Relation between the Area of Triangle (Coordinate Geometry) and Eulars Phi Function and Some Results

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Abstract: ***In this current paper author applied Eulars phi function to find the area of triangle and vice versa.***

Keywords: Relatively prime, triangle, determinant and etc.

1. Introduction

In this current paper author describe how to find the area of triangle ABC and PQR whose vertices are A (p^{a-1},q^{b-1}), B(p^a,q^b) , C(p^{a+1},q^{b+1}) and P(0,0), Q$(p_{r-1}-1,p_r-1)$, R$\{p_{r-1}(p_{r-1}-1)$, $p_r(p_r-1)$ }.

1) Statement

Let n be a positive integer such that $n=(p^a . q^b)$ where p,q are the primes (q>p) and a,b are the positive integers.

then, phi(n) = 2xArea of triangle ABC/(q-p)
vertices of triangle should be of the form A(p^{a-1}, q^{b-1}) , B(p^a, q^b) ,
C(p^{a+1},q^{b+1})

Proof:
LHS: when $n=p^a.q^b$ i.e
using prime factorization method , we get

$$\text{phi}(n) = p^a.q^b.(1-1/p).(1-1/q)$$
$$= p^{a-1}. q^{b-1}.(p-1). (q-1) \quad \text{--------------(1)}$$

RHS: 2x Area of triangle ABC /(q-p)
$=\{p^{a-1}(q^b-q^{b+1}) + q^{b-1}(p^{a+1}-p^a) + (p^aq^{b+1}-p^{a+1}q^b)\} /(q-p)$
$=\{p^{a-1}q^b(1-q) + q^{b-1}p^a(p-1) + p^aq^b(q-p)\} /(q-p)$
$=[\{p^{a-1}q^b(1-q) + q^{b-1}p^a(p-1)\} /(q-p)] + p^aq^b$
$=[\{p^{a-1}q^b-p^{a-1}q^{b+1} + p^{a+1}q^{b-1}-p^aq^{b-1}\} /(q-p)] + p^aq^b$
$= [\{ (p^{a-1}q^b-p^aq^{b-1}) +(p^{a+1}q^{b-1}-p^{a-1}q^{b+1}) \}/(q-p)] + p^aq^b$
$=[\{p^aq^b(p^{-1}-q^{-1})\}/(q-p)] +[\{p^aq^b(pq^{-1}-qp^{-1}\}]/(q-p)] + p^aq^b$
$=p^{a-1}q^{b-1}-p^{a-1}q^{b-1}(p+q) + p^aq^b$
$=p^aq^b [p^{-1}q^{-1}-\{(p+q)/pq\} + 1]$
$=p^{a-1}q^{b-1} [1-(p+q) +pq]$
$=p^{a-1}q^{b-1}[(1-q) + p(q-1)]$
$=p^{a-1}q^{b-1}[(1-q)-p(1-q)]$
$=p^{a-1}q^{b-1}[(1-q)(1-p)]$
or, 2xArea of triangle ABC/(q-p)$=p^{a-1}q^{b-1}(p-1)(q-1)$ ----------(2)

Hence from (1) and (2), we get
phi(n) =2x Area of triangle ABC/(q-p)
phi(n) when $n=p^aq^b$ in determinant form

$$\text{phi}(n)= \begin{vmatrix} p^{a-1} & q^{b-1} & 1 \\ p^a & q^b & 1 \\ p^{a+1} & q^{b+1} & 1 \end{vmatrix} /(q-p)$$

or,

$$(n)= \begin{vmatrix} p^a & q^b & 1 \\ p^{a+1} & q^{b+1} & 1 \\ p^{a+2} & q^{b+2} & 1 \end{vmatrix} /[pq.(q-p)]$$

Example: Find phi (50) using area of triangle.
SOL: phi(50) can also be written as phi(2.5^2)
vertices of triangle are A$(2^{1-1},5^{2-1})$, B$(2^1,5^2)$, C$(2^{1+1},5^{2+1})$
or, A (1,5) , B(2,25), C(4,125)
phi(n) = 2.Area of Triangle ABC/(q-p)
Area of triangle ABC = 30
phi(50) = (2 x 30)/(5-2)=20

Example: Find the Area of triangle ABC whose vertices are A (2,1) , B(4,5) , c (8,25) using Eulars phi function

SOL: A $(2^{2-1}, 5^{1-1})$, B $(2^2,5^1)$, C $(2^{2+1}, 5^{1+1})$
Here, 2 and 5 are the primes . (5>2).

Area of triangle ABC = [phi$(p^a.q^b)$. (q-p)]/2
phi$(p^a.q^b)$ = phi$(2^2.5)$ = 8
Area of triangle ABC =[8 X (5-2)] /2 = 12 .

NOTE: In phi(n) when $n=p^aq^b$
(1) a and b can be equal
Example: phi(36)= phi(2^23^2) (a=b=2)

(2) a can be greater than b
Example: phi(675)= phi(3^35^2) (a>b)

(3) a can be smaller than b
Example: phi(1372)= phi(2^27^3) (a<b)
but p and q should not be equal. (q>p)

Eulars PHI Function Using Both Determinants (2x2 order and 3x3 order determinant)

Let

$$D_1 = \begin{vmatrix} p^{a-1} & q^{b-1} \\ p^a & q^b \end{vmatrix}$$

and

$$D_2= \begin{vmatrix} p^{a-1} & q^{b-1} & 1 \\ p^a & q^b & 1 \\ p^{a+1} & q^{b+1} & 1 \end{vmatrix}$$

then, $phi(p^a.q^b)= (D_2.p^{a-1}.q^{b-1})/D_1$

NOTE: $p^{a-1}q^{b-1}/D_1 = 1/(q-p)$

(B): Relation between Eulars Phi function and area of triangle

phi(n) when $n= p_1^{e_1}.p_2^{e_2}.........p_r^{e_r}$
($e_1,e_2,.......e_r$ are the powers of $p_1,p_2,..........p_r$)

Statement: Let n be a positive integer such that $n= p_1^{e_1}.p_2^{e_2}..........p_{r-1}^{e_{r-1}}.p_r^{e_r}$.
where $p_1,p_2,.....p_{r-1},p_r$ are the primes.And $e_1,e_2,.......e_{r-1},e_r$ are the positive integers.
then, phi(n)=(2x Area of triangle PQR) x $(p_1-1).(p_2-1).......(p_{r-2}-1)$ x $(p_1^{e_1-1})$.
$(p_2^{e_2-1})......(p_r^{e_r-1})$.
vertices of triangle should be of the form A(0,0), $B(P_{r-1}-1,p_r-1)$,
$C\{p_{r-1}(p_{r-1}-1),p_r(p_r-1)\}$

Proof:
LHS: using prime factorization method we get,
$phi(p_1^{e_1}.p_2^{e_2}.........p_{r-1}^{e_{r-1}}.p_r^{e_r})$
$=p_1^{e_1-1}.p_2^{e_2-1}........p_{r-1}^{e_{r-1}-1}.p_r^{e_r-1}$ x $(p_1-1).(p_2-1)......(p_{r-2}-1).(p_{r-1}-1).(p_r-1)$. ----------------------(1)

RHS:
[(2 x Area of triangle PQR) x $(P_1-1).(P_2-1).......(P_{r-2}-1)$. $p_1^{e_1-1}.p_2^{e_2-1}......$
$p_r^{e_r-1}$] $/(p_r-p_{r-1})$ ------------------(let it be *)

2 x Area of triangle PQR=$\{(p_{r-1}-1)p_r(p_r-1) - p_{r-1}(p_{r-1}-1)(p_r-1)\}$

Put the value of above i.e 2 x Area of triangle PQR in (*), we get
$[\{(p_{r-1}-1)p_r(p_r-1)-p_{r-1}(p_{r-1}-1)(p_r-1)\}$ x $(p_1-1)(p_2-1).......(p_{r-2}-1)$ x $p_1^{e_1-1}.p_2^{e_2-1}$
$.........p_r^{e_r-1}]$ $/(p_r-p_{r-1})$. -------------------------------(2)
Put p_{r-1}=s and p_r=t in $[\{(p_{r-1}-1)p_r(p_r-1)-p_{r-1}(p_{r-1}-1)(p_r-1)\}]$ $/(p_r-p_{r-1})$, we get

$\{(s-1)t(t-1)-s(s-1)(t-1)\}$ / $(t-s)=(s-1)(t-1)(t-s)$ /($t-s)=(s-1)(t-1)=(p_{r-1}-1)(p_r-1)$
Put the above in (2), we get
$p_1^{e_1-1}.p_2^{e_2-1}.......p_r^{e_r-1}$ x $(p_1-1)(p_2-1).....(p_{r-2}-1)(p_{r-1}-1)(p_r-1)$
Hence proved.

NOTE: $P_1,P_2,....P_r$ are the first,second ,.......r^{th} prime.

Part (A) is only for two primes (p,q) and part (B) is for two or more than two primes $(p_1,p_2,.....p_r)$.

In part (B) if $n= p_1^{e_1}.p_2^{e_2}$ then $(p_1-1).(p_2-1)......(p_{r-2}-1)=1$

The Eulars PHI Function Using Area of Triangle can be Computed by Using the Following Steps
$phi(p_1^{e_1}.p_2^{e_2}........p_{r-1}^{e_{r-1}}.p_r^{e_r})$
STEP(1): Compute the area of triangle PQR whose vertices should be of the form P(0,0), $Q(p_{r-1}-1 , p_r-1)$, $R\{P_{r-1}(p_{r-1}-1), p_r(p_r-1)\}$
here, p_{r-1} and p_r are the second last and last prime.

STEP(2): Multiply the step (1) by 2

STEP(3): Obtain the difference of p_r and p_{r-1} (last prime and second last prime) i.e p_r-p_{r-1} and divide step (2) by (p_r-p_{r-1})

STEP(4): Multiply step (3) by $p_1^{e_1-1}.p_2^{e_2-1}......p_{r-1}^{e_{r-1}-1}.p_r^{e_r-1}$

STEP(5): Multiply step (4) by $(p_1-1).(p_2-1).......(p_{r-2}-1)$.
p_{r-2} is the third last prime (excluding last two primes).

EXAMPLE: Using the above steps find phi (3465)
STEP(1): phi(3465) = $phi(3^2.5.7.11)$
vertices of triangle PQR are P(0,0) , Q(6,10), R(42, 110).
Area of triangle PQR = 120
STEP(2): 2 x 120 = 240
STEP(3): 240 / (11-7) =60
STEP(4): 60 x 3^{2-1} x 5^{1-1} x 7^{1-1} x 11^{1-1} = 180
STEP(5): 180 x (3-1) x (5-1) = 1440
Therefore, phi(3465) = 1440

On Finding the Area of Triangle PQR using Eulars PHI Function
STEP(1): First check whether the vertices of triangle PQR is of the form P(0,0) , $Q(P_{r-1}-1 ,p_r-1)$, $R\{p_{r-1}(p_{r-1}-1), p_r(p_r-1)\}$ if yes then go through the following steps.

STEP(2): Take phi(n)

STEP(3): Take the difference of last and second last prime i.e p_r and p_{r-1}-and multiply it by step (2).

STEP(4): Multiply half by step (3).

STEP(5): Divide step (4) by $(p_1-1)(p_2-1)....(p_{r-2}-1)$ x $p_1^{e_1-1}.p_2^{e_2-1}......p_r^{e_r-1}$.
p_{r-2} is the third last prime (Excluding last two primes) .

EXAMPLE: Using the above steps find the area of triangle whose vertices are P(0,0), Q(6,10), R(42,110)
Sol; STEP (1): P(0,0), Q{(7-1), (11-1)}, R{7(7-1), 11(11-1)}.

STEP (2): Let phi(n)= $(2.3^2.7^2.11)$=2520

STEP (3): P_r-P_{r-1}=11-7=4
4x2520=10080

STEP (4): 1/2x10080 =5040.

STEP (5): 5040/{(2-1).(3-1).3.7} =120

Therefore, Area of triangle is 120.

Note: (1)In phi(n) we can take any prime which comes before P_{r-1} (Second last prime)In above example we can take any prime which comes before 7.

(2)Power of the primes which we choose should be any positive integers.

(3)We can take any power of P_r (last prime) and P_{r-1} (Second last prime) also.

To find the area of triangle of above example we can also use:

$phi(2^e{}_1.3^e{}_2.5^e{}_3.7^e{}_4.11^e{}_5)$
$phi(2^e{}_1.5^e{}_2.7^e{}_3.11^e{}_4)$
$phi(3^e{}_1.7^e{}_2.11^e{}_3)$, $phi(7^e{}_1.11^e{}_2)$ and etc.

It is necessary to take last two primes which are in the vertices of triangle .(STEP:1)

Note: phi(p.q)= 2xArea of triangle ABC/(q-p)= 2xArea of triangle PQR/(q-p).
Where, p and q are the primes (q>p).
Vertices of triangle ABC and PQR are A(1,1),B(p,q),C(p^2,q^2) and P(0,0), Q[(p-1),(q-1)] R[p(p-1),q(q-1)].
In above case, Area of triangle ABC=Area of triangle PQR.

EXAMPLE : Find the area of triangle ABC whose vertices are A(1,1) , B(3,7) , C (9 , 49)
OR
Area of triangle PQR whose vertices are P(0,0) , Q(2,6) , R (6 , 42) using
phi function

SOL :(1) Area of triangle ABC = [phi(p.q) x (q-p)] /2
Area of triangle ABC =[phi(3.7) x (7-3)] /2 = 24
(2) Area of triangle PQR =[phi(2+1, 6+1) x4] /2 =24
Area of triangle ABC = Area of triangle PQR .

2. Result

(1) If n,(n+2) are the primes and (n+1) is composite then, phi(n!), phi[(n+1)!], phi[(n+2)!] are in geometric progression having common ratio (n+1) i.e composite.

Example(1): phi(3!), phi(4!), phi(5!) are in geometric progression having common ratio 4.

(2): phi(5!), phi(6!), phi(7!) are in geometric progression having common ratio 6.

(3): phi(11!) , phi(12!), phi(13!) are in geometric progression having common ratio 12 .

(2) Area of triangle KLM = [LCM{ phi(p^a.q),pq,q^2}/(2.q^2)] x (q-p)
vertices of triangle KLM should be of the form K(p^{a-1},1), L(p^a,q), M(p^{a+1},q^2).
Where p and q are the primes (q>p) and a is the positive integers. (a>1).

(3) phi(p^a) in the form of HCF, LCM and common ratio. Where p is a prime and a is the positive interger (a>1).
phi(p^a) =[{ HCFof divisors of p^a x LCM of divisors of p^a }/ common ratio of divisors of p^a] x (p-1) .
HCF of divisors of p^a is 1 .
[Divisors are in geometric progression].
If p^a have even numbers of divisors, then
phi(p^a) = [divisor occuring at $(n/2)^{th}$ place$]^2$ x (p-1)
Example: phi(27)
Divisors of 27 are 1,3,9,27.
Divisors occuring at $(n/2)^{th}$ place = 3.
phi(27)=3^2.(3-1)=18

Eulars PHI Function in terms of Roots of Quadratic, Cubic, and Biquadratic Equations

(1) Quadratic Equation

(1) phi[$(a+1)^k.(b+1)^L.(c+1)^M$]= I alpha + beta l .$(a+1)^{k-1}.(b+1)^{L-1}.(c+1)^{M-1}.a^2c$
{mod of alpha plus beta}

(2) phi{$(a+1)^k.(b+1)^L.(c+1)^m$}= (alpha.beta).$(a+1)^{k-1}.(b+1)^{L-1}.(c+1)^{m-1}.a^2b$
where a+1, b+1, c+1 are the primes and K,L,M are the positive integers. (a,b,c, should be positive integers).

Example: find phi(693) using product of roots .
Sol: phi(693) = phi(3^2.11.7)
a+1 =3 , b+1=11 , c+1=7
a=2 , b =10 , c=6
$2x^2+10x+6 =0$
(alpha . beta) =3
phi(693) = 4.10.3.3 = 360
NOTE: In $2x^2+10x+6=0$, If we take 2 as common it become $x^2+5x+3=0$. Here a=1, b=5, c=3.
a+1=2, b+1=3, c+1=4
2, 3 are the primes but 4 is not prime
Therefore, we can not take 2 as common.

(2) Cubic Equation
(1) PHI[$(a+1)^K.(b+1)^L.(c+1)^M.(d+1)^N$]= l alpha + beta + gamma l . $(a+1)^{K-1}.(b+1)^{L-1}.(c+1)^{M-1}.(d+1)^{N-1}.a^2cd$
(2) PHI[$(a+1)^K.(b+1)^L.(c+1)^M.(d+1)^N$]= l alpha . beta. gamma l . $(a+1)^{K-1}.(b+1)^{L-1}.(c+1)^{M-1}.(d+1)^{N-1}.a^2bc$
(3) PHI[$(a+1)^K.(b+1)^L.(c+1)^M.(d+1)^N$]= (alpha.beta+beta.gamma+gamma.alpha).$(a+1)^{K-1}.(b+1)^{L-1}.(c+1)^{M-1}.(d+1)^{N-1}.a^2bd$

Where a+1, b+1, c+1, d+1, are the primes and K, L, M, N are the positive integers. (a, b, c, d should be positive integers).

Example: If $x^3+2x^2+3x+5=0$, find I alpha + beta + gama I using above equation
Sol : $x^3+2x^2+3x+5 =0$.
a=1, b=2, c=3, d=5 .
a+1=2, b+1=3, c+1=4 , d+1=6.

Here , c+1, d+1 are not primes .

Therefore Let , Multiply both side by 2 , We get
$2x^3+4x^2+6x+10 = 0$
Now, a=2 , b=4, c =6 d =10
a+1 =3 , b+1 = 5 , c+1= 7 , d+1 = 11 (All are primes)

Now, we can apply the equation
i.e., I alpha + beta + gama I = [phi(3.5.7.11)] /(4.6.10) = 2 .

(3) Biquadratic Equation

(1) $phi[(a+1)^{K}.(b+1)^{L}.(c+1)^{M}.(d+1)^{N}.(e+1)^{R}] = \mathrm{l}\ alpha + beta + gamma + delta\ \mathrm{l}\ .\ (a+1)^{K-1}.(b+1)^{L-1}.(c+1)^{M-1}.(d+1)^{N-1}.(e+1)^{R-1}.a^{2}cde$.

(2) $phi[(a+1)^{K}.(b+1)^{L}.(c+1)^{M}.(d+1)^{N}.(e+1)^{R}] = [(alpha + beta)\ .\ (gamma + delta) + alpha.beta + gamma.delta].\ (a+1)^{K-1}.(b+1)^{L-1}.(c+1)^{M-1}.(d+1)^{N-1}.(e+1)^{R-1}.\ a^{2}bde$.

(3) $phi[(a+1)^{K}.(b+1)^{L}.(c+1)^{M}.(d+1)^{N}.(e+1)^{R}] = \mathrm{l}\ (alpha+beta)\ gamma.delta + alpha.beta\ (gamma+delta)\ \ \mathrm{l}\ .\ (a+1)^{K-1}.(b+1)^{L-1}(c+1)^{M-1}.(d+1)^{N-1}(e+1)^{R-1}.a^{2}bce$

(4) $phi[(a+1)^{K}.(b+1)^{L}.(c+1)^{M}.(d+1)^{N}.(e+1)^{R}] = (alpha.beta.gamma.delta)\ .\ (a+1)^{K-1}.(b+1)^{L-1}.(c+1)^{M-1}.(d+1)^{N-1}.(e+1)^{R-1}.a^{2}bcd$.

Where, a+1, b+1, c+1, d+1, e+1 are the primes and K,L,M,N,R are the positive integers (a,b,c,d,e should be positive integers).

3. Benefits

We can find the area of triangle using eulars phi function which is very easy and simple.

4. Acknowledgement

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