

Interpolation Model for Beam Analysis

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Abstract: In finite element method (FEM) the calculations are done by approximating the whole model in many small elements known as finite elements and results are obtained at discrete points called nodes. A continuous result from these discrete set of data is obtained by using interpolation functions. Efficiency and speed of a Finite Element Analysis (FEA) system depends on how efficient the interpolation function is. In this paper authors have derived an interpolation model which is both fast and sufficiently accurate. It can be easily implemented in computer programs for fast and accurate analysis of a beam element. Few basic parameters are required by the interpolation function to describe the material properties, shape & size of the of the beam and deflections at any two points which are used to determine bending moment and shear force at each node of the beam irrespective of number of nodes used

Keywords: Finite Element Method, nodes, interpolation function, beam elements, nodes

1. Introduction

Beams are structural elements primarily used in Civil Engineering and Mechanical Engineering applications. Beams carry loads in direction perpendicular to their neutral axis, thus they withstand load by resisting the bending moment induced by the applied load. Hence while designing a beam structure the bending strength is considered as one of the important parameters. Therefore in the forthcoming discussion we have tried to analyze the bending moments of the beam element at various points. Though the beam is designed by considering the maximum bending moment acting on the beam, but knowledge of moments acting throughout the beam can be very helpful in accounting for material defects and susceptibility of failure in the beam at some other point rather than the point of application of maximum bending moment. This also gives the knowledge of how the structure would behave under the applied load.

In the following discussions we have used Euler-Bernoulli beam theory to analyze the moments in the beam. Euler-Bernoulli beam theory is Timoshenko beam theory applicable for small deflections which is a suitable approximation for practical structures. Following are the assumptions made by Euler-Bernoulli beam theory-

- During the deformation the cross-section of the beam remains planar and is perpendicular to the deformed axis.
- The beam has no rotation about the vertical axis.

These assumptions hold good for most of the practical applications hence Euler-Bernoulli beam theory is widely accepted in engineering practices. The Euler-Bernoulli equation gives relation between the moment experienced by the beam and the curvature of the beam under the applied load. The governing equation based on this beam theory which we have implemented in our model is

$$M = EI \frac{d^2 Y(x)}{dx^2} \quad (1.1)$$

Where,

M = Bending moment in the beam;

$Y(x)$ = Transverse deflection of the beam as the function of x which is the length of the beam from its reference end.

E = Young's modulus of the beam material

I = Moment of inertia of the cross section about the axis to rotation.

The term $\frac{d^2 v(x)}{dx^2}$ is called the curvature of the beam element. From equation 1.1 it is evident that moment is linearly dependent on the curvature.

As already mentioned $v(x)$ is the deflection of the beam, the deflections can be readily found at deflection formulas when the applied force is already known but we want to tackle a situation when the deflections are known and force and moment can be found using the displacements which can be an ideal scenario while analyzing a failed structure (to determine the force and moment under which it failed). Thus for this purpose we have developed an interpolation model. In this model only requirement is angular deflection and deflection along the direction perpendicular to the neutral axis and the user can determine the moments along the length of the beam.

2. Interpolation Model

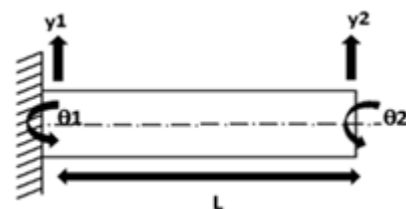


Figure 1

Considering a beam as shown in the Fig.1, the fixed end has degrees of freedom y_1 and θ_1 and the other end has degrees of freedom y_2 and θ_2 and L is the length of the beam.

We consider the fixed end as node 1 and the free end as node 2. The nodal degrees of freedom can be represented in vector form as following-

$$[Q] = [y_1 \quad \theta_1 \quad y_2 \quad \theta_2]^T \quad (2.1)$$

Here we assume that the values of y_1 , θ_1 , y_2 and θ_2 are already known and we have to develop an interpolation function that can fit approximate deflections throughout the length of the beam which can be utilized to find the approximate moments at the nodes without actually knowing the force acting on the beam.

We proceed our calculation by assuming that the interpolating function follows a cubic polynomial equation as a function of x which is the distance of the node from the fixed end which is considered as reference. The polynomial function can be written as-

$$Y = a_0 + a_1x + a_2x^2 + a_3x^3 \quad (2.2)$$

The angular deflection θ is given as-

$$\theta = \frac{dy}{dx} \quad (2.3)$$

$$\text{Or, } \theta = a_1 + 2a_2x + 3a_3x^2 \quad (2.4)$$

Applying boundary conditions,

At node 1; $x=0$, $y=y_1$, $\theta=\theta_1$, applying which we get-

$$a_0 = y_1; a_1 = \theta_1 \quad (2.5)$$

At node 2; $x=L$, $y=y_2$, $\theta=\theta_2$, applying these boundary conditions and values obtained in equation 2.4, in equation 2.2 we get-

$$a_2 = -\frac{(2\theta_1 + \theta_2)}{L} - \frac{3(y_1 - y_2)}{L^2} \quad (2.6)$$

$$a_3 = \frac{2(y_1 - y_2)}{L^3} + \frac{(\theta_1 + \theta_2)}{L^2} \quad (2.7)$$

Substituting the values of a_0 , a_1 , a_2 and a_3 in equation 2.2 and applying suitable calculations we get the following equation-

$$Y = y_1 \left(1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}\right) + \theta_1 \left(x - \frac{2x^2}{L^2} + \frac{x^3}{L^2}\right) + \theta_2 \left(-\frac{x^2}{L^2} + \frac{x^3}{L^2}\right) + y_2 \left(\frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right) \quad (2.8)$$

The preceding equation can again be written in matrix form as-

$$Y = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \end{bmatrix} \quad (2.9)$$

$$\text{Or, } Y = [N] \{Q\}$$

Where Q is displacement vector.

In equation 2.9 N_1 , N_2 , N_3 , N_4 are interpolating functions also called shape functions where-

$$N_1 = \left(1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}\right)$$

$$N_2 = \left(x - \frac{2x^2}{L^2} + \frac{x^3}{L^2}\right)$$

$$N_3 = \left(\frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right)$$

$$N_4 = \left(-\frac{x^2}{L^2} + \frac{x^3}{L^2}\right)$$

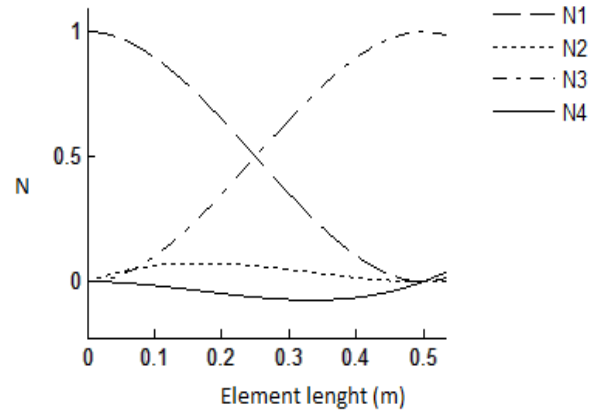


Figure 2

The figure above shows the variation of values of shape function with varying element length. For this particular example the length of the beam (L) is taken as 0.5m.

From Euler-Bernoulli theorem the bending moment is expressed as –

$$M = EI \frac{d^2Y(x)}{dx^2};$$

$$\frac{d^2Y(x)}{dx^2} = \frac{d^2[N]\{Q\}}{dx^2}$$

Since Q is constant that is for our model the values of deflections are known hence-

$$\frac{d^2Y(x)}{dx^2} = \frac{d^2[N]}{dx^2} \{Q\} \quad (2.10)$$

From equation 2.10 equation 1.1 can be written as-

$$M = EI \frac{d^2[N]}{dx^2} \{Q\} \quad (2.11)$$

Therefore, on differentiating the shape functions we get following equations-

$$\frac{d^2N_1}{dx^2} = \frac{12x}{L^3} - \frac{6}{L^2}$$

$$\frac{d^2N_2}{dx^2} = \frac{6x}{L^2} - \frac{4}{L}$$

$$\frac{d^2N_3}{dx^2} = \frac{6}{L^2} - \frac{12x}{L^3}$$

$$\frac{d^2N_4}{dx^2} = \frac{6x}{L^2} - \frac{2}{L}$$

Substituting these values, equation 2.11 can be written as-

$$M = \frac{EI}{L^2} \left(\frac{12x}{L} - 6\right) (6x - 4L) \left(6 - \frac{12x}{L}\right) (6x - 2L) \begin{bmatrix} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \end{bmatrix}$$

$$\text{Or, } M = \frac{EI}{L^2} [B] \{Q\} \quad (2.12)$$

Equation 2.12 can be used to find moment at any distance x from the reference end.

3. Result

Using the equations derived above in a MATLAB program and compared the result as obtained by entering the same parameters in ANSYS.

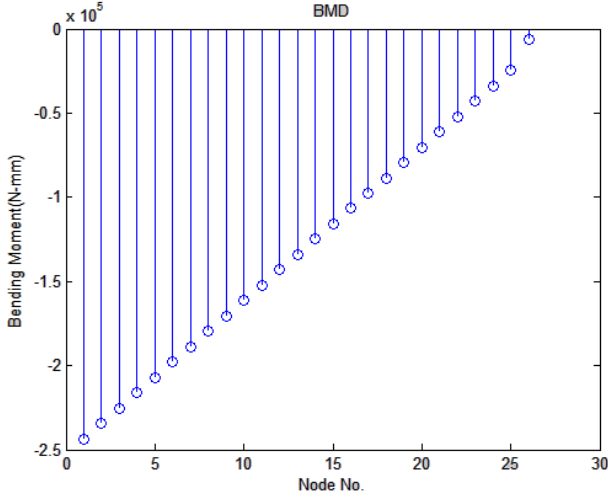
Example-

Cantilever beam of length 5m, moment of inertia $I = 2.67 \times 10^5 \text{ mm}^4$, Young's modulus $E = 2 \times 10^5 \text{ N/mm}^2$, deflections $y_1 = 0$, $\theta_1 = 0$, $y_2 = -38.507 \text{ mm}$, $\theta_2 = -0.0117$ radian,

Entering these parameters in ANSYS and MATLAB program based on the obtained interpolation model we get the-

Maximum bending moment obtained by ANSYS, $M_A = -2.4875 \times 10^5 \text{ N-mm}$

Maximum bending moment obtained by MATLAB, $M_M = -2.4359 \times 10^5 \text{ N-mm}$



Variation of B.M with respect to nodes

Figure 3

On running MATLAB code with specified parameters we get the bending moment diagram as shown in the figure 3.

The figure 4 shows the variation of shear force generated by the MATLAB code.

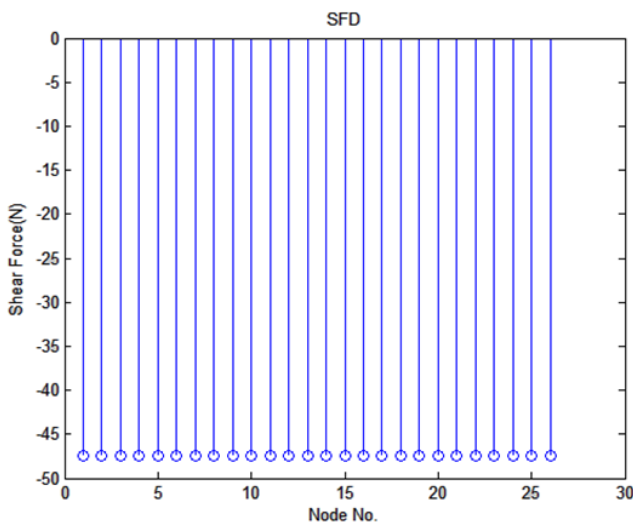
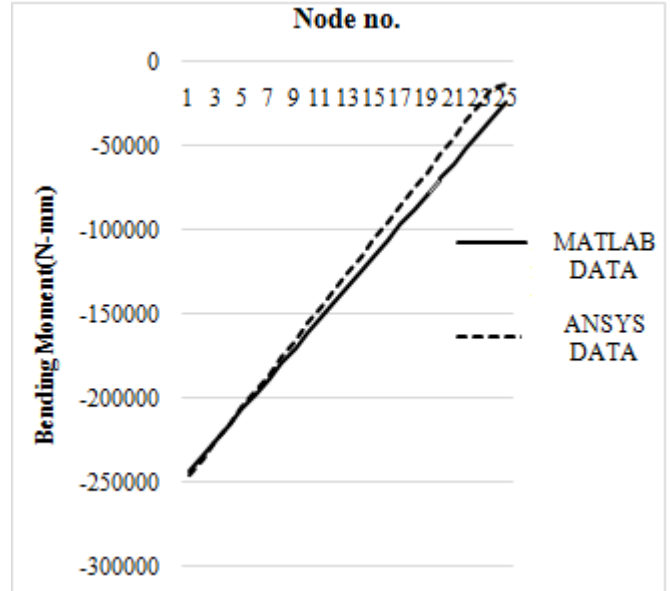


Figure 4

Variation of shear force with respect to nodes

When same parameters are entered in Ansys and simulated, on comparing the results of MATLAB and Ansys, considering the MATLAB result as approximate value and Ansys result as actual value we get following data-



Comparison between MATLAB & ANSYS results

The above graph and table gives comparison between MATLAB code and ansys.

ANSYS DATA (N-mm)	MATLAB DATA(N-mm)	% Difference
-243593.712	-245000	0.577309
-234467.7342	-235000	0.22701
-225341.7563	-225000	0.151661
-216215.7785	-215000	0.562299
-207089.8006	-205000	1.009128
-197963.8228	-195000	1.497154
-188837.8449	-185000	2.032349
-179711.8671	-175000	2.621901
-170585.8892	-165000	3.274532
-161459.9114	-155000	4.000938
-152333.9335	-145000	4.814379
-143207.9557	-135000	5.731494
-134081.9778	-125000	6.773452
-124956	-115000	7.967605
-115830.0222	-105000	9.349927
-106704.0443	-95000	10.9687
-97578.06646	-85000	12.89026
-88452.08862	-75000	15.20833
-79326.11077	-65000	18.05977
-70200.13292	-55000	21.65257
-61074.15508	-45000	26.31908
-51948.17723	-35000	32.62516
-42822.19938	-25000	41.61907
-33696.22154	-15000	55.48462
-24570.24369	-13000	47.09047
-6318.288	-5000	20.86464

4. Conclusion

Comparing ANSYS solution with MATLAB solution the difference between the two is of 2% which is an acceptable value hence the interpolation function works with an acceptable accuracy. It was also found that the equation can be readily applied in the computer program without any further approximation and simplification hence it has a good application in practical scenario.

5. Appendix

Preceding calculations has been done using the MATLAB code as following:-

```
%SFD BMD
%By- Vipul kumar; 5/3/2016
%=====
E=input('enter youngs modulus: ');
I=input('enter moment of inertia of cross section: ');
L=input('enter distance btwn end 1 & end 2: ');
y1=input('enter deflection at end 1: ');
Q1=input('enter angular deflection at end 1: '); %enter the const.
y2=input('enter deflection at end 2: ');
Q2=input('enter angular deflection at end 2: ');
K=input('enter length of the element: ');
n=K/101;
M=zeros(1,26); %creating matrix for 26 nodes
F=zeros(1,26);
for j=0:25
    i=j+1;
    if j==25
        x=n*i;
    else
        x=n*j;
    end
    B1=(12/L)*x-6;
    B2=6*x-4*L;
    B3=6-(12*x)/L; %shape functions
    B4=6*x-2*L;
    b1=-12;
    b2=-6*L;
    b3=12;
    b4=-6*L;
    B=[B1 B2 B3 B4];
    b=[b1 b2 b3 b4];
    Q=[y1;Q1;y2;Q2];
    M(i)=((E*I)/L^2)*(B*Q); %calculating bending moment
    F(i)=((E*I)/L^3)*(b*Q); %calculating shear force
end
stem(M);title('BMD');xlabel('Node No. ');ylabel('Bending Moment(N-mm)');figure; %plotting
bending moment diagram
stem(F);title('SFD');xlabel('Node No. ');ylabel('Shear Force(N)'); %plotting shear force
diagram
```

References

- [1] S. Rao Gunakala,, D.M.G Comissiong and K. Jordan, A Finite Element Solution of the Beam Equation via MATLAB, "International Journal of Applied Science and Technology", Vol. 2 No. 8; October 2012.
- [2] Euler-Bernoulli Beam Equation, http://en.wikipedia.org/wiki/Euler-Bernoulli_beam_equation
- [3] Mihirr Sose and Animesh Dutta, Comparative study of results obtained by analysis of structures using ANSYS, STAAD and MATLAB, "IOSR Journal of Mechanical and Civil Engineering (IOSR-JMCE)", Volume 9, Issue 1 (Sep. - Oct. 2013).
- [4] Yu-xin Jie, Hui-na Yuan, Hou-de Zhou, and Yu-zhen Yu, Bending Moment Calculations for Piles Based on the Finite Element Method, "Journal of Applied Mathematics", Volume 2013 (2013)
- [5] "Finite Element Method with Applications in Engineering", Y. M. Desai, T. I. Eldho , A. H. Shah , Pearson.