Iterated Function System

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Abstract: ***Fractal image compression through IFS is very important for the efficient transmission and storage of digital data. Fractal is made up of the union of several copies of itself and IFS is defined by a finite number of affine transformation which characterized by Translation, scaling, shearing and rotation. In this paper we describe the necessary conditions to form an Iterated Function System and how fractals are generated through affine transformations.***

Keywords: Iterated Function System; Contraction Mapping.

1. Introduction

The exploration of fractal geometry is usually traced back to the publication of the book "*The Fractal Geometry of Nature*" [1] by the IBM mathematician Benoit B. Mandelbrot. Iterated Function System is a method of constructing fractals, which consists of a set of maps that explicitly list the similarities of the shape. Though the formal name Iterated Function Systems or IFS was coined by Barnsley and Demko [2] in 1985, the basic concept is usually attributed to Hutchinson [3]. These methods are useful tools to build fractals and other similar sets. Barnsley et al, 1986 [4] stated their inverse problem: given an object, and an iterated function system that represents that object within a given degree of accuracy. The collage theorem provided the first stepping stone toward solving the inverse problem.However Vrscay [5] have traced the idea back toWilliams [6], who studied fixed points of infinite composition of contractive maps.

2. Preliminaries Mathematical

An Iterated Function Systems is a set of contraction mappings $W = \{w_1, w_2, \dots w_n\}$ acting on a space$X$. Associated with this set of mappings$W$, is a set of probabilities$P = \{P_1, P_2, \dots, P_n\}$. As we will see, these probabilities are used to generate a random walk in the space X. If we start with any point in X and apply these maps iteratively, we will come arbitrarily close to a set of points A in X called the attractor of the IFS. These attractors are very often fractal (for the most part, we will assume attractors are fractal sets, and thus, use the words interchangeably). This forms the basis for creating an algorithm that will approximate the attractor of IFS. We sometime call sets $\{W_k(A)\}$ whose limits are fractals, pre-fractals. These are sets the algorithm will be able to generate. Increasing the number of times we apply the maps will give us a more accurate picture of what the attractor looks like. Creating Fractals using Iterated Function Systems.
In order to understand what iterated function systems are and why the random iteration algorithm works, we need to be familiar with some mathematical concepts. A space X is simply a set of elements (points).

Metric Spaces definition: A space X with a real-valued function $d: X \times X \rightarrow \Re$ is called a metric space (X, d) if d possess the following properties:
1. $d(x,y) \geq 0 \; for \; \forall \; x,y \in X$
2. $d(x,y) = d(y,x) \; \forall \; x,y \in X$
3. $d(x,y) \leq d(x,z) + d(z,y) \forall \; x,y,z \in X$.
(Triangle inequality).
For instance, $\Re$ with $d = |x - y|$ is a metric space. $\Re^2$ with the usual Euclidian distance is also a metric space.

Open Sets definition: A subset S of the metric space (X, d) is open if, for each point $x \in S$, we can find a $r > 0$ so that $\{y \in X: \; d(x,y) < r\}$ is contained in S.

Closed Sets definition: A subset S of the metric space (X, d) is closed if, whenever a sequence $\{x_n\}$ contained in S converges to a limit$x \in X$, then in fact this limit$x \in S$.

Bounded Sets definition: A subset S of the metric space (X, d) is bounded if we can find an $x \in X$ and an $M \in \Re > 0$ so that $d(a, x) \leq M \; \forall \; a \in S$.

Cauchy Sequence definition: A sequence $\{x_n\}$ in X is called a Cauchy sequence if given$\varepsilon > 0$, we can find an $N \in N > 0$ such that $d(x_n, x_m) > N$
Note: A Cauchy sequence need not have a limit in **X**. This stimulates the next definition. **Complete Metric Space**

Definition: A metric space (X, d) is complete if every Cauchy sequences in X converges in$X$.

Compact Sets definition: A subset S of the metric space (X, d) is compact if every sequences in S has a subsequence which converges in$S$. Since we are mostly concerned with metric spaces where the underlying space is $\Re_n$ or C_n, we can state the following :

Theorem: If a subset $S \subset \Re_n$ (or C_n) is closed and bounded, then it is compact. Now that we know what a compact set is, we can define the following space:

Definition: Let X be a complete metric space. Then $H(X)$ consists of the non-empty compact subsets of X. To make $H(X)$ into a metric space, we must find a real valued function $h : H(X)\, x\, H(X) \rightarrow \Re$ with the properties enumerated before. To construct this metric, we need to know what a δ-parallel body $A\delta$ of a set A is:

Definition: Let (X, d) be a complete metric space and $H(X)$ denoting the space whose points are the compact subsets of X *known* as Hausdroff space, other than the empty set. Let $x, y \in X$ and let $A, B \in H(X)$. Then

(1) Distance from the point x to the set B is defined as
$d(x,B) = min\{d(x,y) : y \in B\}$,
(2) Distance from the set A to the set B is defined as
$d(A,B) = max\{d(x,B) : x \in A\}$,
(3) Hausdroff distance from the set A to the set B is defined as
$h(A,B) = d(A,B) \vee d(B,A)$.
Then the function $h(d)$ is the metric defined on the Space $H(X)$.

Contraction Mappings definition: Let S: X → X be a transformation on the metric space (X, d). S is a contraction if $\exists\, s \in \Re$ with $0 \leq s < 1$ such that $d(S(x),S(y)) \leq sd(x,y)\ \forall\, x,y \in X$. Any such number s is called a contractivity factor of S. The following theorem will be very important for later on.

Contraction Theorem: Let S : X → X be a contraction on a complete metric space (X,d). Then S possess exactly one fixed point $x^* \in X$ and moreover for any point x ∈ X, the sequence $\{S_n(x): n = 0,1,2.....\}$ converges to x^*. That is, $lim_{n\to\infty} S_n(x) = x^*$, *for each* $x \in X$

3. Iterated Function Systems

Definition: A (hyperbolic) iterated function system consists of a complete metric space (X, d) together with a finite set of contraction mappings $w_n : X \rightarrow X$, with respective contractivity factor sn, for $n = 1,2,....N$. The abbreviation "IFS" is used for "iterated function systems". The notation for the IFS just announced is $\{X\,; w_n : n = 1,2,...N\}$ and its contractivity factor is $s = max\{s_n : n = 1,2,...N\}$. The following theorem is extremely important and suggests an algorithm for computing the pre-fractals.

Theorem: Let $\{X : T_n, n = 1,2,3,...,N\}$ be a iterated function system with contractivity factors. Then the transformation defined by
$W(B) = \cup_{n=1}^{N} S_n(B)$ *for all* $B \in H(X)$,
is a contraction mapping on the complete metric space $(H(X), h(d))$ with contractivity factor s. That is $h(W(B), W(C)) \leq sh(B, C)$.

Its unique fixed point, which is also called an attractor, $A \in H(X)$, obey $A = W(A) = \bigcup_{n=1}^{N} S_n(A)$ and is given by $A = lim_{n\to\infty} W_{on}(B)$ for any $B \in H(X)$. W_{on} denotes the n-fold composition Of W.

4. Affine Transformation

The use of homogeneous coordinates is the central point of affine transformation which allow us to use the mathematical properties of matrices to perform transformations. So to transform an image, we use a matrix $T \in M_3(R)$ providing the changes to apply

$$T = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ p_x & p_y & 1 \end{bmatrix}$$

The vector $[T_x, T_y]$ represent the translation vector according the canonical vectors. the vector $[P_x, P_y]$ represents the projection vector on the basis. The square matrix composed by the a_{ij} elements is the affine transformation matrix.

An affine transformation $T : R^2 \rightarrow R^2$ is a transformation of the form $T : Ax + B$ defined by

$$T\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \\ 1 \end{pmatrix}$$

Where the parameter $a_{11}, a_{12}, a_{21}, a_{22}$ form the linear part which determines the rotation, skew and scaling and the parameters t_x, t_y are the translation distances in x and y directions, respectively.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = T\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$x' = x\, a_{11} + y\, a_{12} + t_x$
$y' = x\, a_{21} + y\, a_{22} + t_y$

The general affine transformation can be defined with only six parameter:

θ : the rotation angle.
t_x : the x component of the translation vector.
t_y : the y component of the translation vector.
S_x : the x component of the scaling vector.
S_y : the y component of the scaling vector.
Sh_x : the x component of the shearing vector.
Sh_y : the y component of the shearing vector.

In the other words, The Fractal is made up of the union of several copies of itself, where each copy is transformed by a function T_i, such a function is 2D affine transformation, so the IFS is defined by a finite number of affine transformation which characterized by Translation, scaling, shearing and rotation.

5. Conclusion

In this paper, we have reviewed the basic definitions and necessary conditions to generate IFS for fractal image compression. We have also discussed about the different operation such as translation, transvection, rotation and scaling of affine transformation and their effect on fractal image compression.

References

[1] B. B. Mandelbrot, The Fractal Geometry of Nature (Freeman, New York, 1982).
[2] M. F. Barnsley and S. Demko, Iterated function systems and global construction of fractals, Proc. R. Soc. Lond. A 399 (1985) 22243–275.
[3] J. E. Hutchinson, Fractals and self similarity, Indiana Univ. Math. J. 30(5) (1981) 713–747.
[4] M. F. Barnsley, V. Ervin, D. Hadin and J. Lancaster, Solution of an Inverse Problem for Fractals and Other Sets, Proc.Natl Acad.Sci.USA,vol.83,pp.1975-1977,April 1986.
[5] E. R. Vrscay, Iterated function systems: theory, applications and the inverse problem, in Proceedings of the NATO Advanced Study Institute on Fractal Geometry, Montreal, July 1989 (Kluwer Academic Publishers, 1990).
[6] R. E. Williams, Composition of contractions, Bol. Soc. Brasil. Mat. 2 (1971) 55–59.