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The Exponentiated Marshall-Olkin Exponential Distribution

M. M. E. Abd El-Monsef[1], M. O. Alshrani[2]

[1, 2]Faculty of Science, Tanta University

Abstract: ***A new four-parameter model called the Exponentiated Marshall-Olkin Exponential distribution is examined. Several of its mathematical properties including ordinary moments, quantile, generating functions and order statistics are investigated. The maximum likelihood method is used to estimate the model parameters. An application with the remission times of a random sample of bladder cancer patients is given to illustrate the proposed model of lifetime distribution.***

Keywords: Exponentiated Marshall-Olkin family of distributions, Maximum likelihood, Moments

1. Introduction

Many different ways of generating new distributions from classic ones were developed in the last few years. Marshall and Olkin (1997) introduced a method of adding a new parameter to an existing distribution. The resulting new distribution, known as the Marshall–Olkin extended distribution, includes the original distribution as a special case and gives more flexibility to model various types of data. Eugene et al. (2002) specified a class of beta-generated distribution. A family of distributions that arises naturally from the distribution of the order statistics was studied via Jones (2004),and he introduced general properties of the proposed class of distributions. Zografos and Balakrishnan (2009) projected the gamma-generated family of distributions. Subsequently, Cordeiro and de Castro (2011) defined the Kumaraswamy family. Latterly, Alzaatreh et al. (2013) proposed a new technique to derive wider families via using any probability density function (pdf) as a generator.

In this article, we define and study a new four-parameter model called the Exponentiated Marshall-Olkin Exponential (EMOEx) distribution and provide some of its properties. We prove, via an application, that the EMOEx distribution can give better fits than many other distributions.

The new model is generated by applying the exponentiated Marshal-Olkin-G (EMO-G) family (Dias et al., 2016) to the Exponential distribution. Diaset al. (2016) studied general mathematical properties of a new class of continuous distributions with three extra shape parameters called the exponentiated Marshal-Olkin family of distributions. This generator has cumulative distribution function (cdf) defined by:

$$F(x)=\left\{\frac{1-\left[1-G(x;\xi)\right]^{\lambda}}{1-p\left[1-G(x;\xi)\right]^{\lambda}}\right\}^{\alpha}, \qquad (1)$$

where G(x; ξ) is the baseline cdf depending on a parameter vector ξ and α > 0, λ >0 and p < 1 are three additional shape parameters. For each baseline G, the exponentiated

Marshall-Olkin-G (“EMO-G” for short) distribution is defined by the cdf (1).

The density function corresponding to (1) is given by:

$$f(x)=\alpha\lambda(1-p)g(x;\xi)\left[1-G(x;\xi)\right]^{\lambda-1}\frac{\left\{1-\left[1-G(x;\xi)\right]^{\lambda}\right\}^{\alpha-1}}{\left\{1-p\left[1-G(x;\xi)\right]^{\lambda}\right\}^{\alpha+1}}, \qquad (2)$$

where g(x; ξ) is the baseline pdf. This density function will be most tractable when the functions G(x) and g(x) have simple analytic expressions.

The contents of this paper are organized as follows. In Section 2, we define the EMOEx distribution. Shape and some plots for its pdf and hazard rate function (hrf) are displayed in Section 3. A comprehensive account of mathematical properties of the new distribution, include linear representation, the quantile function, the moments, the moment generating function and the order statistics are discussed and provided in Section 4.InSection 5, we demonstrate the maximum likelihood estimates (MLEs) of the unknown parameters and the asymptotic confidence intervals of the unknown parameters. An application of the EMOEx model is presented in Section 6. Finally, Section 7 concludes this paper.

2. The EMOEx Distribution

In this paper, we introduce a special case of the EMO-G family which is the EMOEx model. This case is defined by taking G(x) and g(x) to be the cdf and pdf of the Exponential distribution.

The cdf of the Exponential distribution is given by:

$$G(x,\theta)=1-e^{-x\theta} \qquad ,x>0 \qquad (3)$$

The corresponding pdf is given by

$$g(x,\theta)=\theta e^{-x\theta} \qquad ,x>0 \qquad (4)$$

The new model includes four parameters, referred to as the (EMOEx) distribution, with hope that it will use with many applications in different disciplines, as survival analysis,

reliability, biology and others. The fourth parameter indexed to this distribution makes it more flexible to identify different types of real data than other models.
The new distribution has the cumulative distribution function which is defined by:

$$F(x)=\left\{\frac{1-\left[1-\left(1-e^{-\theta x}\right)\right]^{\lambda}}{1-p\left[1-\left(1-e^{-\theta x}\right)\right]^{\lambda}}\right\}^{\alpha}, \qquad (5)$$

and the corresponding probability density function (pdf) is given by:

$$f(x)=\alpha\lambda(1-p)\left(\theta e^{-\theta x}\right)\left[1-\left(1-e^{-\theta x}\right)\right]^{\lambda-1}\frac{\left\{1-\left[1-\left(1-e^{-\theta x}\right)\right]^{\lambda}\right\}^{\alpha-1}}{\left\{1-p\left[1-\left(1-e^{-\theta x}\right)\right]^{\lambda}\right\}^{\alpha+1}},$$

which it could be simplified into

$$f(x)=\alpha\theta\lambda(1-p)e^{-\theta\lambda x}\frac{\left\{1-e^{-\theta\lambda x}\right\}^{\alpha-1}}{\left\{1-pe^{-\theta\lambda x}\right\}^{\alpha+1}}, \qquad (6)$$

where $\alpha > 0$, $\lambda > 0$ and $p \in (-\infty, 1)$ are shape parameters and $\theta > 0$ is a scale parameter. Hence, we denote by $X \sim$ EMOEx $(\theta, \alpha, \lambda, p)$ a random variable having pdf (6).
The corresponding hazard rate function is:

$$h(x)=\frac{\alpha\theta\lambda(1-p)\mathrm{e}^{-\theta\lambda x}\left[1-\mathrm{e}^{-\theta\lambda x}\right]^{\alpha-1}}{\left[1-p\left(\mathrm{e}^{-\theta\lambda x}\right)\right]^{\alpha}-\left[1-\mathrm{e}^{-\theta\lambda x}\right]^{\alpha}\left[1-p\left(\mathrm{e}^{-\theta\lambda x}\right)\right]}, \qquad (7)$$

3. Shape

In the pdf of new distribution, the parameters $\lambda\ \alpha$ *and* p control the shape distribution, whereas θ controls the scale of the distribution. Figure 1 shows various shapes of the EMOEx pdf for different choices of parameters.

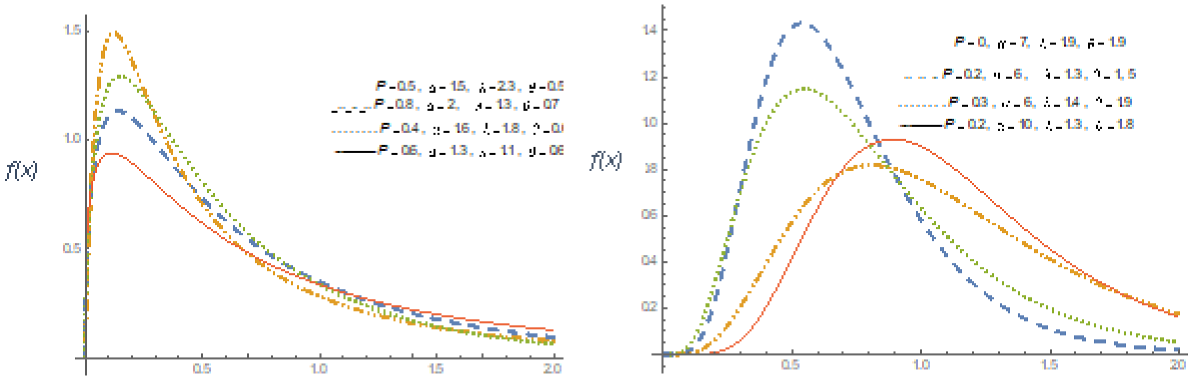

Figure 1: Plots of the EMOEx pdf for varying parameter values

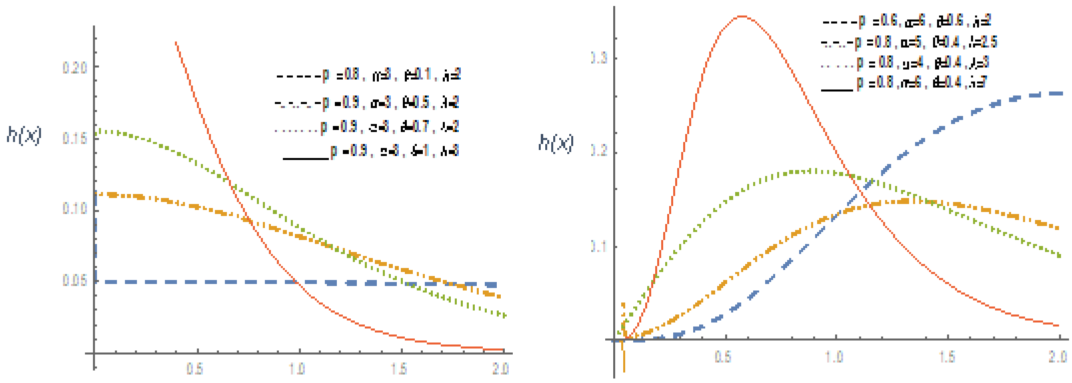

Figure 2: Plots of the EMOEx h(x) for varying parameter values

Figure 2. shows various shapes of the EMOEx h(x) for different choices of parameters.

The critical points of the EMOEx density function are the roots of the equation:

$$\frac{d\ \ln\left[f\left(x\right)\right]}{dx}=\frac{\left\{-1+p\left(\mathrm{e}^{-x\theta}\right)^{\lambda}\left[\left(\mathrm{e}^{-x\theta}\right)^{\lambda}-\alpha\right]+\left(\mathrm{e}^{-x\theta}\right)^{\lambda}\alpha\right\}\theta\lambda}{\left[-1+\left(\mathrm{e}^{-x\theta}\right)^{\lambda}\right]\left[-1+p\left(\mathrm{e}^{-x\theta}\right)^{\lambda}\right]}=0\,, \tag{8}$$

when we solve last equation, there is two solutions which is:

$$x=\frac{\ln\left[2^{\frac{1}{\lambda}}\left(\frac{-\alpha+p\alpha\pm\sqrt{4p+\left(1-p\right)^{2}\alpha^{2}}}{p}\right)^{-1/\lambda}\right]}{\theta}$$

There may be more than one root to (8). Let$\lambda(x)=\frac{d^2\log[f(x)]}{dx^2}$. we have:

$$\lambda\left(x\right)=\frac{d^{2}\ln\left[f\left(x\right)\right]}{dx^{2}}$$

$$=\frac{\left(\mathrm{e}^{-x\theta}\right)^{\lambda}p\left(-1+\left(\mathrm{e}^{-x\theta}\right)^{\lambda}p\left(\left(\mathrm{e}^{-x\theta}\right)^{\lambda}-\alpha\right)+\left(\mathrm{e}^{-x\theta}\right)^{\lambda}\alpha\right)\theta^{2}\lambda^{2}}{\left(-1+\left(\mathrm{e}^{-x\theta}\right)^{\lambda}\right)\left(-1+\left(\mathrm{e}^{-x\theta}\right)^{\lambda}p\right)^{2}}$$

$$+\frac{\left(\mathrm{e}^{-x\theta}\right)^{\lambda}\left(-1+\left(\mathrm{e}^{-x\theta}\right)^{\lambda}p\left(\left(\mathrm{e}^{-x\theta}\right)^{\lambda}-\alpha\right)+\left(\mathrm{e}^{-x\theta}\right)^{\lambda}\alpha\right)\theta^{2}\lambda^{2}}{\left(-1+\left(\mathrm{e}^{-x\theta}\right)^{\lambda}\right)^{2}\left(-1+\left(\mathrm{e}^{-x\theta}\right)^{\lambda}p\right)}$$

$$+\frac{\theta\lambda\left(-\left(\mathrm{e}^{-x\theta}\right)^{2\lambda}p\theta\lambda-\left(\mathrm{e}^{-x\theta}\right)^{\lambda}p\left(\left(\mathrm{e}^{-x\theta}\right)^{\lambda}-\alpha\right)\theta\lambda-\left(\mathrm{e}^{-x\theta}\right)^{\lambda}\alpha\theta\lambda\right)}{\left(-1+\left(\mathrm{e}^{-x\theta}\right)^{\lambda}\right)\left(-1+\left(\mathrm{e}^{-x\theta}\right)^{\lambda}p\right)}$$

The model calculations based on first and second derivatives shows the mode of the density at $x=x_0$ for the EMOEx model has local maximum if $\lambda(x)>0$for all $x<x_0$ and if$\lambda(x)<0$for all$x>x_0$. It corresponds to a local minimum if $\lambda(x)<0$ for all $x<x_0$ and $\lambda(x)>0$ for all $x>x_0$. It gives a point of inflexion if either $\lambda(x)>0$ for all $x\neq x_0$ or $\lambda(x)<0$ for all $x\neq x_0$.

The critical points of the (hrf) of X are obtained from the equation:

$$\frac{d\ \ln\left[h\left(x\right)\right]}{dx}=0$$

There may be more than one root to last equation. Le$\tau(x)=\frac{d^2\log[h(x)]}{dx^2}$.If$x=x_0$is aroot of this equation then it refers to a local maximum if$\tau(x)>0$ for all $x<x_0$ and if $\tau(x)<0$ for all $x>x_0$. It corresponds to a local minimum if $\tau(x)<0$ for all $x<x_0$and $\tau(x)>0$ for all $x>x_0$. It gives a point of inflexion if either $\tau(x)>0$ for all $x\neq x_0$ or $\tau(x)<0$ for all $x\neq x_0$.

4. Mathematical Properties

In this section, we derive some mathematical properties of the new distribution, include linear representation, the quantile function, the moments, and order statistics.

4.1Linear Representation

Here, we originate a suitable linear mixture representation for the cdf and pdf of the EMOEx distribution. The cdf of the EMOEx in (5) can be conveyed as

$$F\left(x\right)=\left\{1-\left[1-\left(1-e^{-\theta x}\right)\right]^{\lambda}\right\}^{\alpha}\left\{1-p\left[1-\left(1-e^{-\theta x}\right)\right]^{\lambda}\right\}^{-\alpha}$$

Applying the binomial expansion defined by

$$\left(1-Z\right)^{-\varphi}=\sum_{i=0}^{\infty}\left(-1\right)^{i}\binom{-\varphi}{i}Z^{i}\,, \tag{9}$$

Now, the cdf of the EMOEx reduces to

$$F\left(x\right)=\sum_{i=0}^{\infty}\left(-1\right)^{i}P^{i}\binom{-\alpha}{i}\left[1-\left(1-e^{-\theta x}\right)\right]^{\lambda i}\left\{1-\left[1-\left(1-e^{-\theta x}\right)\right]^{\lambda}\right\}^{\alpha} \tag{10}$$

Using the generalized binomial expansion defined by

$$\left(1-Z\right)^{\varphi}=\sum_{j=0}^{\infty}\left(-1\right)^{j}\binom{\varphi}{j}Z^{j}\,, \tag{11}$$

Therefore, we can write

$$\left\{1-\left[1-\left(1-e^{-\theta x}\right)\right]^{\lambda}\right\}^{\alpha}=\sum_{j=0}^{\infty}\left(-1\right)^{j}\binom{\alpha}{j}\left[1-\left(1-e^{-\theta x}\right)\right]^{\lambda j},$$

Then, equation (10) reduces to

$$F\left(x\right)=\sum_{i,j=0}^{\infty}\left(-1\right)^{i+j}P^{i}\binom{-\alpha}{i}\binom{\alpha}{j}\left[1-\left(1-e^{-\theta x}\right)\right]^{\lambda\left(i+j\right)} \tag{12}$$

Using (11), we have

$$F(x)=\sum_{i,j,k=0}^{\infty}(-1)^{i+j+k}p^{i}\binom{-\alpha}{i}\binom{\alpha}{j}\binom{\lambda(i+j)}{k}\left[1-e^{-\theta x}\right]^{k} \quad (13)$$

Then, the cdf of the EMOEx reduces to

$$F(x)=\sum_{k=0}^{\infty}b_k G_k(x,\theta)^k, \quad (14)$$

where

$$b_k=\sum_{i,j,k=0}^{\infty}(-1)^{i+j+k}P^{i}\binom{-\alpha}{i}\binom{\alpha}{j}\binom{\lambda(i+j)}{k}$$

and $G_k(x,\theta)^k$ denotes the exponentiated-G ("exp-G") cdf with power parameter k.

The density function of X can be expressed as an infinite linear mixture of exp-G density functions

$$f(x)=\sum_{k=0}^{\infty}b_{k+1}(k+1)g(x,\theta)G_k(x,\theta)^k, \quad (15)$$

where (for $k\geq 0$) $(k+1)g(x,\theta)G_k(x,\theta)^k$ denotes the density function of the random variable $Y_{k+1}\sim$ exp-G(k + 1). Equation (15) discloses that the EMOEx density function is a linear mixture of exp-G density functions. Thus, some of its mathematical properties can be derived directly from those properties of the exp-G distribution. Some structural properties of the exp-G distributions are definite by Mudholkar and Hutson (1996), Gupta and Kundu (2001) and Nadarajah and Kotz (2006), among others.

4.2 Quantile Function

To find the quantile function, we need to solve the equation $F(u)=0$, $0<u<1$. The quantile function of the EMOEx can be written as in the following theorem.

Theorem 1. The quantile function of the EMOEx is given by

$$Q(u)=-\frac{1}{\theta\lambda}ln\left(\frac{1-u^{\frac{1}{\alpha}}}{1-pu^{\frac{1}{\alpha}}}\right), \qquad 0<u<1 \quad (16)$$

where $\theta>0, \lambda>0, \alpha>0$

Proof. The CDF given in eq (5) can be written as

$$F_{Q(u)}=\left[\frac{1-e^{-\theta\lambda Q(u)}}{1-pe^{-\theta\lambda Q(u)}}\right]^{\alpha}=u$$

$$\frac{1-e^{-\theta\lambda Q(u)}}{1-pe^{-\theta\lambda Q(u)}}=u^{\frac{1}{\alpha}}$$

Let

$$Z(u)=-\theta\lambda Q(u), \quad (17)$$

Then eq.(16) becomes

$$1-u^{\frac{1}{\alpha}}=e^{Z(u)}\left[1-u^{\frac{1}{\alpha}}p\right]$$

Then

$$Z(u)=ln\left[\frac{1-u^{\frac{1}{\alpha}}}{1-u^{\frac{1}{\alpha}}p}\right]$$

Therefore, we can write eq. (16) as

$$-\theta\lambda Q(u)=ln\left[\frac{1-u^{\frac{1}{\alpha}}}{1-u^{\frac{1}{\alpha}}p}\right]$$

Hence,

$$Q(u)=-\frac{1}{\theta\lambda}ln\left(\frac{1-u^{\frac{1}{\alpha}}}{1-pu^{\frac{1}{\alpha}}}\right)$$

Corollary 1. The three quartiles of the EMOEx are given by

$$Q_1=-\frac{1}{\theta\lambda}ln\left(\frac{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}{1-p\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}\right); \text{ Median}$$

$$Q_2=-\frac{1}{\theta\lambda}ln\left(\frac{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}{1-p\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}\right) \quad \text{and}$$

$$Q_3=-\frac{1}{\theta\lambda}ln\left(\frac{1-\left(\frac{3}{4}\right)^{\frac{1}{\alpha}}}{1-p\left(\frac{3}{4}\right)^{\frac{1}{\alpha}}}\right)$$

Proof. The proof comes directly by putting u = 0.25, 0.5, and 0.75 in the quantile function derived in Theorem 1.

4.3 Order Statistics

Let $X_1 \dots, X_n$ be a random sample of size $n$from the EMOEx distribution and $X_{(1)} \dots, X_{(n)}$ be the consistent order statistics. Then, the pdf of the ith order statistic $Xi{:}n$, say $fi{:}(x)$, is given by

$$f_{i:n}=\frac{f(x)}{B(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^{j}\binom{n-1}{j}F(x)^{i+j-1}, \quad (18)$$

Using equations (5) and (6), we can write

$$f(x)F(x)^{i+j-1}=\alpha\lambda(1-p)\left(\theta e^{-\theta x}\right)\left[1-\left(1-e^{-\theta x}\right)\right]^{\lambda-1}$$

$$\times\left\{1-\left[1-\left(1-e^{-\theta x}\right)\right]^{\lambda}\right\}^{\alpha(i+j)-1}\left\{1-p\left[1-\left(1-e^{-\theta x}\right)\right]^{\lambda}\right\}^{-\alpha(i+j)-1}$$

Applying the expansions (9) and (11), we have

$$\left\{1-p\left[1-\left(1-e^{-\theta x}\right)\right]^{\lambda}\right\}^{-\alpha(i+j)-1}=\sum_{r=0}^{\infty}(-1)^{r}(p)^{r}\binom{-\alpha(i+j)-1}{r}\left[1-\left(1-e^{-\theta x}\right)\right]^{\lambda r}$$

and

$$\left\{1-\left[1-\left(1-e^{-\theta x}\right)\right]^{\lambda}\right\}^{\alpha(i+j)-1}=\sum_{m=0}^{\infty}(-1)^m\binom{\alpha(i+j)-1}{m}\left[1-\left(1-e^{-\theta x}\right)\right]^{\lambda m}$$

after some simplifications, we have

$$f(x)F(x)^{i+j-1}=\alpha\lambda(1-p)\left(\theta e^{-\theta x}\right)\sum_{r,m,k=0}^{\infty}(-1)^{r+m+k}p^r\binom{-\alpha(i+j)-1}{r}\binom{\alpha(i+j)-1}{m}\binom{\lambda(r+m+k+1)-1}{k}\left[1-e^{-\theta x}\right]^k$$

when we insert the last equation in equation (18), we achieve

$$f_{i:n}(x)=\sum_{k=0}^{\infty}\phi_k\, g(x,\theta)G_k(x,\theta)^k, \qquad (19)$$

Where $g(x,\theta)$denotes pdf of the Exponential distribution, $G_k(x,\theta)^k$ denotes the exponentiated-G ("exp-G") cdf with power parameter k, and

$$\phi_k=\sum_{r,m=0}^{\infty}\sum_{j=0}^{n-i}\frac{(-1)^{r+m+k}p^r(1-p)}{B(i,n-i+1)}\binom{n-i}{j}\binom{-\alpha(i+j)-1}{r}\binom{\alpha(i+j)-1}{m}\binom{\lambda(r+m+k+1)-1}{k}$$

4.4Moments and the Moment Generating Function

Theorem 2. The rth moments E(X^r) of a EMOEx random variable X, is given by

$$E\left(X^r\right)=(1-p)\sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^j\sum_{k=0}^{\infty}\binom{-\alpha-1}{k}p^k(-1)^k\frac{\alpha\Gamma(r+1)}{\lambda^r\theta^r(r+1+j)^{r+1}}, \qquad (20)$$

Proof

$$E\left(X^r\right)=\int_0^{\infty}X^rf(x)dx,$$

$$E\left(X^r\right)=\int_0^{\infty}X^re^{-\theta\lambda x}\left(1-e^{-\theta\lambda x}\right)^{\alpha-1}(1-p)\left(1-e^{-\theta\lambda x}p\right)^{-\alpha-1}\theta\lambda\alpha\, dx$$

$$=\theta\lambda\alpha(1-p)\int_0^{\infty}X^re^{-\theta\lambda x}\left(1-e^{-\theta\lambda x}\right)^{\alpha-1}\left(1-e^{-\theta\lambda x}p\right)^{-\alpha-1}dx, \qquad (21)$$

Using the following binomial series expansion of$(1-e-\theta\lambda x)\alpha-1$given by

$$\left(1-e^{-\theta\lambda x}\right)^{\alpha-1}=\sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^j\left(e^{-\theta\lambda x}\right)^j$$

using the following binomial series expansion of$(1-e^{-\theta\lambda x}\ p)^{-\alpha-1}$given by

$$\left(1-e^{-\theta\lambda x}p\right)^{-\alpha-1}=\sum_{k=0}^{\infty}\binom{-\alpha-1}{k}\left(e^{-\theta\lambda x}\right)^k(-1)^k(p)^k$$

Eq. (21) takes the following form

$$E(X^r)=\alpha\theta\lambda(1-p)\int_0^{\infty}X^re^{-\theta\lambda x}\sum_{j=0}^{\infty}\binom{\alpha-1}{j}\left(e^{-\theta\lambda x}\right)^j(-1)^j$$

$$.\sum_{k=0}^{\infty}\binom{-\alpha-1}{k}\left(e^{-\theta\lambda x}\right)^k(-1)^k(p)^k\,dx$$

$$=\alpha\theta\lambda(1-p)\sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^j\sum_{k=0}^{\infty}\binom{-\alpha-1}{k}(-1)^k(p)^k$$

$$.\int_0^{\infty}X^re^{-\theta\lambda x}e^{-\theta\lambda jx}e^{-\theta\lambda kx}dx$$

$$=\alpha\theta\lambda(1-p)\sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^{j}\sum_{k=0}^{\infty}\binom{-\alpha-1}{k}(-1)^{k}(p)^{k}.\int_{0}^{\infty}X^{r}e^{-\theta\lambda x(k+1+j)}dx$$

using integrate by parts we get

$$E(X^{r})=(1-p)\sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^{j}\sum_{k=0}^{\infty}\binom{-\alpha-1}{k}p^{k}(-1)^{k}\frac{\alpha\Gamma(r+1)}{\lambda^{r}\theta^{r}(r+1+j)^{r+1}}$$

Theorem 3. Let X have an EMOEx distribution. Then the moment generating function of X, $M_x(t)$, is:

$$M_{x}(t)=\left\{\alpha(1-p)\sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^{j}\sum_{k=0}^{\infty}\binom{-\alpha-1}{k}(-1)^{k}(p)^{k}\right\}{}_1\psi_0\left[\begin{matrix}(1,1)\\ -\end{matrix},t/\theta\right] \quad (22)$$

where${}_1\psi_0$is the Wright-generalised hypergeometric function.

Proof

$$M_{x}(t)=\int_{0}^{\infty}e^{tx}f(x)dx,$$

$$M_{x}(t)=\int_{0}^{\infty}e^{tx}e^{-\theta\lambda x}\left(1-e^{-\theta\lambda x}\right)^{\alpha-1}(1-p)\left(1-e^{-\theta\lambda x}p\right)^{-\alpha-1}\theta\lambda\alpha dx \quad (23)$$

using the binomial expansion equation (23) reduces to

$$M_{x}(t)=\alpha\theta\lambda(1-p)\sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^{j}\sum_{k=0}^{\infty}\binom{-\alpha-1}{k}(-1)^{k}(p)^{k}\times\int_{0}^{\infty}e^{tx-(k+1+j)\theta\lambda x}dx$$

using Taylor series expansions the above integral reduces to

$$M_{x}(t)=\alpha\theta\lambda(1-p)\sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^{j}\sum_{k=0}^{\infty}\binom{-\alpha-1}{k}(-1)^{k}(p)^{k}\times\sum_{m=0}^{\infty}\frac{t^{m}}{m!}\int_{0}^{\infty}x^{m}e^{-(k+1+j)\theta\lambda x}dx \quad (24)$$

The integral form of $M_x(t)$in equation (24) can then be obtained by using the Wright-generalized hypergeometric function as

$${}_1\psi_0\left[\begin{matrix}(\beta_1,B_1),\ldots,(\beta_p,B_p)\\ (\delta_1,D_1),\ldots,(\delta_q,D_q)\end{matrix},x\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\beta_j+B_jn)}{\prod_{j=1}^{q}(\delta_j+D_jn)}\frac{x^{n}}{n!}$$

Finally, we have

$$M_{x}(t)=\alpha(1-p)\sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^{j}\sum_{k=0}^{\infty}\binom{-\alpha-1}{k}(-1)^{k}(p)^{k}\sum_{m=0}^{\infty}\frac{\left(\frac{t}{\theta}\right)^{m}}{m!}\Gamma(m+1), \quad (25)$$

Then equation (25) yields the following representation of the Wright-generalised hypergeometric function

$$M_{x}(t)=\left\{\alpha(1-p)\sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^{j}\sum_{k=0}^{\infty}\binom{-\alpha-1}{k}(-1)^{k}(p)^{k}\right\}{}_1\psi_0\left[\begin{matrix}(1,1)\\ -\end{matrix},t/\theta\right]$$

5. Stress-strength Analysis

There are applications (every physical component possess characteristic strength) which survive due to their strength. These applications receive an assured level of stress and sustain. But if a higher level of stress is applied then their strength is unable to sustain and they collapse. Suppose Y represents the 'stress' which is applied to a certain appliance and X represents the 'strength' to sustain the stress, then the stress-strength reliability is denoted by R= P(Y<X), if X,Y are assumed to be random.

Suppose X and Y are random variables independently distributed as $X\sim EMOEx(\theta_1,\alpha_1,\lambda_1,p_1)$ and $Y\sim EMOEx(\theta_2,\alpha_2,\lambda_2,p_2)$with pdf's

$$f(x)=\alpha_1\lambda_1(1-p_1)\left(\theta_1e^{-\theta_1x}\right)[1-\left(1-e^{-\theta_1x}\right]^{\lambda_1-1}\frac{\left\{1-[1-\left(1-e^{-\theta_1x}\right]^{\lambda_1}\right\}^{\alpha_1-1}}{\left\{1-p_1[1-\left(1-e^{-\theta_1x}\right]^{\lambda_1}\right\}^{\alpha_1+1}},$$

and

$$f(y)=\alpha_2\lambda_2(1-p_2)\left(\theta_2 e^{-\theta_2 y}\right)[1-\left(1-e^{-\theta_2 y}\right]^{\lambda_2-1}\frac{\left\{1-[1-\left(1-e^{-\theta_2 y}\right]^{\lambda_2}\right\}^{\alpha_2-1}}{\left\{1-p_2[1-\left(1-e^{-\theta_2 y}\right]^{\lambda_2}\right\}^{\alpha_2+1}}$$

Therefore,

$$R=\int_0^\infty\int_0^x f(x).f(y)\,dxdy, \qquad (26)$$

using the binomial series, we obtain

$$\int_0^x f(y)dy=L_2.\int_0^x e^{-\theta_2\lambda_2 y(k_2+1+j_2)}dy, \qquad (27)$$

where

$$L_2=\alpha_2\theta_2\lambda_2(1-p_2)\sum_{j_2=0}^\infty\binom{\alpha_2-1}{j_2}(-1)^{j_2}\sum_{k_2=0}^\infty\binom{-\alpha_2-1}{k_2}(-1)^{k_2}(p_2)^{k_2}$$

$$\int_0^x f(y)dy=\frac{L_2}{-\theta_2\lambda_2(k_2+1+j_2)}\left[e^{-\theta_2\lambda_2 y(k_2+1+j_2)}\right]_0^x$$

$$=\frac{L_2}{-\theta_2\lambda_2(k_2+1+j_2)}\left[e^{-\theta_2\lambda_2 x(k_2+1+j_2)}-1\right] \quad (28)$$

when we insert the last equation in equation (12), we achieve

now, from (13) we have

$$R=\frac{L_2}{-\theta_2\lambda_2(k_2+1+j_2)}\int_0^\infty\left[e^{-\theta_2\lambda_2 x(k_2+1+j_2)}-1\right]f(x)dx,$$

Then, by using the binomial series, we obtain

$$R=\frac{L_1L_2}{-\theta_2\lambda_2(k_2+1+j_2)}\left[\int_0^\infty e^{-\theta_1\lambda_1 x(k_1+1+j_1)-\theta_2\lambda_2 x(k_2+1+j_2)}-e^{-\theta_1\lambda_1 x(k_1+1+j_1)}dx\right]$$

$$=\frac{L_1L_2}{-\theta_2\lambda_2(k_2+1+j_2)}\left[\frac{1}{-\theta_1\lambda_1(k_1+1+j_1)-\theta_2\lambda_2(k_2+1+j_2)}e^{-x[\theta_1\lambda_1(k_1+1+j_1)-\theta_2\lambda_2(k_2+1+j_2)]}\right.$$
$$\left.-\frac{1}{-\theta_1\lambda_1(k_1+1+j_1)}e^{-\theta_1\lambda_1 x(k_1+1+j_1)}\right]$$

Finally, we have

$$R=\frac{L_1L_2}{-\theta_2\lambda_2(k_2+1+j_2)}\left[-\frac{(1+j_2+k_2)\theta_2\lambda_2}{(1+j_1+k_1)\theta_1\lambda_1\left((1+j_1+k_1)\theta_1\lambda_1+(1+j_2+k_2)\theta_2\lambda_2\right)}\right], \qquad (29)$$

where

$$L_1=\alpha_1\theta_1\lambda_1(1-p_1)\sum_{j_1=0}^\infty\binom{\alpha_1-1}{j_1}(-1)^{j_1}\sum_{k_1=0}^\infty\binom{-\alpha_1-1}{k_1}(-1)^{k_1}(p_1)^{k_1}$$

and

$$L_2=\alpha_2\theta_2\lambda_2(1-p_2)\sum_{j_2=0}^\infty\binom{\alpha_2-1}{j_2}(-1)^{j_2}\sum_{k_2=0}^\infty\binom{-\alpha_2-1}{k_2}(-1)^{k_2}(p_2)^{k_2}$$

Thus the reliability measure is determined by only on the slope parameters $(\alpha_1,\theta_1,\lambda_1,p_1)$ and $(\alpha_2,\theta_2,\lambda_2,p_2)$. This could be used as a measure of the difference between two populations as well as the effectiveness of one medicine over other.

6. Quantile based Shannon entropy

In latest years, there has been a great concentration in the measurement of uncertainty of probability distributions. Suppose that X is a nonnegative continuous random variable representing the lifetime of a component with cumulative distribution function (CDF)$F(t)=P(X\le t)$and survival function$\bar{F}(t)=P(X>t)=1-F(t)$. The measure of uncertainty defined by Shannon (1948) was

$$\xi(X)=\xi(f)=-\int_0^\infty \ln f(X).f(X)dX=-E(\ln f(X)), (30)$$

where $f(X)$is the probability density function of X. Last equation provides the expected uncertainty contained in$f(X)$about the predictability of an outcome of X, which is identified as Shannon entropy measure.

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Many properties of the entropy function can be used as an alternative tool in modeling statistical data. Sometimes the quantile based approach is better in expressions of tractability. New models and characterizations that are unresolvable in the distribution function approach can be resolved with the aid of quantile approach.

According to Sunoj and Sankaran, the Shannon entropy in (30) can be written in terms of quantile function as

$$\xi(X)=\int_0^1 (\ln(q(u))du, \qquad (31)$$

where $q(u)=\dot{Q}(u)$ and $\dot{Q}(u)$ represents quantile function of distribution.

Evidently, by knowing either $Q(u)$or $q(u)$, the expression for $\xi(X)$ is quite simple to compute.

Thus, in order to obtain the Shannon entropy of EMOEx distribution we can use last equation and the $Q(u)$that has given in eq (31) as

$$q(u)=Q(u)=-\frac{\left(1-pu^{\frac{1}{\alpha}}\right)\left[\frac{pu^{-1+\frac{1}{\alpha}}\left(1-u^{\frac{1}{\alpha}}\right)}{\left(1-pu^{\frac{1}{\alpha}}\right)^2\alpha}-\frac{u^{-1+\frac{1}{\alpha}}}{\left(1-pu^{\frac{1}{\alpha}}\right)\alpha}\right]}{\left(1-u^{\frac{1}{\alpha}}\right)\theta\lambda}$$

Therefore, we can obtain

$$\xi(X)=\int_0^1 (\ln(q(u))du$$

$$=1+\gamma+\frac{1}{\alpha}-\frac{{}_2F_1[1,\alpha,1+\alpha,p]}{\alpha}+\ln\left[\frac{1}{\alpha\theta\lambda}\right]+\psi^{(0)}(\alpha), (32)$$

where
$\gamma$is Euler's constant γ, with numerical value $\simeq 0.577216$.
${}_2F_1[1,\alpha,1+\alpha,p]$ is the hypergeometric function.
$\psi^{(0)}(\alpha)$is the n^{th} derivative of the digamma function.

7. Estimation

The maximum likelihood method is the most commonly employed for approaches for parameter estimation in the literature. The maximum likelihood estimators (MLEs) can be used when constructing confidence intervals for the model parameters. In large samples, the normal approximation for these estimators could be easily treated either analytically or numerically. Therefore, we consider the estimation of the unknown parameters of this model from complete samples only by maximum likelihood. In this section, we find the MLEs of the parameters of the new model from complete samples.

Let $x_1, \ldots, x_n$ be the observed values from the EMOEx distribution with parameters p, α, λ and θ. Let $\Theta = (p,\alpha,\lambda,\theta)^T$be the $r\times 1$ parameter vector. The log-likelihood function for Θ, say $\ell = \ell(\Theta)$, is given by

$$\ell = n\ln\left[\alpha\lambda\theta(1-p)\right]-\theta\lambda\sum_{i=1}^n x_i+(\alpha-1)\sum_{i=1}^n \ln\left[1-e^{-\theta\lambda x_i}\right]-(\alpha+1)\sum_{i=1}^n \ln\left[1-p\,e^{-\theta\lambda x_i}\right], \qquad (33)$$

The maximized log-likelihood can be solving directly by using the NLMIXED procedure in SAS.
d = $\left(\frac{\partial l}{\partial p},\frac{\partial l}{\partial \alpha},\frac{\partial l}{\partial \lambda},\frac{\partial l}{\partial \theta}\right)^T$are given by

$$\frac{\partial\ell}{\partial p}=-\frac{n}{1-p}-(1+\alpha)\sum_{i=1}^n -\frac{\left(e^{-\theta x_i}\right)^\lambda}{1-p\left(e^{-\theta x_i}\right)^\lambda}=0$$

$$\frac{\partial\ell}{\partial \alpha}=\frac{n}{\alpha}+\sum_{i=1}^n \ln\left[1-\left(e^{-\theta x_i}\right)^\lambda\right]-\sum_{i=1}^n \ln\left[1-p\left(e^{-\theta x_i}\right)^\lambda\right]=0$$

$$\frac{\partial\ell}{\partial \lambda}=\frac{n}{\lambda}+(\alpha-1)\sum_{i=1}^n -\frac{\left(e^{-\theta x_i}\right)^\lambda \ln\left[e^{-\theta x_i}\right]}{1-\left(e^{-\theta x_i}\right)^\lambda}-(\alpha+1)\sum_{i=1}^n -\frac{\left(e^{-\theta x_i}\right)^\lambda p\ln\left[e^{-\theta x_i}\right]}{1-p\left(e^{-\theta x_i}\right)^\lambda}-\theta\sum_{i=1}^n x_i \;=0$$

and

$$\frac{\partial\ell}{\partial \theta}=\frac{n}{\theta}-\lambda\sum_{i=1}^n x_i+(\alpha-1)\sum_{i=1}^n \frac{\left(e^{-\theta x_i}\right)^\lambda \lambda x_i}{1-\left(e^{-\theta x_i}\right)^\lambda}-(\alpha+1)\sum_{i=1}^n \frac{\left(e^{-\theta x_i}\right)^\lambda p\lambda x_i}{1-p\left(e^{-\theta x_i}\right)^\lambda}=0$$

For interval estimation of the parameters of the EMOEx distribution, we necessitate the 4×4unit observed information matrix is

$$I_n(\Theta)=-\begin{pmatrix} I_{p,p} & I_{p,\alpha} & I_{p,\lambda} & I_{p,\theta} \\ I_{\alpha,p} & I_{\alpha,\alpha} & I_{\alpha,\lambda} & I_{\alpha,\theta} \\ I_{\lambda,p} & I_{\lambda,\alpha} & I_{\lambda,\lambda} & I_{\lambda,\theta} \\ I_{\theta,p} & I_{\theta,\alpha} & I_{\theta,\lambda} & I_{\theta,\theta} \end{pmatrix}$$

the ingredients of the 4×4 information matrix $I_n(\Theta)$are given by

$$\frac{\partial^2\ell}{\partial p^2}=-\frac{n}{(1-p)^2}-(\alpha+1)\sum_{i=1}^n \frac{\left(e^{-x\theta}\right)^{2\lambda}}{\left(1-p\left(e^{-\theta\lambda x}\right)\right)^2},$$

$$\frac{\partial^2\ell}{\partial \alpha^2}=-\frac{n}{\alpha^2},$$

$$\frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{n}{\lambda^2} + (\alpha - 1)\sum_{i=1}^{n}\left(-\frac{\left(e^{-\theta x_i}\right)^{2\lambda} \ln\left[e^{-\theta x_i}\right]^2}{\left(1 - e^{-\theta \lambda x_i}\right)^2} - \frac{e^{-\theta \lambda x_i} \ln\left[e^{-\theta x_i}\right]^2}{1 - e^{-\theta \lambda x_i}}\right)$$

$$-(\alpha + 1)\sum_{i=1}^{n}\left(-\frac{\left(e^{-\theta x_i}\right)^{2\lambda} p^2 \ln\left[e^{-\theta x_i}\right]^2}{\left(1 - p\left(e^{-\theta \lambda x_i}\right)\right)^2} - \frac{e^{-\theta \lambda x_i} p \ln\left[e^{-\theta x_i}\right]^2}{1 - p\left(e^{-\theta \lambda x_i}\right)}\right)$$

$$\frac{\partial^2 \ell}{\partial \theta^2} = -\frac{n}{\theta^2} + (\alpha - 1)\sum_{i=1}^{n}\left(-\frac{\left(e^{-\theta x_i}\right)^{2\lambda} \lambda^2 x_i^{\,2}}{\left(1 - \left(e^{-\theta x_i}\right)^{\lambda}\right)^2} - \frac{\left(e^{-\theta x_i}\right)^{\lambda} \lambda^2 x_i^{\,2}}{1 - \left(e^{-\theta x_i}\right)^{\lambda}}\right)$$

$$-(\alpha + 1)\sum_{i=1}^{n}\left(-\frac{\left(e^{-\theta x_i}\right)^{2\lambda} p^2 \lambda^2 x_i^{\,2}}{\left(1 - p\left(e^{-\theta x_i}\right)^{\lambda}\right)^2} - \frac{\left(e^{-\theta x_i}\right)^{\lambda} p \lambda^2 x_i^{\,2}}{1 - p\left(e^{-\theta x_i}\right)^{\lambda}}\right),$$

$$\frac{\partial^2 \ell}{\partial p \partial \alpha} = -\sum_{i=1}^{n} - \frac{e^{-\theta \lambda x_i}}{1 - p\left(e^{-\theta \lambda x_i}\right)},$$

$$\frac{\partial^2 \ell}{\partial p \partial \lambda} = -(\alpha + 1)\sum_{i=1}^{n}\left(-\frac{\left(e^{-\theta x_i}\right)^{2\lambda} p \ln\left[e^{-\theta x_i}\right]}{\left(1 - p\left(e^{-\theta x_i}\right)^{\lambda}\right)^2} - \frac{\left(e^{-\theta x_i}\right)^{\lambda} \ln\left[e^{-\theta x_i}\right]}{1 - p\left(e^{-\theta x_i}\right)^{\lambda}}\right),$$

$$\frac{\partial^2 \ell}{\partial p \partial \theta} = -(\alpha + 1)\sum_{i=1}^{n}\left(\frac{\left(e^{-\theta x_i}\right)^{2\lambda} p \lambda x_i}{\left(1 - p\left(e^{-\theta x_i}\right)^{\lambda}\right)^2} + \frac{\left(e^{-\theta x}\right)^{\lambda} \lambda x_i}{1 - p\left(e^{-\theta x_i}\right)^{\lambda}}\right),$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \lambda} = \sum_{i=1}^{n} - \frac{\left(e^{-\theta x_i}\right)^{\lambda} \ln\left[e^{-\theta x_i}\right]}{1 - \left(e^{-\theta x_i}\right)^{\lambda}} - \sum_{i=1}^{n} - \frac{\left(e^{-\theta x_i}\right)^{\lambda} p \ln\left[e^{-\theta x_i}\right]}{1 - p\left(e^{-\theta x_i}\right)^{\lambda}},$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \theta} = \sum_{i=1}^{n} \frac{\left(e^{-\theta x_i}\right)^{\lambda} \lambda x_i}{1 - \left(e^{-\theta x_i}\right)^{\lambda}} - \sum_{i=1}^{n} \frac{\left(e^{-\theta x_i}\right)^{\lambda} p \lambda x_i}{1 - p\left(e^{-\theta x_i}\right)^{\lambda}},$$

and

$$\frac{\partial^2 \ell}{\partial \lambda \partial \theta} = -\sum_{i=1}^{n} x_i + (\alpha - 1)\sum_{i=1}^{n}\left(\frac{\left(e^{-\theta x_i}\right)^{\lambda} x_i}{1 - \left(e^{-\theta x_i}\right)^{\lambda}} + \frac{\left(e^{-\theta x_i}\right)^{2\lambda} \lambda \ln\left[e^{-\theta x_i}\right] x_i}{\left(1 - \left(e^{-\theta x_i}\right)^{\lambda}\right)^2} + \frac{\left(e^{-\theta x_i}\right)^{\lambda} \lambda \ln\left[e^{-\theta x_i}\right] x_i}{1 - \left(e^{-\theta x_i}\right)^{\lambda}}\right)$$

$$-(\alpha + 1)\sum_{i=1}^{n}\left(\frac{\left(e^{-\theta x_i}\right)^{\lambda} p x_i}{1 - p\left(e^{-\theta x_i}\right)^{\lambda}} + \frac{\left(e^{-\theta x_i}\right)^{2\lambda} p^2 \lambda \ln\left[e^{-\theta x_i}\right] x_i}{\left(1 - p\left(e^{-\theta x_i}\right)^{\lambda}\right)^2} + \frac{\left(e^{-\theta x_i}\right)^{\lambda} p \lambda \ln\left[e^{-\theta x_i}\right] x_i}{1 - p\left(e^{-\theta x_i}\right)^{\lambda}}\right)$$

The 4×4 unitobserved information matrix $I_n(\Theta) = \{I_{ns}\}$ for $n, s = \alpha, \lambda, p, \theta$ under standard regularity conditions, the multivariate normal $N_4(0, \mathrm{I}(\hat{\theta})^{-1})$ distribution can be used to construct approximate confidence intervals for the model parameters. Here, $I(\hat{\theta})$ is the total observed information matrix evaluated at $\hat{\theta}$. Then, approximate $100(1-\phi)\%$ confidence intervals for the model parameters can be determined in the usual way of the first-order asymptotic theory.

8. Simulation Study

Here, a simulation study is completed to consider the average bias and average mean square error (MSE) of the simulated estimates. The equation f(x) − u = 0, where u is an observation from the uniform distribution (0,1), and F(x) is cumulative distribution function of EMOEx, used to complete the simulation study by generating random samples following EMOEx. The simulation experiment was repeated 5000 times each with sample sizes, 25, 50, 100, 150, 200, 400, 800 for $(\alpha, \theta, \lambda, p)$ = (0.8, 0.8, 1.5, 0.5) and (1.5, 0.5, 2, 0.8). The following measures are calculated:

(i) Average bias of $\hat{\alpha}, \hat{\theta}, \hat{\lambda}$ and $\hat{p}$ of the parameters $\alpha, \theta, \lambda, p$ are, respectively:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}-\alpha), \qquad \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}-\theta),$$
$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda}-\lambda), \quad and \ \frac{1}{N}\sum_{i=1}^{N}(\hat{p}-p)$$

The MSEs of $\hat{\alpha}, \hat{\theta}, \hat{\lambda}$ and $\hat{p}$ of the parameters $\alpha, \theta, \lambda, p$ are, respectively:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}-\alpha)^2, \qquad \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}-\theta)^2,$$
$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda}-\lambda)^2, \quad and \ \frac{1}{N}\sum_{i=1}^{N}(\hat{p}-p)^2$$

From Table 1, it can be established that the MSE and the average bias decrease as the sample size increases.

Table 1: Bias and MSE for the parameters $\alpha, \theta, \lambda, p$

n	α = 0.8		p = 0.5		λ = 1.5		θ = 0.8	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
25	0.03156	0.03806	0.86648	0.02079	0.17144	0.07073	0.04876	0.03772
50	0.01378	0.01848	0.14809	−0.00884	0.08046	0.03636	0.02288	0.01939
100	0.00642	0.00920	0.00970	−0.01044	0.03900	0.01930	0.01109	0.01029
150	0.00433	0.00664	0.00607	−0.00722	0.02623	0.01313	0.00746	0.00700
200	0.00320	0.00539	0.00446	−0.00588	0.01942	0.00828	0.00552	0.00442
400	0.00161	0.00249	0.00221	−0.00260	0.00969	0.00468	0.00275	0.00249
800	0.00080	0.00130	0.00109	−0.00132	0.00475	0.00249	0.00135	0.00133
n	α = 1.5		p = 0.8		λ = 2		θ = 0.5	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
25	0.11213	0.06281	0.01240	−0.00754	0.27232	0.07874	0.01702	0.01968
50	0.04859	0.03436	0.00233	−0.00603	0.11853	0.03072	0.00740	0.00768
100	0.02259	0.01726	0.00097	−0.00333	0.05739	0.01551	0.00358	0.00387
150	0.01524	0.01245	0.00063	−0.00230	0.03877	0.01028	0.00242	0.00257
200	0.01126	0.01010	0.00046	−0.00189	0.02875	0.00573	0.00179	0.00143
400	0.00569	0.00467	0.00023	−0.00084	0.01451	0.00359	0.00090	0.00089
800	0.00283	0.00244	0.00011	−0.00045	0.00710	0.00181	0.00044	0.00045

9. Application

In this section, we use the remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang (2003).

Table 2: The remission times (in months) of a random sample of 128 bladder cancer patients (Lee and Wang, 2003)

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23
3.52	4.98	6.97	9.02	13.29	0.40	2.26	3.57	5.06	7.09
9.22	13.80	25.74	0.50	2.46	3.64	5.09	7.26	9.47	14.24
25.82	0.51	2.54	3.70	5.17	7.28	9.74	14.76	26.31	0.81
2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64	3.88	5.32
7.39	10.34	14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66
15.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01
1.19	2.75	4.26	5.41	7.63	17.12	46.12	1.26	2.83	12.63
4.33	5.49	7.66	11.25	17.14	79.05	1.35	2.87	5.62	7.87
11.64	17.36	1.40	3.02	4.34	5.71	7.93	11.79	18.10	1.46
4.40	5.85	8.26	11.98	19.13	1.76	3.25	4.50	6.25	8.37
12.02	2.02	3.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36
6.76	12.07	21.73	2.07	3.36	6.93	8.65	22.69		

The estimated values of the parameters, −2 log-likelihood statistic, Akaike information criterion (AIC), Bayesian information criterion (BIC), and the Kolmogorov–Smirnov (KS) statistics and the associated p-values are presented in Table 3, for EMOEx and other alternatives including Exponential, Extension Exponential-Geometric (EEG), Inv Gaussian, Frechet and Log-normal distributions.

Table 3: Fitted estimates for different distributions

Distribution	MLEs	-2Log	AIC	BIC	KS	p-Value
EMOEx	$\hat{\alpha} = 1.707$	426.79	861.58	872.988	0.04784	0.9313
	$\hat{\theta} = 0.339$					
	$\hat{\lambda} = 0.271$					
	$\hat{p} = 0.598$					
EEG	$\hat{\alpha} = 0.1$	452.115	908.23	913.934	0.0846	0.31834
	$\hat{\theta} = 0.1$					
Exponential	$\hat{\theta} = 0.10677$	459.677	921.355	924.207	0.08462	0.30123
Inv. Gaussian	$\hat{\theta} = 0.1$	492.639	989.278	994.982	0.1060	0.1044
	$\hat{\theta} = 9.3656$					

Frechet	$\hat{\alpha} = 1.0673$	481.992	967.985	973.689	0.1350	0.0169
	$\hat{\theta} = 3.3383$					
Log-normal	$\hat{\alpha} = 1.0731$	447.555	899.11	904.814	0.3070	0.000041
	$\hat{\theta} = 1.7535$					

From table 2 and table 3, we observe that the EMOEx distribution is a competitive distribution compared with other distributions. In fact, based on the values of the −2 log, AIC, BIC, and KS criterion, we observe that the EMOEx distribution provides the best fit for the data sets among all the models considered. The results in Table 3 have been examined using the Q–Q plots. From the Q–Q plots in Figure 1, we can achieve that EMOEx provides better fits than other distributions considered in this paper.

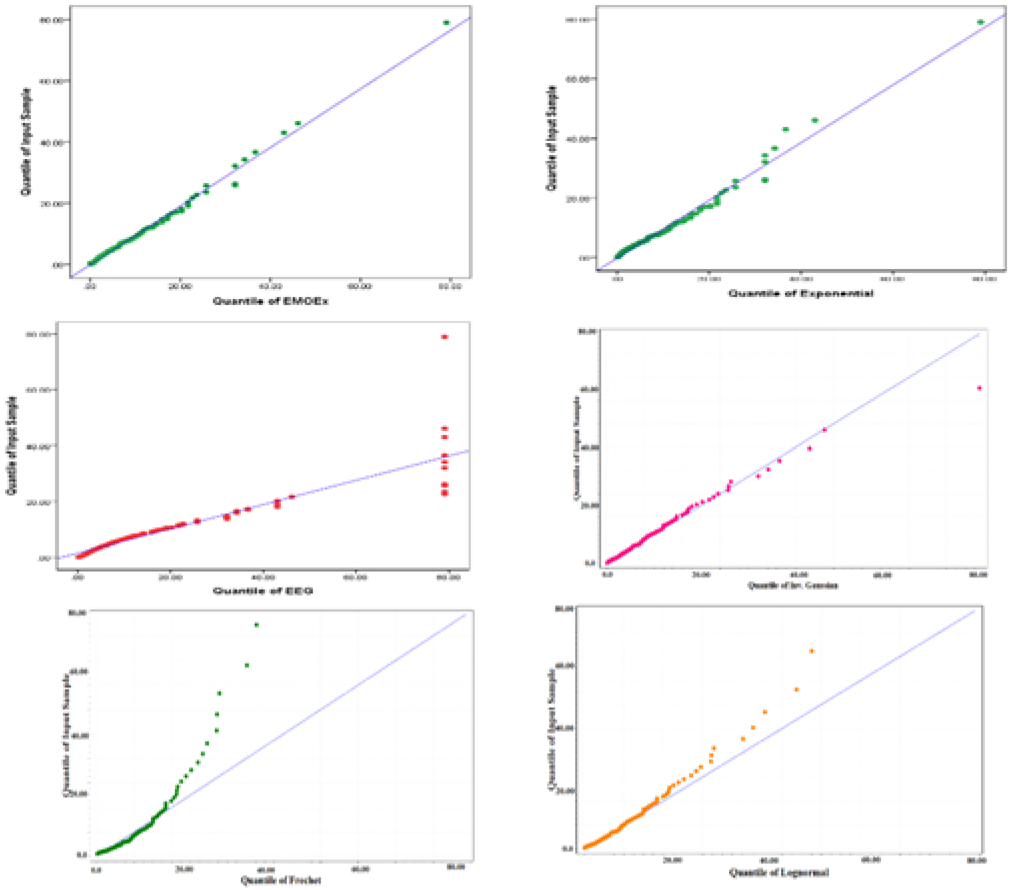

Figure 3: Q-Q plot for the given data set

10. Conclusion

In this paper, we studied a new four-parameter model named the exponentiated Marshall- Olkin exponential (EMOEx) distribution. The EMOEx density function is a linear mixture of G-exponential densities. We derived explicit expressions for its mathematical properties including the ordinary moments, quantile, generating function, order statistics, stress-strength analysis, quantile based Shannon entropy. The model parameters are estimated by maximum likelihood. The new distribution applied to real data sets provides better fits than some other competitive models.

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