Extended Bolzano Weierstrass Theorem

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Abstract: In this paper we have proved the Extended Bolzano Weierstrass theorem i.e. "Let \( (a_n)_{n=1}^{\infty} \) be an bounded sequence. Define the set \( G = \{ x \in \mathbb{R} \mid x < a_n \} \) for infinitely many terms \( a_n \). Then \( \exists \) a subsequence \( (a_{n_k}) \) converges to a supremum of the set \( G \)."

Keywords: Convergence, Bounded, supremum, Completeness

1. Introduction

We define the convergence of a sequence, bounded sequence and the supremum. Then in Theorem 2.1 we showed that "\( (a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty} \) are three real sequences such that \( a_n \leq b_n \leq c_n \forall n \). If \( (a_n)_{n=1}^{\infty} \) and \( (c_n)_{n=1}^{\infty} \) both converges to the same limit then \( (b_n)_{n=1}^{\infty} \) also converges to the same limit." Then in Theorem 2.2 we showed that "Supremum of a bounded set is the least element of the set of upper bounds of that set.”

Then Finally we proved the Extended Bolzano Weierstrass theorem in which we defined a set for each bounded sequence whose supremum is the limit of the subsequence of that sequence.

Notation: \( B(a, \epsilon) \) denotes an neighbourhood of a which is an open ball with centre \( a \) and radius \( \epsilon > 0 \) in \( \mathbb{R} \).

1.1 Convergence of Sequence

A Real sequence \( (x_n)_{n=1}^{\infty} \) is said to be converges to a real number \( x \) if for any \( \epsilon > 0 \) there exist a natural number \( m \in \mathbb{N} \) such that
\[
|x_n - x| < \epsilon \quad n \geq m
\]
OR
\[
\text{A Real sequence } (x_n)_{n=1}^{\infty} \text{ is said to be converges to a real number } x \text{ if the sequence } (x_n)_{n=1}^{\infty} \text{ is eventually in each neighbourhood of } x.[3]
\]

1.2 Bounded Sequence

A Sequence \( (x_n)_{n=0}^{\infty} \) is said to be bounded if there exist a real number \( M > 0 \) such that
\[
|x_n| \leq M \forall n \in \mathbb{N}
\]

1.3 Supremum (Least upper bound)

Let \( S \) be any non empty subset of real numbers \( R \). Then a real number \( \alpha \) is said to be the supremum of the set \( S \) if
1. \( x \leq \alpha \forall x \in S \)
2. For any \( \epsilon > 0 \) there exist an element \( x \in S \) such that
\[
\alpha - \epsilon < x
\]

1.4 Least element of a set

Let \( S \) be any subset of \( R \) then a real number \( \alpha \) is said to be a least element of the set \( S \) if
1. \( \alpha \leq x \forall x \in S \)
2. \( \alpha \in S \)

1.5 Completeness Property

Every non empty bounded above subset of real numbers have a least upper bound (i.e. supremum).

1.6 Properties of order relation in \( R \)

1. Law of Trichotomy:- For any two numbers \( a, b \) in \( R \) only one of the following holds
   \[
a > b, a = b, a < b
\]
2. Transitive:- For all \( a, b, c \in R \) we have
   \[
a < b \text{ and } b < c \Rightarrow a < c
\]
3. Compatibility of order relation with addition :- For all \( a, b, c \in R \) we have
   \[
a < b \Rightarrow a + c < b + c
\]

2. Related Theorems

Theorem 2.1. Let \( (a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty} \) are three real sequences such that \( a_n \leq b_n \leq c_n \forall n \in \mathbb{N} \). If \( (a_n)_{n=1}^{\infty} \) and \( (c_n)_{n=1}^{\infty} \) both converges to the same limit then \( (b_n)_{n=1}^{\infty} \) also converges to the same limit.[2]

Proof. Let \( (a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty} \) and \( (c_n)_{n=0}^{\infty} \) are any sequence of real numbers such that
\[
a_n \leq b_n \leq c_n \forall n
\]
Suppose \( (a_n)_{n=0}^{\infty} \) and \( (c_n)_{n=0}^{\infty} \) converges to the same limit say \( \alpha \).

Then for any \( \epsilon > 0 \) there exist a natural number \( n_1 \) and \( n_2 \) such that
\[
|a_n - \alpha| < \epsilon \forall n \geq n_1
\]
and
\[
|c_n - \alpha| < \epsilon \forall n \geq n_2
\]
\[
\Rightarrow -\epsilon < a_n - \alpha < \epsilon \forall n \geq n_1 \text{ and } -\epsilon < c_n - \alpha < \epsilon \forall n \geq n_2
\]
Now let \( N = \max\{n_1, n_2\} \). Then as
\[
a_n \leq b_n \leq c_n \forall n \Rightarrow a_n \leq b_n \leq c_n \forall n
\]
By the Compatibility of order relation with addition in $\mathbb{R}$ (1.4) we have
$$a_n - \alpha \leq b_n - \alpha \leq c_n - \alpha \quad \forall \ n \geq N$$

Then by (1) we have
$$-\epsilon < a_n - \alpha \leq b_n - \alpha \leq c_n - \alpha < \epsilon \quad \forall \ n \geq N$$

$$\rightarrow b_n - \alpha | \leq \epsilon \quad \forall \ n \geq N$$

The sequence $(b_n)_{n=0}^{\infty}$ also converges to the same limit $\alpha$.

**Theorem 2.2:** Supremum of a bounded subset of Real numbers is the least element of the set of all the upper bound of that set

**Proof.** Let $S$ be any bounded subset of $\mathbb{R}$ Then By the Completeness property $S$ have a least upper bound say $\alpha$

Define a set
$$T = \{ a \in \mathbb{R} : a \text{ is an upper bound of } S \}$$

By the definition of least upper bound, clearly $\alpha$ is a upper bound of $S$
$$\Rightarrow \alpha \in T$$

Suppose if possible $\alpha$ is not a lower bound of $T$
$$\Rightarrow \exists \ a \in T \text{ such that } a < \alpha$$

Let $\epsilon = \alpha - a > 0$.

Then by the definition $\exists b \in S$ such that
$$\alpha - \epsilon < b \Rightarrow a < b$$

which contradicts that $a$ is a upper bound of $S$
$$\Rightarrow a \text{ is a lower bound of } T$$

$$\therefore \alpha \text{ is a least element of } T$$

3. Results and Discussion

**Theorem 3.1.** [BolzanoWeierstrass theorem ]

Let $(a_n)_{n=0}^{\infty}$ be any sequence of real numbers which is bounded . Then there exists a subsequence $(a_{nk})_{k=0}^{\infty}$ of this sequence which converges to a real number.

**Theorem 3.2.** [Extended Bolzano Weierstrass theorem ]

Let $(a_n)_{n=1}^{\infty}$ be an bounded sequence . Define the set
$$G = \{ x \in \mathbb{R} : x < a_n \text{ for infinitely many terms } a_n \}$$

Then $\exists$ a subsequence $(a_{nk})$ converges to a supremum of the set $G$.

**Proof.** Since $(a_n)$ is an bounded sequence.
$$\therefore \exists \text{ a real } M > 0 \text{ such that } |a_n| \leq M \quad \forall \ n \in \mathbb{N}$$

Clearly $(-\infty, -M) \subseteq G \Rightarrow G \neq \emptyset$

Therefore $G$ is an non empty bounded above subset of real numbers which is bounded above.

Hence By the completeness property of the set of real numbers we have the set $G$ have a least upper bound $\text{let } \text{Sup}(G) = \gamma$

Let $k \in \mathbb{N}$ be arbitrary
$$\Rightarrow x \leq \gamma < \gamma + \frac{1}{k} \quad \forall \ x \in G$$

i.e. $\gamma + \frac{1}{k}$ is an upper bound of $G$ which is strictly greater than $\gamma$

Since the supremum of a subset of a real number is the least element of the set of all the upper bound of that set (Theorem 2.2)
$$\therefore \gamma + \frac{1}{k} \notin G \Rightarrow \gamma + \frac{1}{k} \in G$$

$$\Rightarrow \gamma + \frac{1}{k} < a_n \text{ for almost finitely many } n$$

Therefore, For infinitely many natural numbers we have
$$\alpha + \frac{1}{k} \geq a_m$$

(1)

Now
$$\text{For the same } k, \alpha - \frac{1}{k} \text{ is not an upper bound of } G.$$ Therefore there exist an element $x \in G$ such that
$$\alpha - \frac{1}{k} < x$$

Since $x \in G$ then by the definition of $G$ for infinitely many $n \in \mathbb{N}$ we have
$$\alpha - \frac{1}{k} < x < a_n$$

(2)

i.e there are infinitely many terms of the sequence are strictly less than $\alpha - \frac{1}{k}$ for each natural number $k$.

For each $k \in \mathbb{N}$

Define
$$A_k = \{ a_n : \alpha + \frac{1}{k} \geq a_n \}$$

$$B_k = \{ a_n : \alpha - \frac{1}{k} < a_n \}$$

From (1) and (2) we have each $A_k$ and $B_k$ are infinite for all $k \in \mathbb{N}$

Define
$$n_k = \min \{ m : a_m \in A_k \cap B_k \}$$

Then
$$\alpha - \frac{1}{k} < a_{nk} \leq \alpha + \frac{1}{k} \quad \forall \ k \in \mathbb{N}$$

Since
$$\alpha - \frac{1}{k} \rightarrow \alpha \quad as \quad k \rightarrow \infty$$

and
$$\alpha + \frac{1}{k} \rightarrow \alpha \quad as \quad k \rightarrow \infty$$

Then by Theorem 2.1 we have
$$a_{nk} \rightarrow \alpha \quad as \quad k \rightarrow \infty$$

i.e. there exist a subsequence $(a_{nk})_{k=1}^{\infty}$ of the bounded sequence $(a_n)_{n=1}^{\infty}$ which converges to the least upper bound of $G$. 

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4. Conclusion

We have extended the Bolzano Weierstrass theorem by defining a subset of real numbers $G$ whose supremum always exists for any bounded sequence and also there exists a subsequence which converges to that supremum.

References