The Relationship between Portfolio Size and Risk Based on Investor's Utility

Xinliang Xie[1], Meihua Wang[2]

[1,2] School of Economics and Management, XIDIAN University, 266 Xinglong Section of Xifeng Road, Xi'an, Shaanxi 710126, China

Abstract: ***In actual investment activities, rational investors always allocate their funds to different assets in order to diversify their investment. From the perspective of investor's utility, this paper studies the relationship between portfolio size and risk under different risk aversion levels by repeating stochastic simulations on the components of CSI 300 index in Chinese stock market. Empirical studies show that: Compared with the equal-weighted method, the idea based on the maximization of the investor's utility function can diversify most of the unsystematic risk with fewer assets. The lower limit of the optimal portfolio risk of the risk seeker is higher, and is lower for the risk averter.***

Keywords: investment diversification, portfolio, risk, investor's utility

1. Introduction

Diversified investment is an investment strategy to effectively reduce the unsystematic risk of the portfolio, but excessive diversification will increase investment costs. Therefore, in actual investment activities, rational investors always hope to hold a portfolio of the best size in order to reduce the investment risk and ensure the return as much as possible. This problem is essentially an analysis of the quantitative relationship between portfolio size and risk.

In the existing literatures, some scholars have conducted research on this problem. Evans and Archer (1968) used the equal-weighted method to construct 60 equal-weighted portfolios of 1 to 40 securities from 470 stocks on the 1967 of NYSE. They found that the portfolio size of 8 to 10 was sufficient to spread most of the unsystematic risk when variance was used to represent risk. Fisher and Lorie (1970), taking the stocks of the NYSE as the research object too, found that 16 stocks can disperse most of the unsystematic risk. Campbell et al. (2001) found that, studying the stocks of AMEX, the unsystematic risk which was reduced by 20 randomly selected stocks between 1963 and 1985 required at least 50 stocks between 1986 and 1997. In 1952, Markowitz (1952) put forward the famous M-V model, which transformed the portfolio selection problem into a mathematical programming problem. In actual investment activities, a rational investor will allocate the investment funds according to the M-V model instead of simply and mechanically distributing the funds equally among each stock. Statman (2002) found that the number of stocks needed to reach the optimal level of diversification is more than 120 when using M-V model. In recent years, some scholars have also studied the diversification of investment. Wang et al. (2015) studied the relationship between loan portfolio size and risk diversification for commercial bank. Koumou (2016) revisited the risk reduction and diversification within Markowitz's M-V model.

In the previous research on the relationship between portfolio size and risk, scholars mostly use equal-weighted method or single-objective M-V model (risk minimization under return constraint or return maximization under risk constraint) and draw different conclusions. However, in actual investment activities, the risk aversion level of investors is not consistent, but the optimal portfolio based on M-V model is undifferentiated for all investors, which is obviously unreasonable. Then, this paper, on the basis of existing studies, takes the components of CSI 300 index in Chinese as the research object, and from the perspective of investors' utility function, repeats the simulation of different portfolio sizes respectively to find out the average risk which is faced by investors under the size when they reach the maximum utility. The rest of this paper is organized as follows. In the next section, we introduce the data and methodology used in this paper. In Section 3 we report the empirical results of the equal-weighted method and the utility function maximization model and present the empirical comparison of these two methods. In Section 4 we summarize the conclusions of this paper.

2. Data and Methodology

In this section, we introduce the research methods, including equal-weighted method and utility function maximization model, and the data set used in this paper. We assume that investors are rational and risk averse.

2.1 Data

The data set selected in this paper is the components of CSI 300 index of Chinese stock market. We obtained the weekly return data from the RESSET database which spans from 1/1/2016 to 31/12/2019. The data set finally contains 210 stocks after filtering the stocks with more missing return data.

2.2 Equal-weighted method

Suppose we have N assets to be managed. Then the risk of a portfolio consisting of any K assets can be expressed as:

$$\sigma_K^2 = \frac{1}{K^2}\sum_{i=1}^{K}\sum_{j=1}^{K}\sigma_{ij} \qquad (2\text{-}1)$$

In this paper, for different size K, we selected the portfolio containing K stocks 100 times randomly, and calculated the variance of each selected portfolio. Finally, we calculated the average variance of the 100 random portfolios, as the risk of the portfolio when the size was *K*.

2.3 Utility function maximization model

Many scholars have done a lot of work on the theory of utility function, but there are various forms of utility function, and there is no consensus on which utility function is the most appropriate. Therefore, this paper adopts the quadratic utility function, which is widely used in existing literatures, and the form is as follows:

$$U = r - 0.5A\sigma^2 \quad (2\text{-}2)$$

Where U is the investor's utility; r is the return of the portfolio; σ^2 is the variance of the portfolio; A denotes the level of risk aversion of the investor. And $A > 0$, which denotes that the investor is risk averse. In order to express convenience, we use $b = 0.5A$ to represent the risk aversion coefficient. So:

$$U = r - b\sigma^2 \quad (2\text{-}3)$$

Then the utility function maximization model can be expressed as follows:

$$\begin{aligned} &\max U = r - b\sigma^2 = R'\omega - b\omega'\Sigma\omega \\ &s.t. \quad I'\omega = 1 \quad (2\text{-}4) \\ &(\omega_i \geq 0, i = 1, 2, \ldots, K) \end{aligned}$$

Where $R = [R_1, R_2, \ldots, R_K]'$ is the expected return vector composed of the expected return of K assets. R_i is the average return over a period of time of asset *i*. Σ is the covariance matrix between the assets. $I = [1,1,\ldots,1]'$ is the *K*-dimension column vector. $\omega = [\omega_1, \omega_2, \ldots, \omega_n]'$ denotes the weight vector. $\omega_i \geq 0$ denotes that there are no short positions. In this paper, three different risk aversion coefficient investors are studied ($b = 2、4、8$), which represent investors with low risk aversion level, medium risk aversion level and high risk aversion level respectively. Similarly, we conducted 100 random simulations for different portfolio sizes and calculated the mean variance of the portfolio as the risk of the portfolio under the corresponding size. In this paper, we used Adaptive Genetic Algorithm (AGA) to solve the problem.

3. Empirical result

In this section, we present the empirical results of this paper. We demonstrate the change of the portfolio risk with size when using equal-weighted method and utility function maximization model, and analyze the advantages of utility function maximization model through comparison with the equal-weighted method.

3.1 Result of the Equal-weighted method

Table 1, by using the equal-weight method, shows the quantitative relationship between the degree of risk reduction of the portfolio relative to $K = 1$ and the change of the portfolio size. Column "risk" refers to the mean variance of 100 random portfolios; column "degree of reduction (%)" refers to the degree of risk reduction when $K = 1$ as the benchmark.

Table 1: Relationship between portfolio size and risk (equal-weight method)

Size	*Risk*	*Degree of reduction (%)*	*Size*	*Risk*	*Degree of reduction (%)*
1	0.002416	0.00	51	0.000774	67.98
2	0.001456	39.71	52	0.000779	67.77
3	0.001326	45.12	53	0.000780	67.72
4	0.001157	52.11	54	0.000786	67.46
5	0.001037	57.08	55	0.000769	68.16
6	0.000983	59.31	56	0.000770	68.11
7	0.000948	60.75	57	0.000769	68.15
8	0.000951	60.64	58	0.000771	68.09
9	0.000927	61.62	59	0.000771	68.10
10	0.000882	63.49	60	0.000765	68.35
11	0.000884	63.39	61	0.000768	68.19
12	0.000888	63.22	62	0.000765	68.35
13	0.000875	63.76	63	0.000773	68.01
14	0.000846	64.96	64	0.000772	68.04
15	0.000838	65.31	65	0.000775	67.92
16	0.000840	65.23	66	0.000772	68.04
17	0.000824	65.89	67	0.000770	68.14
18	0.000825	65.83	68	0.000761	68.50
19	0.000831	65.61	69	0.000764	68.39
20	0.000824	65.90	70	0.000764	68.37
21	0.000821	66.01	71	0.000770	68.12
22	0.000829	65.69	72	0.000768	68.23
23	0.000824	65.88	73	0.000769	68.18
24	0.000812	66.37	74	0.000767	68.25
25	0.000803	66.77	75	0.000764	68.37
26	0.000800	66.90	76	0.000768	68.21
27	0.000822	65.97	77	0.000769	68.16
28	0.000809	66.49	78	0.000768	68.22
29	0.000797	67.01	79	0.000766	68.28
30	0.000784	67.53	80	0.000767	68.25
31	0.000795	67.08	81	0.000758	68.63
32	0.000795	67.09	82	0.000772	68.05
33	0.000793	67.17	83	0.000771	68.10
34	0.000775	67.91	84	0.000764	68.39
35	0.000793	67.18	85	0.000770	68.13
36	0.000782	67.65	86	0.000763	68.43
37	0.000790	67.31	87	0.000762	68.47
38	0.000793	67.18	88	0.000760	68.53
39	0.000783	67.58	89	0.000763	68.43
40	0.000779	67.75	90	0.000761	68.49
41	0.000783	67.59	91	0.000766	68.29
42	0.000780	67.69	92	0.000762	68.44
43	0.000779	67.75	93	0.000764	68.38
44	0.000772	68.05	94	0.000766	68.27
45	0.000791	67.24	95	0.000759	68.56
46	0.000779	67.77	96	0.000767	68.25
47	0.000774	67.97	97	0.000763	68.40
48	0.000781	67.66	98	0.000766	68.28
49	0.000776	67.86	99	0.000757	68.67
50	0.000790	67.31	100	0.000751	68.92

It can be seen that the risk of the portfolio decreases with the increase of the size of the portfolio, which means that the larger the portfolio size is, the more diversified the investment strategy is, and the less risk the investor will face. The risk decreased by 63.49% when the size of the portfolio was expanded from 1 to 10, while when the size of the portfolio continued to expand from 10 stocks to 20 stocks, the risk only decreased by 2.41%. If the number of stocks in the portfolio continues to increase, the risk still decreases, but the room for decline becomes smaller; when the portfolio size was increased from 20 to 30, the risk was reduced by only 1.63%. The risk that can be reduced through diversification is unsystematic risk, while the residual risk that has not been dispersed is systematic risk. Therefore, we can roughly infer that the systematic risk accounts for about 32%~33% in Chinese stock market. This is lower than the results of previous studies by scholars. According to the time span of the data sets of other studies, the proportion of systematic risk is constantly decreasing, indicating that Chinese stock market is gradually moving towards standardization and marketization.

3.2 Result of the Utility function maximization model

Table 2 is the quantitative relationship between the degree of risk reduction of the portfolio relative to $K=1$ and the change of the portfolio size based on the utility function maximization model.

Table 2: Relationship between portfolio size and risk (utility function maximization model)

Size	Risk (*Degree of reduction (%)*) ($b=2$)	Risk (*Degree of reduction (%)*) ($b=4$)	Risk (*Degree of reduction (%)*) ($b=8$)
1	0.002416(0.00)	0.002416(0.00)	0.002416(0.00)
2	0.001636(32.27)	0.001523(36.94)	0.001423(41.11)
3	0.001458(39.64)	0.001222(49.40)	0.001151(52.37)
4	0.001436(40.54)	0.001105(54.27)	0.000994(58.85)
5	0.001394(42.28)	0.001084(55.13)	0.000886(63.30)
6	0.001392(42.36)	0.001006(58.37)	0.000784(67.53)
7	0.001355(43.90)	0.000995(58.83)	0.000803(66.75)
8	0.001339(44.57)	0.001008(58.27)	0.000781(67.65)
9	0.001289(46.62)	0.000908(62.41)	0.000756(68.70)
10	0.001308(45.84)	0.000922(61.84)	0.000700(71.02)
11	0.001228(49.18)	0.000922(61.84)	0.000684(71.67)
12	0.001223(49.38)	0.000904(62.58)	0.000672(72.20)
13	0.001374(43.10)	0.000921(61.89)	0.000706(70.76)
14	0.001251(48.22)	0.000904(62.57)	0.000666(72.44)
15	0.001302(46.08)	0.000935(61.29)	0.000695(71.22)
16	0.001320(45.36)	0.000941(61.04)	0.000636(73.67)
17	0.001250(48.24)	0.000887(63.30)	0.000634(73.76)
18	0.001256(47.99)	0.000929(61.55)	0.000639(73.57)
19	0.001309(45.82)	0.000898(62.80)	0.000655(72.88)
20	0.001295(46.38)	0.000877(63.70)	0.000645(73.28)

It can be seen that the risk is no longer significantly reduced when the investor diversifies assets to about 10 stocks. Of course, for investors with different levels of risk aversion, the degree of risk reduction is also different. For investors who are more willing to pursue risk ($b=2$), when they allocate assets to 10 assets or more, the risk that can be reduced is about 46%, at this time the portfolio risk is about 0.0013; for investors with moderate risk aversion ($b=4$), when they allocate assets to 10 assets or more, the risk that can be reduced is about 62%, and the portfolio risk is about 0.0009; for investors who are more willing to avoid risks ($b=8$), when they allocate assets to 10 assets or more, the risk that can be reduced is about 71%, and the portfolio risk is about 0.0007.

3.3 Comparison

Figure 1 shows the trend of the risk with the increasing of the portfolio size when the equal-weighted method and the utility function maximization model are used.

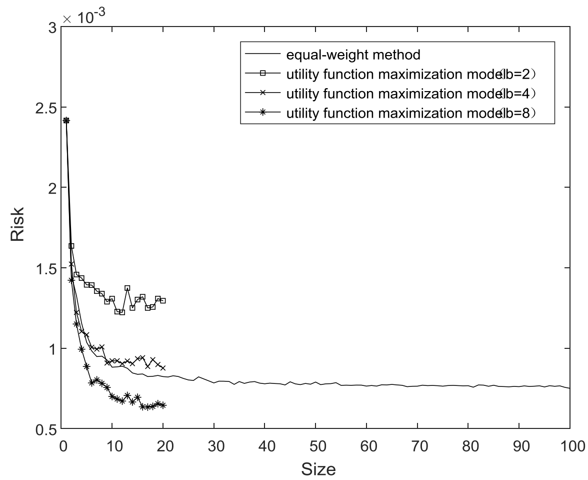

Figure 1: Comparison

By comparison, we can find that the utility function maximization model is more efficient in diversifying the portfolio risk than the equal-weighted method. When investors use the utility function maximization model to allocate their investment funds, they can reduce the portfolio risk to the maximum extent by spreading their funds among 10 stocks, while when using the equal-weight method, it takes 20-30 stocks to reduce the portfolio risk to the maximum extent. In addition, by comparing the performance of the utility function maximization model under different risk aversion coefficients, it can be found that the degree of risk reduction in the utility function maximization model depends on the risk preference of investors. The more investors tend to pursue risks, the greater the lower limit of portfolio risk they can realize. The more investors tend to avoid risk, the lower the risk limit of their portfolio will be.

4. Conclusion

Based on the investor's utility function maximization model, this paper studies the relationship between portfolio size and risk when constructing a portfolio on the data set of the components of CSI 300 index of Chinese stock market. Different from previous studies, we consider investors' risk preference and establish a portfolio selection model aiming at maximizing investors' utility function. Through empirical analysis, we draw the following conclusions:

(1) Diversification can indeed effectively reduce the risk of the investment portfolio. A portfolio containing 20-30 stocks can disperse most of the unsystematic risk. However, with the increase of portfolio size, the degree of risk reduction is limited, so the systematic risk cannot be eliminated by diversification.

(2) Compared with the equity-weighted method, the investor's utility function maximization model can reduce the

portfolio risk more effectively. Compared with the 20-30 stocks in the equity-weighted method, the utility function maximization model only needs 10 stocks to reduce the portfolio risk to the lowest level acceptable to investors.

(3) When investors with different risk aversion levels use the utility function maximization model to allocate their investment funds, the lower limits of portfolio risk that can be reduced are different. The lower limit of the optimal portfolio risk of the risk seeker is higher, and is lower for the risk averter.

References

[1] J. L. Evans, S. H. Archer, "Diversification and the reduction of dispersion: an empirical analysis", Journal of Finance, 23(5):761-767, 1968.
[2] L. Fisher, J. H. Lorie, "Some studies of variability of returns on investments in common stocks", The Journal of Business, 43(2), 99~134, 1970.
[3] J. Y. Campbell, M. Lettau, B. G. Malkiel, Y. Xu, "Have individual stocks become more volatile? An empirical exploration of idiosyncratic risk", The Journal of Finance, 56(1), 1~43, 2001.
[4] H. M. Markowitz, "Portfolio selection", The Journal of Finance, 7（1）:77, 1952.
[5] M. Statman, "How Much Diversification is Enough?", Social ence Electronic Publishing, 2002(Jan):18-28, 2003.
[6] Y. Zhang, S. Zhou, B.S. Shi, "The Relationship between Loan Portfolio Size and Risk Diversification for Commercial Bank", International Research Association of Information and Computer Science, Proceedings of 2015 International Conference on Social Science, 2015.
[7] G. B. Koumou, "Risk reduction and Diversification within Markowitz's Mean-Variance Model: Theoretical Revisit", Social ence Electronic Publishing, 2016.