}{C}\left(\mathrm{a}\,h^{-1}g^{-1}\right)$, put $t = ah^{-1}$

$$\underset{B}{C}\left(\mathrm{a}(\mathrm{g}\,h)^{-1}\right) = \underset{B}{C}\left(tg^{-1}\right) = \underset{B}{C}\left(tg_1^{-1}\right)$$
$$= \underset{Bg_1}{C}(t) = \underset{Bg_1}{C}\left(ah^{-1}\right) = \underset{Bg_1}{C}\left(ah_1^{-1}\right)$$
$$= \underset{Bg_1h_1}{C}(a)$$
$$\underset{B}{C}\left(\mathrm{a}(\mathrm{g}\,h)^{-1}\right) = \underset{Bg_1h_1}{C}(a)$$
$$\underset{Bgh}{C}(a) = \underset{Bg_1h_1}{C}(a)\ \forall a \in \Gamma$$
$$Bgh = Bg_1h_1$$

Which means that the operation is well defined witch completes the proof of the theorem.

Theorem 3.2: Let Γ be a group and $B \in AMG(\Gamma)$, and $G = \Gamma / \mathrm{B} = \{Bg : g \in \Gamma\}$ and let "$\circ$"be an operation on G such that $Bg \circ Bh = B(gh)$ then $<G, 0>$ is the group, moreover this group is isomorphic to $\underset{\zeta(e)}{G} = \Gamma / \underset{\zeta(e)}{B}$ i.e. $\Gamma / B \cong \Gamma / \underset{\zeta(e)}{B}$.

Proof: From the previous theorem (i.e. theorem 3.1) we have shown that the operation is closed and well defined and in our previous note (see [16]),we have shown that the operation "$\circ$" is associative.

Now it is easy to see that B is the identity element of $G = \Gamma / \mathrm{B} = \{Bg : g \in \Gamma\}$

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$B \circ Bg = Be \circ Bg = B(eg) = Bg \ \forall g \in \Gamma$, also for existence of the inverse $Bg \circ Bg^{-1} = B(gg^{-1}) = Be = B \ \forall g \in \Gamma$, and hence $G = \Gamma / \mathrm{B}$ is a group. Since $B \in AMG(\Gamma)$, then $\Gamma / \underset{\zeta(e)}{B}$ is a group (since $\underset{\zeta(e)}{B}$ is normal subgroup of Γ).

Now to prove that Γ / B and $\Gamma / \underset{\zeta(e)}{B}$ are isomorphic, let us consider the mapping $\theta : \Gamma / B \to \Gamma / \underset{\zeta(e)}{B}$ where $\theta(Bg) = \underset{\zeta(e)}{B}(g)$, θ is injection, since if $\theta(Bg) = \theta(Bh)$, thus $\underset{\zeta(e)}{B}(g) = \underset{\zeta(e)}{B}(h)$, and from theorem 2.6, we get $Bg = Bh$, so θ is a injection, θ preserves the operation:

$$\theta(Bg \circ Bh) = \theta\left(\mathrm{B}(gh)\right)$$
$$= \underset{\zeta(e)}{B}(gh) = \underset{\zeta(e)}{B} g . \underset{\zeta(e)}{B} h$$
$$= \theta(Bg).\theta(Bh)$$

The surjection of θ is clear and hence θ is an isomorphism and so $\Gamma / B, \Gamma / \underset{\zeta(e)}{B}$ are isomorphic, hence our required result follows.

4. Conclusion

Cosets and quotient groups in the multi-group perspective and some of their properties were discussed.

References

[1] H. Aktas and N. Cagman, Soft sets and soft groups, Inform. Sci. 177 (2007) 2726–2735.
[2] S. Alkhazdeh, A. R. Salleh and N. Hassan, Soft multiset theory, Appl. Math. Sci. 5(72) (2011) 3561–3573.
[3] W. D. Blizard, Multiset theory, Notre Dame J. Formal Logic, 30 (1989) 36–66.
[4] K. Chakraborty, On bags and fuzzy bags, Adv. Soft Comput. Techniq. Appl. 25 (2000) 201– 212.
[5] K. Chakraborty, Notion of fuzzy IC-bags, Internat. J. Uncertain. Fuzziness Knowledge-Based Systems 12(3) (2004) 327–345.
[6] M. Delgado, M. J. Martin–Bautista, D. Sanchez and M. A. Vila, An extended characterization of fuzzy bags, Int. J. Intell. Syst. 24 (2009) 706–721.
[7] M. Delgado, M. D. Ruiz and D. Sanchez, RL-bags: A conceptual, level-based approach to fuzzy bags, Fuzzy Sets and Systems 208 (2012) 111–128.
[8] YU Feng, Intuitionistic L-fuzzy groups, School of Power Engineering, Nanjing University of Science and Technology, Nanjing 210094, P.R.China.
[9] K. P. Girish and S. J. John, Relations and functions in multiset context, Inform. Sci. 179(6)(2009) 758–768.
[10] B. Li, W. Peizhang and L. Xihui, Fuzzy bags with set-valued statistics, Comput. Math. Appl. 15 (1988) 3–39.
[11] P. Majumdar and S. K. Samanta, Soft multisets, J. Math. Comput. Sci. 2(6) (2012) 1700–1711.
[12] S. Miyamoto, Operations for real-valued bags and bag relations, in ISEA–EUSFLAT, (2009)612–617.
[13] S. Miyamoto, Two generalizations of multisets, in: M. Inuiguchi, S. Tsumito eds., Rough Set theory & Granular computing, Springer, (2003) pp 59–68.
[14] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971) 512–517.
[15] R. R. Yager, On the theory of bags, Internat. J. Gen. Systems 13(1) (1987) 23–37.
[16] S.F. El-Hadidi, Algebraic structures on Multi-groups I, International Journal of Science and Research (IJSR) vol. 6, Issue 5, 2017, 759-767.

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