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Extended Bolzano Weierstrass Theorem

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Abstract: ***In this paper we have proved the Extended Bolzano Weierstrass theorem i.e. "Let (*** $(a_n)_{n=1}^{\infty}$ ***be an bounded sequence . Define the set G = {x ∈ R :x < a_n for infinitely many terms a_n}***
Then ∃ a subsequence (a_{nk}) converges to a supremum of the set G. "

Keywords: Convergence, Bounded, supremum, Completeness

1. Introduction

We define the convergence of a sequence, bounded sequence and the supremeum. Then in Theorem 2.1 we showed that " $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ are three real sequences such that $a_n \leq b_n \leq c_n \ \forall \ n$. If $(a_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ both converges to the same limit then $(b_n)_{n=1}^{\infty}$ also converges to the same limit" Then in Theorem 2.2 we showed that "Supremum of a bounded set is the least element of the set of upper bounds of that set ".

Then Finally we proved the Extended Bolzano Weierstrass theorem in which we defined a set for each bounded sequence whose supremum is the limit of the subsequence of that sequence.

Notation : $B(a, \epsilon)$ **denotes an neighbourhood of a which is an open ball with centre a and radius** $\epsilon > 0$ **in** $\mathbb{R}$ **i.e.**

$$B(a, \epsilon) = \{x \in \mathbb{R} : |x - a| < \epsilon\} \quad [1]$$

1.1 Convergence of Sequence

A Real sequence $(x_n)_{n=1}^{\infty}$ is said to be converges to a real number x if for any >0 there exist a natural number $m \in$ N such that

$$|x_n - x| < \epsilon \ \forall \quad n \geq m$$

OR

A Real sequence $(x_n)_{n=1}^{\infty}$ is said to be converges to a real number x if the sequence $(x_n)_{n=1}^{\infty}$ is eventually is in each neighbourhood of x.[3]

1.2 Bounded Sequence

A Sequence $(x_n)_{n=0}^{\infty}$ is said to be bounded if there exist a real number M >0 such that

$$|x_n| \leq M \ \forall \ n \in \mathrm{N}$$

1.3 Supremum (Least upper bound)

Let S be any non empty subset of real numbers R. Then a real number α is said to be the supremum of the set S if

1. $x \leq \alpha \ \forall \ x \in S$
2. For any $\epsilon > 0$ $\exists$ an element $x \in S$ such that

$$\alpha - \epsilon < x$$

1.4 Least element of a set

Let S be any subset of R then a real number α is said to be a least element of the set S if

1. $\alpha \leq x \ \forall \ x \in S$
2. $\alpha \in S$

1.5 Completeness Property

Every non empty bounded above subset of real numbers have a least upper bound (i.e. supremum).

1.6 Properties of order relation in R

1. Law of Trichotomy:- For any two numbers a, b in R only one of the following holds

$$a > b,\ a = b,\ a < b$$

2. Transitive:- For all $a,b,c \in$ R we have

$$a < b \text{ and } b < c \Rightarrow a < c$$

3. Compatibility of order relation with addition :- For all a, b, $c \in$ R we have

$$a < b \Rightarrow a + c < b + c$$

2. Related Theorems

Theorem 2.1. *Let* $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ *and* $(c_n)_{n=1}^{\infty}$ *are three real sequences such that* $a_n \leq b_n \leq c_n \ \forall \ n \in \mathbb{N}$. *If* $(a_n)_{n=1}^{\infty}$ *and* $(c_n)_{n=1}^{\infty}$ *both converges to the same limit then* $(b_n)_{n=1}^{\infty}$ *also converges to the same limit.[2]*

Proof. Let $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$ and $(c_n)_{n=0}^{\infty}$ are any sequence of real numbers such that

$$a_n \leq b_n \leq c_n \forall \ n$$

Suppose $(a_n)_{n=0}^{\infty}$ and $(c_n)_{n=0}^{\infty}$ converges to the same limit say α.

Then for any $\epsilon > 0$ $\exists$ natural number n_1 and n_2 such that

$$|a_n - \alpha| \leq \epsilon \ \forall \ n \geq n_1$$

$$and$$

$$|c_n - \alpha| \leq \epsilon \ \forall \ n \geq n_2$$

$$\Rightarrow -\epsilon < a_n - \alpha < \epsilon \ \forall \ n \geq n_1 \ and \ \epsilon < c_n - \alpha < \epsilon \ \forall \ n \geq n_2 \quad -(1)$$

Now let $N = max\{n_1,n_2\}$. Then as

$$a_n \leq b_n \leq c_n \forall \ n \Rightarrow a_n \leq b_n \leq c_n \forall \ N$$

By the Compatibility of order relation with addition in R (1.4) we have

$$\Rightarrow a_n - \alpha \le b_n - \alpha \le c_n - \alpha \ \forall \ n \ge N$$

Then by (1) we have

$$-\epsilon < a_n - \alpha \le b_n - \alpha \le c_n - \alpha < \epsilon \ \forall \ n \ge N$$

$$\Rightarrow |b_n - \alpha| \le \epsilon \ \forall \ n \ge N$$

∴ the sequence $(b_n)_{n=0}^{\infty}$ also converges to the same limit α.

Theorem 2.2: *Supremum of a bounded subset of Real numbers is the least element of the set of all the the upper bound of that set*

Proof. Let S be any bounded subset of R Then By the Completeness property S have a least upper bound say α

Define a set

$$T = \{a \in \mathbb{R} : a \text{ is a upper bound of } S\}$$

By the definition of least upper bound, clearly α is a upper bound of S

$$\Rightarrow \alpha \in T$$

Suppose if possible α is not a lower bound of T
$\Rightarrow \exists\ a \in T$ such that
$a < \alpha$ Let
$\epsilon = \alpha - a > 0$,
Then by the definition $\exists\ b \in S$ such that
$\alpha - \epsilon < b \Rightarrow a < b$
which contradicts that a is a upper bound of S
$\Rightarrow \alpha$ is a lower bound of T
∴ α is a least element of T

3. Results and Discussion

Theorem 3.1. {*BolzanoWeierstrass theorem* }
Let $(a_n)_{n=0}^{\infty}$ *be any sequence of real numbers which is bounded . Then there exists a subsequence* $(a_{nk})_{k=0}^{\infty}$ *of this sequence which converges to a real number.*

Theorem 3.2. {*Extended Bolzano Weierstrasstheorem* }
Let $(a_n)_{n=1}^{\infty}$ *be an bounded sequence . Define the set*
G = $\{x \in \mathbb{R} : x < a_n$ *for infinitely many terms* $a_n\}$
Then ∃ *a subsequence* (a_{nk}) *converges to a supremum of the set G.*

Proof. Since $(a_n)_{n=1}^{\infty}$ is an bounded sequence.
∴ ∃ a real $M > 0$ such that
$|a_n| \le M \ \forall \ n \in \mathbb{N}$
Clearly $(-\infty, -M) \subseteq G \Rightarrow G \neq \emptyset$

Therefore G is an non empty bounded above subset of real numbers which is bounded above.

Hence By the completeness property of the set of real numbers we have the set G have a least upper bound *let* $Sup(G) = \gamma$
Let $k \in \mathbb{N}$ be arbitrary

$$\Rightarrow x \le \gamma < \gamma + \frac{1}{k} \quad \forall \quad x \in G$$

i.e. $\gamma + \frac{1}{k}$ is an upper bound of G which is strictly greater than γ

Since the supremum of a subset of a real number is the least element of the set of all the upper bound of that set (Theorem 2.2)

$$\therefore \ \gamma + \frac{1}{k} \notin G \quad \Rightarrow \quad \gamma + \frac{1}{k} \in G^c$$

$\Rightarrow \gamma + \frac{1}{k} < a_n$ for atmost finitely many n

Therefore, For infinitely many natural numbers we have

$$\alpha + \frac{1}{k} \ge a_m \tag{1}$$

Now
For the same k, $\alpha - \frac{1}{k}$ is not an upper bound of G.

Therefore there exist an element $x \in G$ such that

$$\alpha - \frac{1}{k} < x$$

Since $x \in G$ then by the definition of G for infinitely many $n \in \mathbb{N}$ we have

$$\alpha - \frac{1}{k} < x < a_n \tag{2}$$

i.e there are infinitely many terms of the sequence are stictly less than $\alpha - \frac{1}{k}$ for each natural number k.

For each $k \in \mathbb{N}$
Define

$$A_k = \{a_n : \alpha + \frac{1}{k} \ge a_n\}$$

$$B_k = \{a_n : \alpha - \frac{1}{k} < a_n\}$$

From (1) and (2) we have each A_k and B_k are infinite for all $k \in \mathbb{N}$

Define

$$n_k = \min\{m : a_m \in A_k \cap B_k\}$$

Then

$$\alpha - \frac{1}{k} < a_{nk} \le \alpha + \frac{1}{k} \qquad \forall \ \ k \in \mathbb{N}$$

Since

$$\alpha - \frac{1}{k} \longrightarrow \alpha \qquad as \qquad k \longrightarrow \infty$$

and

$$\alpha + \frac{1}{k} \longrightarrow \alpha \qquad as \qquad k \longrightarrow \infty$$

Then by Theorem 2.1 we have

$$a_{nk} \longrightarrow \alpha \qquad as \qquad k \longrightarrow \infty$$

i.e. there exist a subsequence $(a_{nk})_{k=1}^{\infty}$ of the bounded sequence $(a_n)_{n=1}^{\infty}$ which converges to the least upper bound of G.

4. Conclusion

We have extended the Bolzano Weierstrass theorem by defining a subset of real numbers G whose supremum always exists for any bounded sequence and also there exists a subsequence which converges to that supremum.

References

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[2] Robert G Bartle and Donald R Sherbert. *Introduction to real analysis*, volume 2. Wiley New York, 2000.
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