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New Modified Adomian Decomposition Method for Solving Second-Order Boundary Value Problems with Neumann Conditions

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Abstract: ***In this paper, a modification of the Adomian Decomposition Method called New Modified Adomian Decomposition Method is proposed. The proposed method is based on the Modified Adomian Decomposition Method and the inverse linear operator theorem. Some examples are presented to show the ability of the method for linear and nonlinear second- order boundary value problems with Neumann conditions.***

Keywords: Neumann boundary conditions, New Modified Adomian Decomposition Method, Boundary Value Problems

1. Introduction

A Boundary Value Problem (BVP) is a system of ordinary differential equations with solutions and derivative value specified at more than one point. Most commonly at just two points (boundaries) defining a two-point boundary value problem. BVPs arise in science and engineering and most of them do not have exact solutions. As a result semi-analytical and numerical methods are used to solve BVPs. The analysis of literature shows that not much is done on solutions of second-order two-point BVPs with Neumann boundary conditions. [1] Solved linear second order two-point boundary prob- lems with Neumann boundary conditions using Quadratic Spline, Cubic Polynomials and Nonpolynomial Spline. [2] used Polynomial Spline approach to solve linear and nonlin- ear second-order two-point boundary problems with Neumann boundary conditions. [3] and [4] solved second order BVPs with Neumann conditions using Advanced Adomian Decomposition Method (AADM).

In this paper, we solve second-order BVPs using a new method called New Modified Adomian Decompositon Method (NMADM). It is based on the traditional Adomian De- composition Method (ADM), the Modified Adomian Decomposition Method (MADM) and the inverse linear operator theorem found in [3].

2. Analysis of ADM, MADM and NMADM

Consider the general form of a two-point nonlinear second-order BVP:

$$y^{tt}(x) + h(x,y) = g(x), \qquad a \le x \le b, \qquad (1)$$

with Neumann boundary conditions

$$y^{t}(a) = \delta_1$$
$$y^{t}(b) = \delta_2, \qquad (2)$$

where $h(x,y)$ is a linear or nonlinear function of y and $g(x)$ are continuous functions defined in the interval $x \in [a,b]$ and δ_1, δ_2 are real constants. In operator form [5 − 8], equation (1) can be written as:

$$Ly = g(x) - Ny, \qquad (3)$$

where $L = \frac{d^2}{dx^2}$ is the linear operator and $Ny = h(x, y)$ is the nonlinear operator.

The ADM expresses the solution of (1) into infinite series as follows:

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \qquad (4)$$

and the nonlinear term $h(x, y)$ is decomposed into:

$$Ny = \sum_{n=0}^{\infty} A_n \qquad (5)$$

where the A'_{n^s} are called Adomian polynomials and are calculated by the formular,

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i y^i\right)\right]_{\lambda=0}, n = 0,1,2,\cdots \qquad (6)$$

2.1 ADM

Applying the inverse operator $L^{-1}[\cdot] = \int_2^x . \int_0^2 [.]dxdx$ on both sides equation (3) together with equation (2), we obtain the following:

$$y(x) = k_1 + k_2 x + L^{-1}g(x) - L^{-1}Ny \qquad (7)$$

where k_1 and k_2 are constants of integration. Substituting equations (4) and (5) into equation (7) yields,

$$\sum_{n=0}^{\infty} y_n(x) = k_1 + k_2 x + L^{-1}g(x) - L^{-1}\left[\sum_{n=0}^{\infty} A_n\right] \qquad (8)$$

From equation (8), we can write the following recursive:

$y_0 = k_1 + k_2 x + L^{-1}g(x)$

$y_{n+1} = -L^{-1}A_n$, $n = 0, 1, 2, \ldots.$

Thus, the n-term approximation of the solution is given by

$\emptyset_n(x) = \sum_{i=0}^{n-1} yi(x)$

with $y(x) = \lim_{x \to \infty} \emptyset_n(x)$

2.2 MADM

Rewriting equation (8) by inserting the following expression [9 − 11],

$$L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] - pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right]$$

We get the following:

$$\sum_{n=0}^{\infty} y_n(x) = k_1 + k_2 x + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] - pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] + L^{-1}g(x) - L^{-1}\left[\sum_{n=0}^{\infty} A_n\right], \quad (10)$$

where p is an artificial parameter and a_n', $n = 0,1,2,...,$ are unknown coefficients. From equation (10), we obtain the following recursive:

$$y_0 = k_1 + k_2 x + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right],$$

$$y_1 = -pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] + L^{-1}g(x) - L^{-1}A_0,$$

$$y_{n+1} = -L^{-1}A_n, \; n = 1,2,\cdots.$$

To avoid the calculations of A_n, $n = 1,2,\cdots$, we determine a_n, for $n = 1,2,\cdots$ such that $\mu_1 = 0$. Consequently, $\mu_2 = u_3$ =…0. We set $p = 1$ and find the solution of equation (1) as:

$$y(x) = k_1 + k_2 x + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right]$$

2.3 NMADM

Operating on the left hand side of equation (3) with the inverse operator

$L_{xx}^{-1}[\cdot] = \int_\Omega^x ds \int_a^s [\cdot]dt + \frac{1}{\Omega}\int_0^\Omega ds \left(s \int_b^s [\cdot]dt\right)$ [4]

and introducing $L^{-1}[.]$, we obtain the following equation

$$y(x) = (x-\Omega)y'(a) + \frac{\Omega}{2}y'(b) + \frac{1}{\Omega}\int_0^\Omega y(x)dx + L^{-1}g(x) - L^{-1}Ny. \quad (11)$$

Substituting equations (4) and (5) and inserting equation (9) in equation (11), we obtain the following:

$$\sum_{n=0}^{\infty} y_n(x) = (x-\Omega)y'(a) + \frac{\Omega}{2}y'(b) + \frac{1}{\Omega}\int_0^\Omega y(x)dx + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] - pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] + L^{-1}g(x) - L^{-1}\left[\sum_{n=0}^{\infty} A_n\right] \quad (12)$$

From equation (12), the iterations are then determined in the following recursive way:

$$y_0 = (x-\Omega)y'(a) + \frac{\Omega}{2}y'(b) + \frac{1}{\Omega}\int_0^\Omega y(x)dx + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right]$$

$$y_1 = -pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] + L^{-1}g(x) - L^{-1}A_0$$

$$y_{n+1} = -L^{-1}A_n, \; n = 1,2,\cdots.$$

Thus calculating the coefficients a_n, for $n = 0,1,2,\cdots$ and setting $p = 1$ as $\Omega \to 0$, we write the solution to equation (1) as,

$$y(x) = x\delta_{-1} + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] \quad (13)$$

3. Numerical Illustrations

Example 1. [4]

Consider the following linear second-order two-point BVP

$$y'' + y = -1, \quad 0 \le x \le 1 \quad (14)$$

$$y'(0) = \frac{1-\cos(1)}{\sin(1)} = y'(1)$$

In operator form, equation (14) can be written as,

$$L(y'') = L(-y-1) \quad (15)$$

By using equation (12), we have the following:

$$\sum_{n=0}^{\infty} y_n(x) = (x-\Omega)y'(a) + \frac{\Omega}{2}y'(b) + \frac{1}{\Omega}\int_0^\Omega y(x)dx + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] - pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] + L^{-1}1 - L^{-1}\left[\sum_{n=0}^{\infty} y_n\right] \quad (16)$$

Thus,

$$y_0 = (x-\Omega)u'(a) + \frac{\Omega}{2}u'(b) + \frac{1}{\Omega}\int_0^\Omega u(x)dx + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right]),$$

$$= \left(\frac{1-\cos(1)}{\sin(1)}\right)x + \frac{a_0 x^2}{2} + \frac{a_1 x^3}{6} + \frac{a_2 x^4}{12} + \frac{a_3 x^5}{20} + \frac{a_4 x^6}{30} + \frac{a_5 x^7}{42} + \cdots.$$

And

$$y_1 = -pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] - L^{-1}(1) - L^{-1}(y_0),$$
$$= -\frac{x^2}{2} - \left(\frac{1-\cos(1)}{6\sin(1)}\right)x^3 - \frac{a_0x^4}{24} - \frac{a_1x^5}{120} - \frac{a_2x^6}{360} - \frac{a_3x^7}{840} - \frac{a_4x^8}{1680} - \frac{a_5x^9}{30240} - \cdots - \frac{a_0px^2}{2}$$
$$- \frac{a_1px^3}{6} - \frac{a_2px^4}{12} - \frac{a_3px^4}{20} - \frac{a_4px^6}{30} - \frac{a_5px^7}{42}.$$

For $y_1 = 0$ and $p = 1$, it can be shown that:

$a_0 = -1,\quad a_1 = \frac{\cos(1)-1}{\sin(1)},\quad a_2 = \frac{1}{2},$

$a_3 = \frac{1-\cos(1)}{6\sin(1)},\quad a_4 = \frac{-1}{24},\quad a_5 = \frac{\cos(1)-1}{120\sin(1)}$

Using equation (13), the solution to equation (14) is therefore given by,

$$y(x) = \frac{x - x\cos(1)}{\sin(1)} + \frac{x^3\cos(1)-x^3}{6\sin(1)} + \frac{x^5 - x^5\cos(1)}{120\sin(1)} + \frac{x^7\cos(1)-x^7}{5040\sin(1)} - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + 1 - 1,$$
$$= \left(\frac{1-\cos(1)}{\sin(1)}\right)\sin x + \cos x - 1$$

Example 2. [4]

Consider the following linear second-order two-point BVP,

$$y'' + xy = (3 - x - x^2 + x^3)\sin(x) + 4x\cos(x), \quad 0 \le x \le 1 \tag{17}$$
$$y'(0) = -1, \quad u'(1) = 2\sin(1).$$

In operator form, equation (17) can be written as,

$$L(y'') = (3 - x - x^2 + x^3)\sin(x) + 4x\cos(x) - xy \tag{18}$$

Using equation (12) we write equation (18) as follows:

$$\sum_{n=0}^{\infty} y_n(x) = (x-\Omega)y'(a) + \frac{\Omega}{2}y'(b) + \frac{1}{\Omega}\int_0^{\Omega} y(x)dx + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right]$$
$$- pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] + L^{-1}\left[(3 - x - x^2 + x^3)\sin(x) + 4x\cos(x)\right] \tag{19}$$
$$- L^{-1}\left[\sum_{n=0}^{\infty} xA_n\right]$$

Thus,

$$y_0 = (x-\Omega)(-1) + \frac{\Omega}{2}(2\sin(1)) + \frac{1}{\Omega}\int_0^{\Omega} u(x)dx + \int_0^x\int_0^x (a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3$$
$$+ a_4x^4 + a_5x^5 + \cdots)dxdx,$$
$$= -x + \frac{a_0x^2}{2} + \frac{a_1x^3}{6} + \frac{a_2x^4}{12} + \frac{a_3x^5}{20} + \frac{a_4x^6}{30} + \frac{a_5x^7}{42} + \cdots.$$

And

$$y_1 = -pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] - L^{-1}\left[(3 - x - x^2 + x^3)\sin x + 4x\ cosx\right] - L^{-1}(xA_0),$$
$$= -p\int_0^x\int_0^x (a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots)dxdx$$
$$- \int_0^x\int_0^x [(3 - x - x^2 + x^3)\sin x + 4x\cos x]dxdx - \int_0^x\int_0^x (xy_0)dxdx,$$

Using the Taylor series of cos x and sin x of order 10 we find y_1 as follows:

$$y_1 = -\frac{a_1px^3}{6} - \frac{a_2px^4}{12} - \cdots + \frac{7}{6}x^3 - \frac{1}{12}x^4 - \frac{7}{40}x^5 - \cdots + \frac{1}{12}x^4 - \frac{1}{40}a_0x^5 + \cdots.$$

For $y_1 = 0$ and $p = 1$, it can be shown that:

$a_0 = -0,\ a_1 = 7,\ a_2 = 0,\ a_3 = -\frac{2}{7},\ a_4 = 0,\ a_5 = \frac{43}{120}$

Using equation (13), the solution to equation (17) is therefore given by:

$$y = -x + \int_0^x\int_0^x \left(7x - \frac{7}{2}x^2 + \frac{43}{120}x^5 + \cdots\right)dxdx$$
$$= -x + \frac{7}{6}x^3 - \frac{7}{40}x^5 + \frac{43}{720}x^7 + \cdots$$
$$= -x + \frac{7}{3!}x^3 - \frac{21}{5!}x^5 + \frac{43}{7!}x^7 + \cdots$$

Example 3. [4]

Consider the following nonlinear second-order two-point BVP,

$$y'' - y^2 = 2\pi^2\cos(2\pi x) - \sin^4(\pi x),\ 0 \le x \le 1$$
$$y'(0) = 0 = y'(1).$$

We rewrite equation (20) as follows:

$$y'' = 2\pi^2\cos(2\pi x) - \sin^4(\pi x) + y^2$$

Using NMADM we find y_0 and y_1 as follows:

$$y_0 = (x-\Omega)(0) + \frac{\Omega}{2}(0) + \frac{1}{\Omega}\int_0^{\Omega} u(x)dx + \int_0^x \int_0^x (a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3$$

$$+a_4x^4 + a_5x^5 + \cdots)dxdx,$$

$$= \frac{a_0x^2}{2} + \frac{a_1x^3}{6} + \frac{a_2x^4}{12} + \frac{a_3x^5}{20} + \frac{a_4x^6}{30} + \frac{a_5x^7}{42} + \cdots .$$

$$y_1 = -pL^{-1}\left[\sum_{n=0}^{\infty} a_nx^n\right] - L_{xx}^{-1}\left[2\pi^2\, cos(2\pi x) - \sin^4(x\pi)\right] + L^{-1}[A_0(y^2)],$$

$$= -p\int_0^x \int_0^x [a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots]\, dxdx$$

$$+ \int_0^x \int_0^x \left[2\pi^2\, cos(2\pi x) - \sin^4(x\pi)\right] dxdx - \int_0^x \int_0^x y_0^2 dxdx,$$

Using the Taylor series of $\cos(2\pi x)$ and $\sin^4(\pi x)$ of order 10 we find y_1 as follows:

$$y_1 = -\frac{a_1px^3}{6} - \frac{a_2px^4}{12} - \cdots + \pi^2x^2 - \frac{1}{3}\pi^4x^4 - \frac{1}{30}\pi^4x^6 + \cdots + \frac{1}{120}a_0^2x^6 + \cdots .$$

For $y_1 = 0$ and $p = 1$, it can be shown that:

$a_0 = 2\pi^2,\ a_1 = 0,\ a_2 = -4\pi^4,\ a_3 = 0,\ a_4 = \frac{4}{3}\pi^6,\ a_5 = 0$

Using equation (13), the solution to equation (20) is therefore given by:

$$y = \int_0^x \int_0^x \left(2\pi^2 - 4\pi^4x^2 + \frac{4}{3}\pi^6x^6 + \cdots\right) dxdx$$

$$= \pi^2x^2 - \frac{1}{3}\pi^4x^4 + \frac{2}{45}\pi^6x^6 + \cdots$$

$$= (\pi x)^2 - \frac{1}{3}(\pi x)^4 + \frac{2}{45}(\pi x)^4 + \cdots$$

Example 4. [4]

Consider the following nonlinear second two-point BVP:

$$y'' + e^{-2y} = 0, \quad 0 \le x \le 1$$

$$y'(0) = 1, \quad u'(1) = \frac{1}{2}.$$

We rewrite equation (21) as follows:

$$y'' = -e^{-2y}, \quad 0 \le x \le 1$$

Solution by NMADM

$$y_0 = (x-\Omega)(1) + \frac{\Omega}{2}\left(\frac{1}{2}\right) + \frac{1}{\Omega}\int_0^{\Omega} u(x)dx + \int_0^x \int_0^x (a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3$$

$$+a_4x^4 + a_5x^5 + \cdots)dxdx,$$

$$= x + \frac{a_0x^2}{2} + \frac{a_1x^3}{6} + \frac{a_2x^4}{12} + \frac{a_3x^5}{20} + \frac{a_4x^6}{30} + \frac{a_5x^7}{42} + \cdots .$$

$$y_1 = -p\int_0^x \int_0^x [a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots]\, dxdx - \int_0^x \int_0^x e^{-2y_0} dxdx.$$

Using the Taylor series of e^{y0} of order 10, we evaluate y_1 as follows:

$$y_1 = -\frac{a_1px^3}{6} - \frac{a_2px^4}{12} - \frac{a_3px^4}{20} - \cdots - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{12}a_0x^4 + \cdots$$

For $y_1 = 0$ and $p = 1$, it can be shown that

$a_0 = -1,\ a_1 = 2,\ a_2 = -3,\ a_3 = 4,\ a_4 = -5,\ a_5 = 6.$

Using equation (13), the solution to equation (21) is therefore given by:

$$y = x + \int_0^x \int_0^x (-1 + 2x - 3x^2 + 4x^3 - 5x^4 + 6x^5 + \cdots)\, dxdx$$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \frac{1}{7}x^7 + \cdots .$$

4. Conclusion

The modified Adomian decomposition method has been modified by incorporating the inverse linear operator theorem. The resulting method is the new Adomian decomposi- tion method. The new method has been proved to be effective and efficient in solving second-order boundary value problems with Neumann conditions. It is demonstrated through four examples that the method has the ability of solving second-order BVPs with Neumann conditions. The results obtained confirm with the results obtained in [4] where a different method was applied. It is important to note that unlike in traditional ADM, NMADM requires the calculation of u_0 and u_1 only. Further, for nonlinear terms, it requires A_0 only. It is also observed that using NMADM, the solutions are obtained in closed form.

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