

Common Fixed Point for Intimate Mappings in Banach Space

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Abstract: In this paper, common fixed point theorems for intimate mapping in Banach space are obtained. The presented results in this paper generalize and improve the corresponding results of Savita [6].

Keywords: Intimate mapping, compatible mapping of type (A).

1. Introduction

The concept of compatible maps of type (A) introduced by Jungck, Murthy and Cho [2] in metric and Banach space by motivating the concept of compatible maps see Jungck [3]. Recently Sahu, Dhagat and Shrivastava [5] generalized the concept of intimate mapping in metric space. In this paper, we present a common fixed point theorem for intimate mappings in Banach space.

Definition 1

Let A and S be mapping of Banach space X into itself. Then $\{A, S\}$ is said to be compatible pair of type (A) if

$\lim_{n \rightarrow \infty} \|ASx_n - SSx_n\| = \lim_{n \rightarrow \infty} \|SAx_n - AAsx_n\| = 0$
whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = t$, for some $t \in X$.

Definition 2

Let A and S be mapping of Banach space X into itself. Then $\{A, S\}$ is said to be S -intimate if and only if

$$\alpha \|SAx_n - Sx_n\| \leq \alpha \|ASx_n - Ax_n\|,$$

where $\alpha = \limsup$ or $\liminf$, $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = t$, for some $t \in X$.

Example: Let $X = [0, 1]$ and A, S are self mappings on X defined as follows $Ax = 2/(x+2)$ and $Sx = 1/(1+x)$ for all x in $[0, 1]$. Now the sequence $\{x_n\}$ in X defined by $x_n = 1/n$, $n \in \mathbb{N}$. Then we have $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = 1$.

Again $\|ASx_n - Ax_n\| \rightarrow 1/3$ and $\|SSx_n - Sx_n\| \rightarrow 1/2$ as $n \rightarrow \infty$. The clearly, we have

$\lim_{n \rightarrow \infty} \|ASx_n - Ax_n\| \leq \lim_{n \rightarrow \infty} \|SSx_n - Sx_n\|$. This $\{A, S\}$ is A -intimate.

But $\{A, S\}$ is not compatible mapping of type (A).

Proposition 1: If the pair $\{A, S\}$ is compatible of type (A) then it is both A and S -intimate.

Proof: Since

$$\|ASx_n - Ax_n\| \leq \|ASx_n - SSx_n\| + \|SSx_n - Sx_n\|$$

For $n \in \mathbb{N}$ therefore,

$$\alpha \|ASx_n - Ax_n\| \leq \alpha \|ASx_n - SSx_n\| + \alpha \|SSx_n - Sx_n\|$$

$$\alpha \|ASx_n - Ax_n\| \leq \alpha \|SSx_n - Sx_n\|$$

whenever $\{x_n\}$ is a sequence in Banach space X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = t$ for some $t \in X$.

Thus the pair $\{A, S\}$ is A -intimate.

Similarly we have show that the pair $\{A, S\}$ is S -intimate. But its converse need not be true. See example.

Proposition 2: Let A and S be a mappings of Banach space X into itself. If the pair $\{A, S\}$ is S -intimate and $At = St = p \in X$,

Then $\|Sp - p\| \leq \|Ap - p\|$.

Proof: Suppose $x_n = t$ for all $n \geq 1$, so $Ax_n = Sx_n = At = St = p$. Since the pair $\{A, S\}$ is S -intimate, then

$$\|SAx_n - Sx_n\| = \lim_{n \rightarrow \infty} \|SAx_n - Sx_n\|$$

$$\leq \lim_{n \rightarrow \infty} \|AAx_n - Ax_n\|$$

$$= \|AAx_n - Ax_n\|$$

$$\Rightarrow \|Sp - p\| \leq \|Ap - p\|$$

LEMMA 1: (Singh and Meade 1977). For every $t > 0$, $\gamma(t) < t$ if and only if $\lim_{n \rightarrow \infty} \gamma^n(t) = 0$ where γ^n denotes the n times composition of γ .

We now suppose that A, B, S and T be the mapping from Banach space X into itself such that

- (1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$
- (2) $\|Ax - By\| \leq c \max \{ \|Sx - Ty\|, \|Sx - Ax\|, \|Ty - By\|, [\|Sx - By\| + \|Ty - Ax\|]/2 \}$, for all x, y in X and $0 < c < 1$.

Let x_0 be an arbitrary point in X . Then by (1) there exists a point x_1 in X such that $Ax_0 = Tx_1$ and then a point x_2 in X such that $Bx_1 = Sx_2$ and soon, we obtain a sequence $\{y_n\}$ in X such that

$$(3) \begin{cases} y_{2n} = Sx_{2n} = Bx_{2n-1} \\ y_{2n+1} = Tx_{2n+1} = Ax_{2n} \text{ for } n = 1, 2, 3, \dots \end{cases}$$

Now prove the following lemma:

LEMMA 2: suppose that A, B, S and T be the mapping from Banach space X into itself satisfying condition (1) and (2). Then the sequence $\{y_n\}$ as defined by (3) is a Cauchy sequence.

Proof: Using (2) and (3), we get

$$\begin{aligned} \|y_{2n+1} - y_{2n}\| &= \|Ax_{2n} - Bx_{2n-1}\| \\ &\leq c \max \{ \|Sx_{2n} - Tx_{2n-1}\|, \|Sx_{2n} - Ax_{2n}\|, \\ &\quad \|Tx_{2n-1} - Bx_{2n-1}\|, [\|Sx_{2n} - Bx_{2n-1}\| \\ &\quad + \|Tx_{2n-1} - Ax_{2n}\|]/2 \} \\ &= c \max \{ \|y_{2n} - y_{2n-1}\|, \|y_{2n} - y_{2n+1}\|, \\ &\quad \|y_{2n-1} - y_{2n}\|, [\|y_{2n} - y_{2n}\| \\ &\quad + \|y_{2n-1} - y_{2n+1}\|]/2 \} \\ &= c \max \{ \|y_{2n} - y_{2n-1}\|, \|y_{2n} - y_{2n+1}\|, \\ &\quad \|y_{2n-1} - y_{2n}\|, [\|y_{2n-1} - y_{2n}\| \\ &\quad + \|y_{2n} - y_{2n+1}\|]/2 \} \\ \text{If } \|y_{2n+1} - y_{2n}\| &> \|y_{2n} - y_{2n-1}\| \end{aligned}$$

Then $\|y_{2n+1} - y_{2n}\| \leq c \|y_{2n+1} - y_{2n}\|$ which is contradiction.

Thus

$$\|y_{2n+1} - y_{2n}\| \leq c \|y_{2n} - y_{2n-1}\|$$

Similarly we have

$$\|y_{2n+2} - y_{2n+1}\| \leq c \|y_{2n+1} - y_{2n}\|$$

$$\text{Now } \|y_{n+1} - y_n\| \leq c \|y_n - y_{n-1}\|$$

$$\leq c^2 \|y_{n-1} - y_{n-2}\| \dots \dots$$

$$\dots \dots \leq c^n \|y_1 - y_0\|$$

For every integer $p > 0$, we get

$$\begin{aligned} \|y_n - y_{n+p}\| &\leq \|y_n - y_{n+1}\| + \|y_{n+1} - y_{n+2}\| \\ &\quad \dots \dots \|y_{n+p-1} - y_{n+p}\| \\ &\leq (1 + c + c^2 + \dots \dots c^{p-1}) \|y_n - y_{n+1}\| \\ &\leq \{c^p/(1-c)\} \|y_n - y_{n+1}\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus $\|y_n - y_{n+p}\| \rightarrow 0$.

Therefore $\{y_n\}$ is a Cauchy sequence. Now, we shall prove our main result:

THEOREM: Let A, B, S and T be mapping for a Banach space X into itself satisfying (1), (2), (3) and following condition:

(4) The pair $\{A, S\}$ is S - intimate and $\{B, T\}$ is T - intimate.

(5) $S(X)$ is complete.

Then A, B, S and T have a unique common fixed point in X .

PROOF: Define the sequence $\{y_n\}$ (as above by lemma 2) is a Cauchy sequence and has a limit u in X . Since $S(X)$ is

complete and $\{Sx_{2n}\}$ is Cauchy sequence. Then it converges to a point $p = Su$ for some u in X . Also $\{Ax_{2n}\}$, $\{Bx_{2n-1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ are subsequence of $\{y_n\}$, this subsequence converge to p . Hence Ax_{2n} , Sx_{2n} , Bx_{2n-1} , $Tx_{2n+1} \rightarrow p$.

Now from (2), we get

$$\begin{aligned} \|Au - Bx_{2n+1}\| &\leq c \max \{ \|Su - Tx_{2n+1}\|, \\ &\quad \|Su - Au\|, \|Tx_{2n+1} - Bx_{2n+1}\|, \\ &\quad [\|Su - Bx_{2n+1}\| + \|Tx_{2n+1} - Au\|]/2 \}, \text{ as } n \rightarrow \infty \\ \|Au - p\| &\leq c \max \{ \|Su - p\|, \|Su - Au\|, \\ &\quad \|p - p\|, [\|Su - p\| + \|p - Au\|]/2 \} \\ \|Au - p\| &\leq c \max \{ \|p - p\|, \|p - Au\|, \\ &\quad \|p - p\|, [\|p - p\| + \|p - Au\|]/2 \} \\ \|Au - p\| &\leq c \|Au - p\| \end{aligned}$$

which is contradiction. So, $Au = p = Su$.

Since $A(X) \subseteq T(X) \exists v \in X$ such that $Tv = p$. Hence for (2), we have

$$\begin{aligned} \|p - Bv\| &= \|Au - Bv\| \\ &\leq c \max \{ \|Su - Tv\|, \|Su - Au\|, \|Tv - Bv\|, [\|Su - Bv\| \\ &\quad + \|Tv - Au\|]/2 \} \\ \|p - Bv\| &\leq c \|p - Bv\| \end{aligned}$$

which is contradiction. So, $Bv = p = Tv$.

Since $Au = Su = p$ and the pair $\{A, S\}$ is S - intimate. Then by Proposition 2, we have

$$\|Sp - p\| \leq \|Ap - p\|,$$

Suppose $Ap \neq p$ then from (2), we get

$$\begin{aligned} \|Ap - p\| &= \|Ap - Bv\| \\ &\leq c \max \{ \|Sp - Tv\|, \|Sp - Ap\|, \\ &\quad \|Tv - Bv\|, [\|Sp - Bv\| + \|Tv - Ap\|]/2 \} \\ &= c \max \{ \|Sp - p\|, \|Sp - Ap\|, \\ &\quad \|p - p\|, [\|Sp - p\| + \|p - Ap\|]/2 \} \\ &\leq c \max \{ \|Ap - p\|, [\|Ap - p\| + \|p - Ap\|]/2, \\ &\quad 0, [\|Ap - p\| + \|p - Ap\|]/2 \} \\ &\leq c \|Ap - p\| \end{aligned}$$

Which is contradiction. So, $Ap = p$ and also $Sp = p$.

Hence $Ap = Sp = p$.

Similarly, we get $Bp = Tp = p$.

UNIQUENESS:

Let us consider q is another common fixed point A, B, S and T such that $p \neq q$, therefore

$$\begin{aligned} \|p - q\| &= \|Ap - Bq\| \\ &\leq c \max \{ \|Sp - Tq\|, \|Sp - Ap\|, \\ &\quad \|Tq - Bq\|, [\|Sp - Bq\| + \|Tq - Ap\|]/2 \} \\ &= c \max \{ \|p - q\|, \|p - p\|, \|q - q\|, \\ &\quad [\|p - q\| + \|q - p\|]/2 \} \\ &\leq c \|p - q\| \end{aligned}$$

a contradiction. This show that $p = q$.

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