

# Some Results on Multidimensional Fractional Integral Operators

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**Abstract:** *In the present paper the certain multidimensional fractional integral operators involving a general class of polynomials for multidimensional Laplace transform are derived. The fractional integral operators studied here are general in nature, as by taking suitable values of the coefficients involved the general class of polynomials can be reduced to the classical orthogonal polynomials, the Bessel polynomials, and several other classes of generalized hypergeometric polynomials.*

**Keywords:** Fractional operators, general class of polynomial, multidimensional Laplace transform.

## 1. INTRODUCTION

The fractional integral operators play important role in several problems of mathematical physics and applied mathematics. Due to their importance various generalization of these operators are introduced and studied by several authors. A detailed study of these operators, their properties and applications was carried out by Miller and Ross [1], Samko et.al. [2], Debnath and Bhatta [3] and, Kilbas et.al. [4]. Ross [5] obtained the fractional integral formulae with the help of series expansion method. Banerji and Choudhary [6], Raina and Ladda [7], Raina [8] and, Miyakoda [9] have studied and defined fractional calculus operators using series expansion method. The use of two dimensional Laplace transform is common in the solutions of various types of partial differential equations, difference equations and in the study of useful properties of various special functions. Laplace and Fourier transforms in three and more variables has also inspired the development and applications of operational methods. This paper attempt to obtain certain fractional calculus formulae, involving multidimensional fractional integral operators and multidimensional Laplace Transform.

## 2. SOME DEFINITIONS

In this section definitions of the functions under consideration are given.

### 2.1 General Class of Polynomials

A general class of polynomials introduced by Srivastava [10] is

$$S_V^U [x] = \sum_{j=0}^{\lfloor V/U \rfloor} \frac{(-V)_{Uj} A_{V,j}}{j!} x^j \quad \dots(1.1)$$

where  $U$  is an arbitrary positive integer and the coefficients  $A_{V,j} (V, j \geq 0)$  are arbitrary constants, real or complex.

### 2.2 Multi-dimensional Fractional Integral Operators

Fractional integral operators defined in [11] are

$$\begin{aligned} & \mathbb{I}_{\mu_1, \dots, \mu_r}^{z, m, n; \rho, h} f(x_1, \dots, x_r) \\ &= \prod_{i=1}^r \frac{x_i^{-h_i - \mu_i}}{\Gamma(\mu_i)} \int_0^{x_1} \dots \int_0^{x_r} \prod_{i=1}^r (x_i - t_i)^{-\mu_i - 1} t_i^{h_i} \\ & \cdot S_n^m \left[ z \prod_{i=1}^r \left( 1 - \frac{t_i}{x_i} \right)^{\rho_i} \right] f(t_1 \dots t_r) dt_1 \dots dt_r \quad \dots(1.2) \end{aligned}$$

where

$$\operatorname{Re}(\mu_i + \rho_i j) > 0, (i = 1, \dots, r; j = 0, 1, 2, \dots, [n/m])$$

and

$$\begin{aligned} & \mathbb{K}_{\mu_1, \dots, \mu_r}^{z, m, n; \rho, h} f(x_1, \dots, x_r) \\ &= \prod_{i=1}^r \frac{x_i^{h_i}}{\Gamma(\mu_i)} \int_{x_r}^{\infty} \dots \int_{x_1}^{\infty} \prod_{i=1}^r (t_i - x_i)^{\mu_i - 1} t_i^{-h_i - \mu_i} \\ & \cdot S_n^m \left[ z \prod_{i=1}^r \left( 1 - \frac{x_i}{t_i} \right)^{\rho_i} \right] f(t_1, \dots, t_r) dt_1 \dots dt_r \quad \dots(1.3) \end{aligned}$$

where

$$\operatorname{Re}(\mu_i) > -\operatorname{Re}(\rho_i j), (i = 1, \dots, r; j = 0, 1, 2, \dots, [n/m])$$

The fractional integral operators defined by equations (1.2) and (1.3) can be considered as the generalizations of the repeated forms of the corresponding one dimensional operators defined by Erdelyi [12] and Kober [13].

### 2.3 Multi-dimensional Laplace Transform

The multidimensional Laplace transform of the function

$f(x_1, \dots, x_r)$  is defined as

$$(\mathcal{L}f)s = \mathcal{L} \{f(x_1, \dots, x_r); s_1, \dots, s_r\}$$

$$= \int_0^\infty \dots \int_0^\infty \exp\left(-\sum_{i=1}^r x_i s_i\right) f(x_1, \dots, x_r) dx_1 \dots dx_r$$

Where  $\Re(s_i) > 0, i = 1, \dots, r$

### 2.4 Mellin Transform

The multidimensional Mellin transform [14] of the function

$f(x_1, \dots, x_r)$  is defined as

$$(\mathcal{M}f)s = \int_0^\infty \dots \int_0^\infty x_1^{s_1-1} \dots x_r^{s_r-1} f(x_1, \dots, x_r) dx_1 \dots dx_r$$

Where  $x_1, \dots, x_r > 0$

## 3. MAIN RESULTS

### Theorem 1

Let  $f(t_1, \dots, t_r)$  be such that the multidimensional Laplace transform represented as

$$g(s_1, \dots, s_r) = \mathcal{L} \{f(t_1, \dots, t_r); s_1, \dots, s_r\}$$

exists then for  $\Re(s_i) > 0, i = 1, \dots, r$

$$I_{\mu_1, \dots, \mu_r}^{z; m, n; \rho, h} [g_1(s_1, \dots, s_r)]$$

$$= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{\Gamma(\mu_i + \rho_i k)}{\Gamma \mu_i}$$

$$\sum_{j=0}^{\infty} \frac{(-1)^j s_i^j}{j!} \frac{\Gamma(h_i + j + 1)}{\Gamma(h_i + j + \mu_i + \rho_i k + 1)} \mathcal{M}[f(j+1)]$$

**Proof:**

$$\text{Let } \Delta_1 = I_{\mu_1, \dots, \mu_r}^{z; m, n; \rho, h} [g_1(s_1, \dots, s_r)]$$

$$= s_1^{-h_1 - \mu_1} \dots s_r^{-h_r - \mu_r} s_1^{h_1} \dots s_r^{h_r}$$

$$\cdot \frac{1}{\Gamma \mu_1 \dots \Gamma \mu_r} \int_0^{s_1} \dots \int_0^{s_r} (s_1 - t_1)^{\mu_1 - 1} \dots (s_r - t_r)^{\mu_r - 1}$$

$$\cdot S_n^m \left[ z \left(1 - \frac{t_1}{s_1}\right)^{\rho_1} \dots \left(1 - \frac{t_r}{s_r}\right)^{\rho_r} \right] t_1^{h_1} \dots t_r^{h_r} g(t_1 \dots t_r) dt_1 \dots dt_r$$

$$= \prod_{i=1}^r \frac{s_i^{-h_i - \mu_i}}{\Gamma \mu_i} \int_0^{s_1} \dots \int_0^{s_r} \prod_{i=1}^r t_i^{h_i} (s_i - t_i)^{\mu_i - 1}$$

$$\cdot S_n^m \left[ z \prod_{i=1}^r \left(1 - \frac{t_i}{s_i}\right)^{\rho_i} \right] g_1(t_1, \dots, t_r) dt_1 \dots dt_r$$

$$= \prod_{i=1}^r \frac{s_i^{-h_i - \mu_i}}{\Gamma \mu_i} \int_0^{s_1} \dots \int_0^{s_r} \prod_{i=1}^r t_i^{h_i} (s_i - t_i)^{\mu_i - 1}$$

$$\cdot \left[ \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k \prod_{i=1}^r \left(1 - \frac{t_i}{s_i}\right)^{\rho_i k}}{k!} \right]$$

$$\cdot \left[ \int_0^\infty \dots \int_0^\infty \exp\left(-\sum_{i=1}^r x_i t_i\right) f(x_1, \dots, x_r) dx_1 \dots dx_r \right] \cdot dt_1 \dots dt_r$$

where  $\Re(x_i) > 0$

$$\Delta_1 = \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{s_i^{-h_i - \mu_i - \rho_i k}}{\Gamma \mu_i} \int_0^\infty \dots \int_0^\infty f(x_1, \dots, x_r)$$

$$\cdot \left[ \int_0^{s_1} \dots \int_0^{s_r} \prod_{i=1}^r t_i^{h_i} (s_i - t_i)^{\mu_i + \rho_i k - 1} e^{-x_i t_i} dt_1 \dots dt_r \right] dx_1 \dots dx_r$$

$$= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{s_i^{-h_i - \mu_i - \rho_i k}}{\Gamma \mu_i} \left[ \int_0^\infty \dots \int_0^\infty f(x_1, \dots, x_r)$$

$$\cdot \sum_{j=0}^{\infty} \frac{(-x_i t_i)^j}{j!} dt_1 \dots dt_r \right] dx_1 \dots dx_r$$

$$= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{s_i^{-h_i - \mu_i - \rho_i k}}{\Gamma \mu_i} \left[ \int_0^\infty \dots \int_0^\infty f(x_1, \dots, x_r)$$

$$\prod_{i=1}^r \sum_{j=0}^{\infty} \frac{(-x_i)^j}{j!} \int_0^{s_i} \dots \int_0^{s_r} t_i^{h_i + j} (s_i - t_i)^{\mu_i + \rho_i k - 1} dt_1 \dots dt_r \right] dx_1 \dots dx_r$$

$$\text{Using } \int_0^u x^{\nu-1} (u-x)^{\mu-1} dx = u^{\mu+\nu-1} B(\nu, \mu)$$

$$\Delta_1 = \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{s_i^{-h_i - \mu_i - \rho_i k}}{\Gamma \mu_i}$$

$$[15] \left[ \int_0^\infty \dots \int_0^\infty f(x_1, \dots, x_r) \sum_{j=0}^{\infty} \frac{(-x_i)^j}{j!} s_i^{h_i + j + \mu_i + \rho_i k} \right]$$

$$\cdot B(h_i + j + 1, \mu_i + \rho_i k)$$

$$= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{1}{\Gamma(\mu_i)} \sum_{j=0}^{\infty} \frac{(-1)^j s_i^j}{j!}$$

$$\frac{\Gamma(h_i + j + 1) \Gamma(\mu_i + \rho_i k)}{\Gamma(h_i + j + \mu_i + \rho_i k + 1)} \int_0^\infty \dots \int_0^\infty x_i^j f(x_1, \dots, x_r) dx_1 \dots dx_r$$

$$= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{\Gamma(\mu_i + \rho_i k)}{\Gamma \mu_i} \sum_{j=0}^{\infty} \frac{(-1)^j s_i^j}{j!} \frac{\Gamma(h_i + j + 1)}{\Gamma(h_i + j + \mu_i + \rho_i k + 1)} \mathfrak{m}[f : (j+1)]$$

**Theorem 2**

Let  $f(t_1, \dots, t_r)$  be such that the multidimensional Laplace transform represented as

$$g(s_1, \dots, s_r) = \mathfrak{L} \{ f(t_1, \dots, t_r); s_1, \dots, s_r \} \text{ exists then for } \Re(s_i) > 0, i = 1, \dots, r$$

$$\mathcal{K}_{\mu_1, \dots, \mu_r}^{z; m, n; \rho, h} g(s_1, \dots, s_r)$$

$$= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{\Gamma(\mu_i + \rho_i k)}{\Gamma(\mu_i)} \sum_{j=0}^{\infty} \frac{(-1)^j s_i^j}{j!} \frac{\Gamma(h_i - j)}{\Gamma(h_i - j + \mu_i + \rho_i k)} \mathfrak{m}[f : (j+1)]$$

**Proof:**

$$\text{Let } \Delta_2 = \mathcal{K}_{\mu_1, \dots, \mu_r}^{z; m, n; \rho, h} g(s_1, \dots, s_r)$$

$$= \prod_{i=1}^r \frac{s_i^{h_i}}{\Gamma \mu_i} \int_{s_i}^{\infty} \dots \int_{s_r}^{\infty} \prod_{i=1}^r t_i^{-h_i - \mu_i} (t_i - s_i)^{\mu_i - 1}$$

$$\cdot \left[ \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} (-n)_{mk} A_{n,k} \frac{z^k \prod_{i=1}^r \left(1 - \frac{s_i}{t_i}\right)^{\rho_i k}}{k!} \right] \cdot \left[ \int_0^{\infty} \dots \int_0^{\infty} \exp\left(-\sum_{i=1}^r x_i t_i\right) f(x_1, \dots, x_r) dx_1 \dots dx_r \right] dt_1 \dots dt_r$$

$$= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{s_i^{h_i}}{\Gamma \mu_i} \int_0^{\infty} \dots \int_0^{\infty} f(x_1, \dots, x_r) \cdot \left[ \int_{s_1}^{\infty} \dots \int_{s_r}^{\infty} \prod_{i=1}^r t_i^{-h_i - \mu_i - \rho_i k} (t_i - s_i)^{\mu_i + \rho_i k - 1} e^{-x_i t_i} dt_1 \dots dt_r \right] dx_1 \dots dx_r$$

$$= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{s_i^{h_i}}{\Gamma \mu_i} \int_0^{\infty} \dots \int_0^{\infty} f(x_1, \dots, x_r) \cdot \left[ \int_{s_1}^{\infty} \dots \int_{s_r}^{\infty} \prod_{i=1}^r t_i^{-h_i - \mu_i - \rho_i k} (t_i - s_i)^{\mu_i + \rho_i k - 1} \sum_{j=0}^{\infty} \frac{(-x_i t_i)^j}{j!} dt_1 \dots dt_r \right] dx_1 \dots dx_r$$

$$= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{s_i^{h_i}}{\Gamma \mu_i} \left[ \int_0^{\infty} \dots \int_0^{\infty} f(x_1, \dots, x_r) \sum_{j=0}^{\infty} \frac{(-x_i)^j}{j!} \int_{s_1}^{\infty} \dots \int_{s_r}^{\infty} \prod_{i=1}^r t_i^{j - h_i - \mu_i - \rho_i k} (t_i - s_i)^{\mu_i + \rho_i k - 1} dt_1 \dots dt_r \right] dx_1 \dots dx_r$$

where  $\text{Re}(h_i + \mu_i + \rho_i k - j) > \text{Re}(\mu_i + \rho_i k) > 0$

$$\text{Using } \int_u^{\infty} x^{-\nu} (x-u)^{\mu-1} dx = u^{\mu-\nu} B(\nu-\mu, \mu) \text{ [15]}$$

$$\Delta_2 = \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{s_i^{h_i}}{\Gamma \mu_i} \left[ \int_0^{\infty} \dots \int_0^{\infty} f(x_1, \dots, x_r) \sum_{j=0}^{\infty} \frac{(-x_i)^j}{j!} s_i^{h_i} B(h_i - j, \mu_i + \rho_i k), \right]$$

$$= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{1}{\Gamma \mu_i} \sum_{j=0}^{\infty} \frac{(-1)^j s_i^j}{j!} \frac{\Gamma(h_i - j) \Gamma(\mu_i + \rho_i k)}{\Gamma(h_i - j + \mu_i + \rho_i k)} \int_0^{\infty} \dots \int_0^{\infty} x_i^j f(x_1, \dots, x_r) dx_1 \dots dx_r$$

$$= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{\Gamma(\mu_i + \rho_i k)}{\Gamma(\mu_i)} \sum_{j=0}^{\infty} \frac{(-1)^j s_i^j}{j!} \frac{\Gamma(h_i - j)}{\Gamma(h_i - j + \mu_i + \rho_i k)} \mathfrak{m}[f : (j+1)]$$

**4. CONCLUSION**

In this paper two theorems on certain multidimensional fractional integral operators involving a general class of polynomials for multidimensional Laplace transform are derived. The fractional integral operators studied here are general in nature, since by taking suitable values of the coefficients involved the general class of polynomials can be reduced to the classical orthogonal polynomials, the Bessel polynomials, and several other classes of generalized hypergeometric polynomials. These results can be further extended by special choices of multidimensional transforms. The one and two-dimensional analogues of Theorems 1 and 2 can be deduced easily. The results derived in this section are general in nature and will find useful applications in the theory of fractional calculus and integral transform.

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