

# A Study on Best Proximity Point (A Brief Review)

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**Abstract:** *In this survey article, we collected the historical developmental of best proximity points theory. Some important results from beginning up to now are incorporated in this paper.*

**Keywords:** best proximity points theory, cyclic contraction, cyclic  $\phi$ -contraction.

## 1. Introduction

In the present era, cyclic contraction mappings and best proximity point are among the popular topics in the fixed point theory and have received considerable interest. The Banach contraction principle is a fundamental result in fixed point theory. One kind of generalization of the Banach contraction principle is the notation of cyclical maps.

The Banach contraction principle [8] states that, if  $(X, d)$  is a complete metric space and  $T : X \rightarrow X$  is a contraction mapping (i.e.,  $d(Tx, Ty) \leq \alpha d(x, y)$  for all  $x, y \in X$ , where  $\alpha$  is a nonnegative number such that  $\alpha < 1$ ). Then  $T$  has a unique fixed point. This principle has been generalized in many ways.

The existence and convergence results for best proximity points are obtained for cyclic contraction maps and these results are generalized by generalizing the notion of cyclic contraction to the different class of cyclic contractions. Also in different spaces like different metric spaces, uniformly convex Banach spaces, reflexive Banach spaces etc.

## 2. Historical development

In 2003, Kirk et al.[18] generalized the Banach contraction principle by using two closed subsets of a complete metric space, and they first gave the notion of cyclic contraction mappings.

Later in 2006, Eldred and Veeramani [10] proved some existence results for best proximity points of cyclic contraction maps in uniformly convex Banach space. They raised a question about the existence of a best proximity point for a cyclic contraction map in a reflexive Banach space.

In 2009, Al-Thagafi and Shahzad [3] gave a positive answer to this question by introducing the new class of maps called cyclic  $\phi$ -contraction, which contains the cyclic contraction mappings as a subclass. In fact they solved the problem for cyclic  $\phi$ -contraction maps. But they proved some results about best proximity points of weakly continuous cyclic contraction maps satisfying the proximal property on

reflexive (and strictly convex) Banach Spaces. In this way they raised another question for cyclic  $\phi$ -contraction maps. Many of the authors have provided a positive answer to the question of Al-Thagafi and Shahzad like, Sh. Rezapour, M. Derafshpour and N. Shahzad [28], Al-Abkar and M. Gabeleh [1], H. K. Pathak and N. Shahzad [25], Sh. Rezaapour and M. Derafshpour [9].

The notion of cyclic Meir-keeler contraction was introduced by C. D. Bari et.al. [7] in 2008. They proved the existence result for a best proximity point with cyclic Meir-keeler contraction in the framework of uniformly convex Banach space. With every cyclic contraction mappings being a cyclic Meir-keeler contraction their main result is a generalization of Eldred Veeramani's theorem. In 2009, S. Karpagam and S. Agrawal [19] generalized the cyclic meir keeler contraction to the p-cyclic meir-keeler contraction. Then Suzuki et.al [45] extended Eldred and Veeramani's theorem to metric spaces with the property UC.

In 2010, C.Vetro [46] generalized the cyclic  $\phi$ -contraction maps to the p-cyclic  $\phi$ -contraction maps and then existence and convergence results for best proximity points are obtained. Moreover, they generalize the results of Al. Thagafi and Shahzad [3]. In [20], S. Karpagam and S. Agrawal introduced a notion of cyclic orbital Meir-keeler contraction and they generalized the best proximity point result due to C. D. Bari et.al.[7].

Recently, Pathak and Shahzad [25] introduced a new concept of C-proximity point and a new class of maps, called cyclic C'-contraction maps, as subclasses. Convergence and existence results of Best C-proximity points for cyclic C' contraction maps are also obtained. M. A. Petric [27] was introduced the notion of weak cyclic Kannan contraction. They also gave some existence and convergence results for best proximity points for weak cyclic Kannan contractin in the setting of uniformly convex Banach space.

## 2.1 Definitions

**Definition 2.1. ([10]).** Let  $A$  and  $B$  be nonempty subsets of a metric space  $X$  and  $T : A \cup B \rightarrow A \cup B$  be a cyclic contraction map then a point  $x \in A \cup B$  is said to be a Best Proximity Point if  $d(x, Tx) = \text{dist}(A, B)$  where  $\text{dist}(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$ .

**Definition 2.2. ([10]).** Let  $A$  and  $B$  be nonempty subsets of a metric space  $X$ . A map  $T : A \cup B \rightarrow A \cup B$  is said to be a cyclic contraction map if it satisfies:

1.  $T(A) \subseteq B$  and  $T(B) \subseteq A$ .

2. For some  $k \in (0, 1)$ ,

$d(Tx, Ty) \leq kd(x, y) + (1 - k)\text{dist}(A, B)$ , for all  $x \in A, y \in B$  where  $\text{dist}(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$ .

**Definition 2.3 ([7]).** Let  $(X, d)$  be a metric space let  $A$  and  $B$  be nonempty subsets of  $X$ . Then a mapping  $T : A \cup B \rightarrow A \cup B$  is called a cyclic Meir-Keeler contraction if the following are satisfied:

1.  $T(A) \subseteq B$  and  $T(B) \subseteq A$ .

2. for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$d(x, y) < d(A, B) + \varepsilon + \delta \Rightarrow d(Tx, Ty) < d(A, B) + \varepsilon$  for all  $x \in A, y \in B$ .

**Definition 2.4 ([3]).** Let  $A$  and  $B$  be nonempty subsets of a metric space  $(X, d)$  and let  $T : A \cup B \rightarrow A \cup B$  such that  $T(A) \subseteq B$  and  $T(B) \subseteq A$ . The map  $T$  is said to be cyclic  $\varphi$ -contraction if  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  is strictly increasing map and

$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B))$

for all  $x \in A, y \in B$ .

### 3. Cyclic contractions in metric spaces

Let  $A$  and  $B$  be nonempty closed subsets of a complete metric space  $X$ . A generalized version of mappings  $T : A \cup B \rightarrow A \cup B$  satisfying  $T(A) \subseteq B$  and  $T(B) \subseteq A$  were the subject of [18].

Suppose there exists two nonempty closed subsets  $A$  and  $B$  of  $X$  such that the mapping  $F : A \cup B \rightarrow A \cup B$  satisfies:

1.  $F(A) \subseteq B$  and  $F(B) \subseteq A$

2.  $d(Fx, Fy) \leq kd(x, y), \forall x \in A, y \in B$ , where  $k \in (0, 1)$ .

Then it readily follows that for any  $x \in A \cup B$ ,

$d(Fx, Fy) \leq kd(x, F(x))$

and this again implies that  $\{Fn(x)\}$  is a Cauchy sequence.

Consequently  $\{Fn(x)\}$  converges to some point  $z \in X$ .

However, in view of (2) an infinite number of terms of the sequence  $\{Fn(x)\}$  lie in  $A$  and an infinite number of terms lie in  $B$ . Therefore  $z \in A \cap B$ , so  $A \cap B$  is a contraction mapping.

Since Banach contraction mapping principle applies to  $F$  on  $A \cap B$  we have the following result.

**Theorem 3.1 (Kirk et.al.[18]).** Let  $A$  and  $B$  be two nonempty closed subsets of a complete metric space  $X$ , and suppose  $F : X \rightarrow X$  satisfies (1) and (2) above. Then  $F$  has a unique fixed point in  $A \cap B$ .

In order to extend this to the case when  $A \cap B = \emptyset$ , in 2006 A. Eldred and P. Veeramani [10] introduced the generalization of (2) which does not entail  $A \cap B$  to be nonempty and ask, not for the existence of a fixed point of  $F$ , but for a best proximity point; that is, a point  $x$  in  $A \cup B$  such

that  $d(x, Tx) = \text{dist}(A, B)$ , where  $\text{dist}(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$ .

In 2006, A.A.Eldred and P.Veeramani [10] gave the following existence results for Best Proximity Points with cyclic contraction maps in metric space.

**Proposition 3.2 (Eldred, Veeramani[10]).** Let  $A$  and  $B$  be nonempty subsets of a metric space  $X$ . Suppose  $T : A \cup B \rightarrow A \cup B$  is a cyclic contraction map. Then starting with any  $x_0$  in  $A \cup B$ , we have  $d(x_n, Tx_n) \rightarrow \text{dist}(A, B)$ , where  $x_{n+1} = Tx_n, n=0,1,2,\dots$

**Proposition 3.3 (Eldred, Veeramani[10]).** Let  $A$  and  $B$  be nonempty closed subsets of a complete metric space  $X$ . Let  $T : A \cup B \rightarrow A \cup B$  be a cyclic contraction map, let  $x_0 \in A$  and define  $x_{n+1} = Tx_n$ . Suppose  $\{x_{2n}\}$  has a convergent subsequence in  $A$ . Then there exists  $x$  in  $A$  such that  $d(x, Tx) = \text{dist}(A, B)$ .

### 4. Cyclic $\phi$ -contractions in metric spaces

In 2009, M.A.Al-Thagafi and Naseer Shahzad [3] introduced a new class of maps, called cyclic  $\varphi$ -contractions which contains the cyclic contraction maps as a subclass. For such maps they obtained convergence and existence results of best proximity points.

Now we have the following results for cyclic  $\varphi$ -contraction maps.

**Theorem 4.1. (Thagafi, Shahzad [3]).** Let  $A$  and  $B$  be nonempty subsets of a metric space  $X$  and let  $T : A \cup B \rightarrow A \cup B$  be a cyclic  $\varphi$ -contraction map. For  $x_0 \in A \cup B$ , define  $x_{n+1} := Tx_n$  for each  $n_0$ . Then  $d(x_n, x_{n+1}) \rightarrow d(A, B)$  as  $n \rightarrow \infty$ .

In 2010, Sh. Rezapour, M. Derafshpour and N. Shahzad [28] were generalized the proposition 2.2.4 of [10] for cyclic  $\varphi'$ -contraction.

**Theorem 4.2. (Shahzad et al.[28]).** Let  $\varphi : [0, \infty) \rightarrow [0, \infty)$  be a strictly increasing unbounded map. Also, let  $A$  and  $B$  be nonempty subsets of a metric space  $(X, d)$ ,  $T : A \cup B \rightarrow A \cup B$  a cyclic  $\varphi$ -contraction map,  $x_0 \in A \cup B$  and  $x_{n+1} = Tx_n$  for all  $n_0$ . Then, the sequences  $\{x_{2n}\}$  and  $\{x_{2n+1}\}$  are bounded.

In 2010, A.Abkar and M.Gabeleh [1] gave the existence results of best proximity points for cyclic  $\varphi'$ -contraction maps in metric spaces. Indeed they proved two existence theorems on best proximity points in hyperconvex spaces as well as in ultrametric spaces.

### 5. p-Cyclic $\phi$ -contractions in metric spaces

In 2010, Calogero Vetro [46] introduced a new class of mappings, called p-cyclic  $\varphi$ -contractions, which contains the p-cyclic contraction mappings as a subclass and he was also obtained the convergence and existence results for best proximity points.

We begin with some basic definitions and concepts related to the main results of p-cyclic  $\varphi$ -contraction maps.

**Definition 5.1 ([10]).** Let  $A_1, \dots, A_p$  be nonempty subsets of a metric space  $(X, d)$ . A p-cyclic mapping  $T$  on  $\bigcup_{i=1}^p A_i$  is called a contraction mapping if there exists  $k \in ]0, 1[$  such that  $d(A_i, A_{i+1}) \leq kd(x, y) + (1-k)d(A_i, A_{i+1})$  for all  $x \in A_i, y \in A_{i+1}, i = 1, \dots, p$ .

**Definition 5.2 ([3]).** Let  $(X, d)$  be a metric space and let  $A_1, \dots, A_p$  be nonempty subsets of  $X$ . A p-cyclic mapping  $T$  on  $\bigcup_{i=1}^p A_i$  is called a  $\varphi$ -contraction if there exists a strictly increasing function  $\varphi : [0, \infty) \rightarrow [0, \infty)$  such that  $d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + (d(A_i, A_{i+1}))$

**Theorem 5.3 (Vetro[46]).** Let  $A_1, A_2, \dots, A_p$  be nonempty subsets of a metric space  $(X, d)$  and let  $T : \bigcup_{i=1}^p A_i \rightarrow \bigcup_{i=1}^p A_i$  be a p-cyclic  $\varphi$ -contraction. Then  $d(T^n x, T^{n+1} y) \rightarrow d(A_i, A_{i+1})$  as  $n \rightarrow \infty$  for all  $x, y \in A_i$ .

**Theorem 5.4 (Vetro[46]).** Let  $A_1, A_2, \dots, A_p$  be nonempty subsets of a metric space  $(X, d)$  and let  $T : \bigcup_{i=1}^p A_i \rightarrow \bigcup_{i=1}^p A_i$  be a p-cyclic  $\varphi$ -contraction. If for  $x_0 \in A_i$ , the sequence  $\{T^{pn} x_0\}$  has a subsequence  $\{T^{p n_k} x_0\}$  convergent to a point  $x \in A_i$ , then  $d(x, Tx) = d(A_i, A_{i+1})$ .

## 6. Cyclic Contraction map in uniformly convex Banach space

We begin with the results of A.A.Eldred and P.Veeramani [10] which gives existence uniqueness and convergence for Best proximity points. The following convergence Lemma forms the basis for the main results of [10].

**Lemma 6.1 (Eldred, Veeramani [10]).** Let  $A$  be a nonempty closed and convex subset and  $B$  be a nonempty closed subset of a uniformly convex Banach space. Let  $\{x_n\}$  and  $\{z_n\}$  be sequences in  $A$  and  $\{y_n\}$  be a sequence in  $B$  satisfying :

(i)  $\|z_n - y_n\| \rightarrow \text{dist}(A, B)$ .

(ii) For every  $\varepsilon > 0$  there exists  $N_0$  such that for all  $m > n \geq N_0$ ,  $\|x_m - y_n\| \leq \text{dist}(A, B) + \varepsilon$ .

Then, for every  $\varepsilon > 0$  there exists  $N_1$  such that for all  $m > n \geq N_1$ ,  $\|x_m - z_n\| \leq \varepsilon$ .

**Lemma 6.2 (Eldred, Veeramani [10]).**

Let  $A$  be a nonempty closed and convex subset and  $B$  be a nonempty closed subset of a uniformly convex Banach space. Let  $\{x_n\}$  and  $\{z_n\}$  be sequences in  $A$  and  $\{y_n\}$  be a sequence in  $B$  satisfying :

(i)  $\|x_n - y_n\| \rightarrow \text{dist}(A, B)$ .

(ii)  $\|z_n - y_n\| \rightarrow \text{dist}(A, B)$ .

Then  $\|z_n - y_n\|$  converges to zero.

We have the main result of A.A.Eldred and P.Veeramani [10] as follows:

**Theorem 6.3 (Eldred, Veeramani [10]).** Let  $A$  and  $B$  be nonempty closed and convex subsets of a uniformly convex

Banach space. Suppose  $T : A \cup B \rightarrow A \cup B$  is a cyclic contraction map, then there exists a unique best proximity point  $x$  in  $A$  (that is with  $\|x - Tx\| = \text{dist}(A, B)$ ). Further, if  $x_0 \in A$  and  $x_{n+1} = Tx_n$ , then  $\{x_{2n}\}$  converges to the best proximity point.

Eldred and Veeramani [10] raised a question that, "whether a best proximity point exists when  $A$  and  $B$  are nonempty closed and convex subsets of a reflexive Banach space?" The positive answer of this question was given by Al.Thagafi and Shahzad [3].

## 7. Cyclic Meir Keeler Contractions in uniformly convex Banach space

In 2008, C.D.Bari, T.Suzuki and C.Vetro [7] generalized the theorem 6.3 which is given by Eldred and Veeramani [10] by introducing Cyclic Meir-Keeler contractions in the framework of uniformly convex Banach space. As we seen in chapter 1, the definition of cyclic Meir Keeler contraction being every cyclic contraction mapping is a cyclic Meir-Keeler contraction.

**Theorem 7.1 (Suzuki [7]).** Let  $X$  be a uniformly convex Banach space and let  $A$  and  $B$  be nonempty subsets of  $X$ . Suppose that  $A$  is closed and convex. Let  $f : A \cup B \rightarrow A \cup B$  be a cyclic Meir Keeler contraction. Then there exists a unique best proximity point in  $A$ . Further for each  $x \in A$ ,  $\{f^{2n}(x)\}$  converges to the best proximity point.

**Theorem 3.3.2 (Suzuki [7]).** Let  $X$  be a uniformly convex Banach space and let  $A$  and  $B$  be nonempty subsets of  $X$ . Suppose that  $A$  is closed and convex. Let  $f : A \cup B \rightarrow A \cup B$  be a cyclic mapping. Suppose that  $f^{2l+1}$  is a cyclic Meir – Keeler contraction for some  $l \in \mathbb{N}$ . Then there exists a best proximity point  $z$  in  $A$  such that  $\lim_n f^{2n}(x) = z$  holds for all  $x \in A$ . Moreover  $z$  is a unique fixed point of  $f^2$  in  $A$ .

## 8. Cyclic $\phi$ -contractions map in uniformly convex Banach space

In 2009, M.A.Al-Thagafi and N.Shahzad [3] give the existence and convergence results for best proximity point by introducing cyclic  $\varphi$ -contraction map in uniformly convex Banach space. Their results are given as follows.

**Lemma 3.4.1 (Thagafi, Shahzad[3]).** Let  $A$  and  $B$  be nonempty subsets of a uniformly convex Banach space  $X$  such that  $A$  is convex and let  $T : A \cup B \rightarrow A \cup B$  be a map with  $T(A) \subseteq B$  and  $T(B) \subseteq A$ . For  $x_0 \in A$ , define  $x_{n+1} = Tx_n$  for each  $n_0$ . Then  $\|x_{2n+1} - x_{2n}\| \rightarrow 0$  and  $\|x_{2n+3} - x_{2n+1}\| \rightarrow 0$  as  $n \rightarrow \infty$ .

**Theorem 3.4.2 (Thagafi, Shahzad[3]).** Let  $A$  and  $B$  be nonempty subsets of a uniformly convex Banach space  $X$  such that  $A$  is convex and let  $T : A \cup B \rightarrow A \cup B$  be a cyclic  $\varphi$ -contraction map. For  $x_0 \in A$ , define  $x_{n+1} = Tx_n$  for each  $n_0$ . Then for each  $\varepsilon > 0$ , there exists a positive integer  $N_0$  such that for all  $m > n \geq N_0$ ,  $\|x_{2m} - x_{2n+1}\| < d(A, B) + \varepsilon$ .

**Theorem 3.4.3 (Thagafi, Shahzad[3]).** Let  $A$  and  $B$  be nonempty subsets of a uniformly convex Banach space  $X$  such that  $A$  is convex and let  $T : A \cup B \rightarrow A \cup B$  be a cyclic  $\varphi$ -contraction map. For  $x_0 \in A$ , define  $x_{n+1} = Tx_n$  for each  $n_0$ . If  $d(A, B) = 0$ , then  $T$  has a unique fixed point  $x \in A \cap B$  and  $x_n \rightarrow x$  as  $n \rightarrow \infty$ .

**Theorem 3.4.4 (Thagafi, Shahzad[3]).** Let  $A$  and  $B$  be nonempty subsets of a uniformly convex Banach space  $X$  such that  $A$  is closed and convex and let  $T : A \cup B \rightarrow A \cup B$  be a cyclic  $\varphi$ -contraction map. For  $x_0 \in A$ , define  $x_{n+1} = Tx_n$  for each  $n_0$ . Then  $\{x_{2n}\}$  and  $\{x_{2n+1}\}$  are Cauchy sequences.

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