A Note on Fuzzy Ideals of Ring

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Abstract: ***Fuzzy group, fuzzy subgroup and their fuzzification are defined by various researchers. We try to introduce the notion of fuzzy ideals of ring and level subset of the ring and establish some interesting resulfs for fuzzy ideals of ring and have also derived some results on the level subst of a ring R.***

Keywords: Fuzzy ideal, Level subset, Fuzzy right Ideal, Fuzzy left ideal

1. Introduction

In 1971 **Rosenfeld** defined the concept of fuzzy subgroup of a group. Since then several attempts have been made to investigate the properties of fuzzy subgroups. However, the process of fuzzification is relatively slow in ring theory. In this paper our aim is to establish some interesting results for fuzzy ideals of ring and have also derived some results on the level subset of a ring R. Fuzzy subsets of set was introduced by **zadeh** in 1965. Firstly **Rosenfeld** applied this concept to the theory of groupoids and groups. Liu defined and studied the notion of fuzzy ideal of a ring whereas fuzzy ideal of a group were studied by Rosenfeld. Here, we will, derive some results on fuzzy ideals of a ring.

2. Fuzzy Ideal of R

A fuzzy subset μ of a ring R is called a **fuzzy left (right) ideal** of R if and only if.

1) $\mu(x-y) \geq \text{Min}\{\mu(x), \mu(y)\}$
2) $\mu(x, y) \geq \mu(y)$ or $\mu(xy) \geq \mu(x)$ respectively for all x, $y \in R$.

And the fuzzy subset μ of R be called a **fuzzy ideal** of R if for all x, y in R

1) $\mu(x-y) \geq \text{Min}\{\mu(x), \mu(y)\}$
2) $\mu(x\,y) \geq \text{Max}\{\mu(x), \mu(y)\}$

In other words, if μ is a fuzzy left ideal as well as fuzzy right ideal then it will be called as fuzzy ideal.

Theorem 3.1: If μ be a fuzzy left (right) ideal of a ring R, then -

1) $\mu(x) \leq \mu(0)$
2) $\mu(x) = \mu(-x)$

and

3) $\mu(x+y) \geq \text{Min}\{\mu(x), \mu(y)\}\ \forall\, x, y \in R$.

Proof: Let (R, + , **.**) be a ring and μ is a fuzzy left (right) ideal of R.

We have $\mu(0) = \mu(x-x) \geq \text{Min}\{\mu(x), \mu(x)\}\ \forall\, x \in R$.

$= \mu(x)$

$\therefore\ \mu(0) \geq \mu(x)$

i.e. $\mu(x) \leq \mu(0)$ .... (i)

Hence (I) Proved

Further, we have

$\mu(0-x) \geq \text{Min}\{\mu(0), \mu(x)\}$

$= \mu(x)$ (using (i))

i.e. $\mu(-x) \geq \mu(x)$ .... (ii)

On replacing x by – x, we get -

$\mu\{-(-x)\} \geq \mu(-x)$

i.e. $\mu(x) \geq \mu(-x)$ ....(iii)

Hence from (ii) & (iii), we get -

$\mu(x) = \mu(-x),\ \forall\, x \in R$. ....(iv)

Hence (ii) Proved

Now for the proof of (III)rd result

$\mu(x+y) = \mu\{x-(-y)\} \geq \text{Min}\{\mu(x), \mu(-y)\}$

$= \text{Min}\{\mu(x), \mu(y)\}$ (Using (iv)

$\therefore\ \mu(x+y) \geq \text{Min}\{\mu(x), \mu(y)\}\ \forall\, x, y \in R$.

Hence (iii) proved.

Theorem 3.2: If μ is a fuzzy ideal of a ring R, then -

1) $\mu(0) \geq \mu(x) \geq \mu(1),\ \forall\, x \in R$.
2) $\mu(x-y) = \mu(0)$ implies $\mu(x) = \mu(y)$

Proof: Let μ be a fuzzy left (right) ideal of a ring R. Then $\forall$ x, y in R,

We have

1) $\mu(x-y) \geq \text{Min}\{\mu(x), \mu(y)\}$

and

2) $\mu(x\,y) \geq \text{Max}\{\mu(x), \mu(y)\}$

We have,

$\mu(o) = \mu(x-x) = \mu\{x+(-x)\} \geq \text{Min}\{\mu(x), \mu(-x)\}$

$= \text{Min}\{\mu(x), \mu(x)\}$

Using theorem (3.1)

$= \mu(x)$

$\therefore\ \mu(0) \geq \mu(x)$

Further, we have

$\mu(x) = \mu(x, 1) \geq \text{Max}\{\mu(x), \mu(1)\}$

$= \mu(1)$

$\therefore\ \mu(x) \geq \mu(1)$

Thus we get that $\mu(0) \geq \mu(x) \geq \mu(1)$

Hence (i) is proved.

Now, we have

Let x and y are any two distinct elements of R such that $\mu(x - y) = \mu(0)$.

Then we have
$\mu(x) = \mu(x - y + y)$
$= \mu\{(x - y) + y\}$
$\geq \text{Min}\{\mu(x - y), \mu(y))$

Using (iii) of theorem (3.1)
$= \text{Min}\{\mu(0), \mu(y)\}$
$= \mu(y)$ (Using (i) of the theorem (3.1))
$\therefore \mu(x) \geq \mu(y)$ .... (i)

We have
$\mu(y - x) = \mu\{-(x-y)\}$
$= \mu(x - y)$
$= \mu(y - x) = \mu(x - y)$ .... (ii)
$\therefore \mu(y) = \mu(y - x + x)$
$= \mu\{(y - x) + x\}$
$\geq \text{Min}\{\mu(y - x), \mu(x)\}$
$= \text{Min}\{\mu(x - y), \mu(x)\}$
(using equation (ii))
$= \text{Min}\{\mu(0), \mu(x)\}$
$= \mu(x)$.
$\therefore \mu(y) \geq \mu(x)$ .... (iii)

Thus from equation (i) & (iii) we get.
$\mu(x) = \mu(y)$,
Hence (ii) is proved.

Theorem 3.3: μ is a fuzzy ideal of a ring (R, +, .) if and only if
1) $\mu(x + y) \geq \text{Min}\{\mu(x), \mu(y)\}$
2) $\mu(xy) \geq \text{Min}\{\mu(x), \mu(y)\}$

Proof :- Let μ is a fuzzy ideal of a ring (R, + , .). Then $\forall$ x, y $\in$ R, we have
1) $(x - y) \geq \text{Min}\{\mu(x), \mu(y)\}$
and
2) $\mu(xy) \geq \text{Min}\{\mu(x), \mu(y)\}$

On replacing y by – y in (i), we get
$\mu\{x - (-y)\} \geq \text{Min}\{\mu(x), \mu(-y)\}$
i.e. $\mu(x + y) \geq \text{Min}\{\mu(x), \mu(y)\}$
(Since $\mu(-y) = \mu(y)$ in a fuzzy ideal)
$\therefore \mu(x + y) \geq \text{Min}\{\mu(x), \mu(y)\}$
and $\mu(xy) \geq \text{Max}\{\mu(x), \mu(y)\}$ holds

Conversely, we suppose that μ is a fuzzy subset of R and is such that
$\mu(x + y) \geq \text{Min}\{(x), \mu(y)\}$
$\mu(xy) \geq \text{Max}\{\mu(x), \mu(y)\}$

Since R is a ring hence $O \in R$. Therefore, we have
$\mu(0 - x) = \text{Min}\{\mu(0), \mu(x)\}$
$\Rightarrow \mu(-x) \geq \mu(x)$

Again, we have-
$\mu(0 + x) = \mu\{(0 - (-x)\} \geq \text{Min}\{\mu(0), \mu(-x)\}$
$\Rightarrow \mu(x) \geq \mu(-x)$
$\therefore \mu(-x) = \mu(x)$

Finally, we have -
$\mu(x + y) = \mu\{x - (-y)\} \geq \text{Min}\{\mu(x), \mu(-y)\}$
$= \text{Min}\{\mu(x), \mu(y)\}$
$\therefore \mu(x + y) \geq \text{Min}\{\mu(x), \mu(y)\}$
Hence μ is a fuzzy ideal of R.

Theorem 3.4: If μ is a fuzzy left (right) ideal of R and if $0 \leq t \leq \mu(0)$, then μ_t is a left (right) ideal of R.

Proof: Let R is a ring and μ is a fuzzy left ideal of R. We also assume that
$0 \leq t \leq \mu(0)$. Since $\mu(0) \geq t$, hence from the definition of level subset, we get that $0 \in \mu_t$.

Thus we conclude that $\mu \neq \phi$.

Let X and Y are any two elements of μ_t. Hence $\mu(x) \geq t$ and $\mu(y) \geq t$. Since μ is a fuzzy left ideal of R, therefore
$\mu(x - y) \geq \text{Min}\{\mu(x), \mu(y)\}$
$\geq \text{Min}\{t, t\}$
$= t.$
$\therefore \mu(x - y) \geq t \Rightarrow x - y \in \mu_t$.
Thus, we get that for all $x, y \in \mu_t \Rightarrow x - y \in \mu_t$.
Now we assume that $r \in R$. Since A is a fuzzy left ideal, hence
$\mu(r x) \geq \mu(x) \geq t$
i.e. $\mu(r x) \geq t \Rightarrow rx \in \mu_t$.
Hence μ_t is a left ideal of R.

Following the same footsteps as above we can show that μ_t is also a fuzzy right ideal of R. Hence μ_t is a fuzzy ideal of R.

3. Conclusion

Fuzzy ideal of a ring is quite a new branch of Mathematics and is an area of high research potential. Several research papers related to fuzzy group. Fuzzy subgroup etc are published but the process of fuzzification is relatively slow in Ring theory. We have try to fuzzify the fuzzy ideal of Ring and proved some interesting results related to Ring theory with the help of fuzzy ideals in Iucid manner, which can provide the direction for further research in this area.

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