APPLICATION OF THE GENERALIZED DIFFERENCE OPERATOR OF THE n^{TH} KIND

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Abstract:

In this paper, the authors extend the theory of the generalized difference operator Δ_l to the Generalized difference operator of the n^{th} kind $L = \{l_1, l_2, l_3, \ldots, l_n\}$ for the positive reals $l_1, l_2, l_3, \ldots, l_n$ and obtain some interesting results on n^{th} kind . Also by defining its inverse, we establish a few formulae for the sum of third partial sums of products of several powers of the arithmetic progressions in number theory. Appropriate examples are provided to illustrate the results.

Keywords: Generalized difference operator; Generalized polynomial factorial; inverse operator.

1 Introduction

The theory of difference equations is based on the operator Δ defined as

$$\Delta u(k) = u(k+1) - u(k), k \in \mathbb{N} = \{0, 1, 2, \ldots\} \tag{1}$$

Even though many authors $[1-3]$ have suggested the definition of the difference operator Δ as

$$\Delta_l u(k) = u(k+l) - u(k), l \in (0, \infty) \tag{2}$$

no significant progress took place on this line. But,in 2006 [3] ,by taking the definition of Δ as given in [2], the theory of difference equations was developed in a difference direction and many interesting results were obtained in number theory [4].

The theory was then extended for real $l \in (0, \infty)$ and

$$\Delta_{-l} u(k) = u(n-l) - u(k)$$

and based on this definition they studied the qualitative properties of a particular difference equation and no one else has handled this operator.

In this paper, we develop theory for Δ_L, the generalized difference operator of n th kind and obtain some significant results, relations and formule in number theory.

Throughout this paper, we make use of the following assumptions:

1. q is the positive integer and l is a positive real,
2. $[x]$ denotes the integer part of x,

3. $nC_r = \frac{n!}{(n-r)!r!}$ where $0! = 1, n! = 1, 2, \ldots, r,$

4. $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}(a) = \{a, a+1, \ldots\}$

5. $L = (l_1, l_2, \ldots, l_n)$

6. $0(L) = \{\phi\}$

7. $1(L) = \{\{l_1\}, \{l_2\}, \ldots, \{l_n\}\}$

8. $p(L) = \{\{l_1, l_2, \ldots, l_n\}\}$

9. $\mathbb{N}_l(j) = \{j, j+l, \ldots\}$

10. $\wp(L) = \bigcup_{r=0}^{n} r(L)$,the power set of L

2 Preliminaries

In this section, we present some definitions and some results on generalized difference operator and polynomial factorials, which will be useful for subsequent discussion.

Definition 2.1

[4] If $\{u(k)\}, k \in [0, \infty)$ be a real or complex valued function.Then for $l \in (0, \infty)$, the generalized difference operator Δ_l on $u(k)$ is defined as

$$\Delta_l u(k) = u(k+l) - u(k) \tag{3}$$

Definition 2.2

The generalized difference operator of the nth kind for the function $u(k), k \in [0,\infty)$, denoted by Δ_L, is defined as

$$\Delta_L u(k) = \sum_{r=0}^{n} (-1)^{n-r} \left\{ \sum_{A \in \wp = \bigcup r(L)} u\left(k + \sum_{l \in A} l\right) \right\} \tag{4}$$

Definition 2.3

[5] Let n be a positive integer and $l \in (0,\infty)$ Then generalized positive polynomial factorial for k is defined as

$$k_l^n = k\,(k-l)\,(k-2l)\ldots(k-(n-1)\,l) \tag{5}$$

When $l = 1, k_1^n = k\,(k-1)\,(k-2)\ldots(k-(n-1)) = k^{(n)}$

Definition 2.4

If $l \in 0,\infty$ and $n \in \mathbb{N}(1)$, then the inverse operator is such an operator Δ_l^{-1} that if

$$\Delta_l(z(k)) = u(k), then z(k) = \Delta_l^{(-1)}(u(k)) + a_j \tag{6}$$

where a_j is constant.

Definition 2.5

[8] The inverse of the generalized difference operator denoted by Δ_L^{-1} on $u(k)$ is defined as, if $\Delta_L v(k) = u(k)$, then

$$\Delta_L^{-1} u(k) + a_{(n-1)j}\left(\frac{k_l^{(n-1)}}{(n-1)!l^{(n-1)}}\right) + a_{(n-2)j}\left(\frac{k_l^{(n-2)}}{(n-2)!l^{(n-2)}}\right) + \ldots + a_{1j}\left(\frac{k_l^{(1)}}{l}\right) + a_{0j}, \tag{7}$$

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where a_j is a constant.

Remark 2.4

[8] For $n \in N(1)$, if s_i^n and S_i^n are the striling numbers of the first and second kinds respectively, then

$$(k)_l^{(n)} = \sum_{r=1}^{n} s_r^n (l)^{n-r} k^r ; k^n = \sum_{r=1}^{n} s_r^n (l)^{n-r} (k)_l^{(n)} \tag{8}$$

3 Main results

In this section, we derive the formula for the sum of general partial sums of products of several powers of consecutive terms of an arithmetic progression.

Theorem 3.1

If $p \in \mathbb{N}(2), l \in (0, \infty)$ and $k \in (pl, \infty)$, then

$$\underbrace{\Delta^{(-1)}_{l,l,\ldots,l}}_{n-times} u(k) = \sum_{r_p=2}^{r_p^*} \sum_{r_{p-1}=1}^{r_{p-1}^*} \ldots \sum_{r_1=0}^{r_1^*} u(k - r_p l - r_{p-1} l - \ldots - r_1 l)$$

$$+ a_{(p-1)j} \left(\frac{k_l^{(p-1)}}{(p-1)! l^{(p-1)}} \right) + a_{(p-2)j} \left(\frac{k_l^{(p-2)}}{(p-2)! l^{(p-2)}} \right) + \ldots + a_{1j} \left(\frac{k_l^{(1)}}{l} \right) + a_{0j}, \tag{9}$$

where $r_p^* = \left[\frac{k}{l} \right], r_{p-i}^* = r_{p-(i-1)}^* - r_{p-(i-1)}$, for $i = 1, 2, \ldots, p-1$ and $c_{0j}, c_{1j}, \ldots, c_{(p-1)j}$ are constants for all $k \in \mathbb{N}_l(j), j = k - \left[\frac{k}{l} \right] l$ and when $p = 1$,

$$\Delta_l^{-1} u(k) = \sum_{r=1} \left[\frac{k}{l} \right] u(k - rl) + a_{0j}.$$

Proof.

Since

$$\Delta_l \left\{ \sum_{r=1}^{\left[\frac{k}{l}\right]} u(k-rl) \right\} = \sum_{r=1}^{\left[\frac{k}{l}\right]+1} u(k+l-rl) - \sum_{r=1}^{\left[\frac{k}{l}\right]} u(k-rl) = u(k),$$

by definition 2.4, we obtain

$$\sum_{r=1}^{\left[\frac{k}{l}\right]} u(k-rl) = \Delta_l^{-1} u(k) + a_{0j}. \tag{10}$$

Since $\Delta_{l,l}^{-1} = \Delta_l^{-1}\left(\Delta_l^{-1}\right)$, by taking Δ_l^{-1} on both sides of (9) and again applying (9), we get

$$\sum_{r_2=2}^{\left[\frac{k}{l}\right]} \sum_{r_1=0}^{\left[\frac{k}{l}\right]-r_2} u(k-r_2l-r_1l) + \frac{k}{l}a_{1j} + a_{0j} = \Delta_{l,l}^{-1}u(k).$$

Proceeding like this way and using the relation $\Delta_{l,l,l\ldots,l} = \Delta_l\Delta_l\ldots\Delta_l$, we get

$$\underbrace{\Delta_{l,l,\ldots,l}^{(-1)}}_{p-times} u(k) = \sum_{r_p=2}^{r_p^*} \sum_{r_{p-1}=1}^{r_{p-1}^*} \ldots \sum_{r_1=0}^{r_1^*} u(k-r_pl-r_{p-1}l-\ldots-r_1l)$$

$$+ a_{(p-1)j}\left(\frac{k_l^{(p-1)}}{(p-1)!l^{(p-1)}}\right) + a_{(p-2)j}\left(\frac{k_l^{(p-2)}}{(p-2)!l^{(p-2)}}\right) + \ldots + a_{1j}\left(\frac{k_l^{(1)}}{l}\right) + a_{0j},$$

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Lemma 3.2

If p, q are positive integers, l is a real and $p > ql$, then

$$(k-(p-1)l)^q - (p-1)(k-(p-2)l)^q + \ldots + (-1)^{p-1}k^q = \frac{1}{p}\sum_{r=1}^{q} S_r^q l^{q-r} k^{(r)}_{\underbrace{l,l,\ldots,l}_{p-times}}$$

$$\Delta^{(-1)}_{\underbrace{l,l,\ldots,l}_{p-times}} k_l^q = \frac{k^{(q+2p-1)}_{\underbrace{l,l,\ldots,l}_{p-times}}}{p(q+1)(q+2)\ldots(q+2p-1)l^{2p-1}} + a_{(p-1)j}\left(\frac{k_l^{(p-1)}}{(p-1)!l^{(p-1)}}\right)$$

$$+ a_{(p-2)j}\left(\frac{k_l^{(p-2)}}{(p-2)!l^{(p-2)}}\right) + \ldots + a_{1j}\left(\frac{k_l^{(1)}}{l}\right) + a_{0j}, \tag{11}$$

$$k_l^{(q)} = \frac{1}{p}\sum_{r=1}^{q} S_r^q l^{q-r} \Delta^{(-1)}_{\underbrace{l,l,\ldots,l}_{(p-1)-times}} k^{(r)}_{\underbrace{l,l,\ldots,l}_{p-times}} \tag{12}$$

Theorem 3.3

If p, q are positive integers, l is a real and $p > ql$, then

$$\sum_{r_p=2}^{r_p^*}\sum_{r_{p-1}=1}^{r_{p-1}^*}\ldots\sum_{r_1=0}^{r_1^*}(k-r_pl-r_{p-1}l-\ldots-r_1l)_l^{(m)} = \frac{k^{(q+2p-1)}_{\underbrace{l,l,\ldots,l}_{p-times}}}{p(q+1)(q+2)\ldots(q+2p-1)l^{2p-1}}$$

$$+ a_{(p-1)j}\left(\frac{k_l^{(p-1)}}{(p-1)!l^{(p-1)}}\right) + a_{(p-2)j}\left(\frac{k_l^{(p-2)}}{(p-2)!l^{(p-2)}}\right) + \ldots + a_{1j}\left(\frac{k_l^{(1)}}{l}\right) + a_{0j},$$

where a_{ij}'s are obtained by solving p equations by putting $k = (m+a)l + j$ for $a = p-1, p, p+1, \ldots, 2p-2$.

Proof:

The following theorem gives the formula for the sum of $(n-1)$ times partial sums (ie., $\underbrace{\text{partial sums of partial sums of partial sums of}}_{(p-1)-times}$) for products of x_i^{th}, $(i=1,2,\dots,q)$ powers of q- consecutive terms $k^{x_1}(k-l)^{x_2}\dots(k-(q-1)l)^{x_q}$ of an arithmetic progression $k, k-l, \dots, j$, where $j = k - \left[\frac{k}{l}\right] l$.

Theorem 3.4

Let S_r^t be the strilings number of the second kind, $x_1, x_2, \dots, x_q$ are positive integers and $k \in [X_q l + j, \infty)$, where X_q. Then,

$$\sum_{r_p=2}^{r_p^*}\sum_{r_{p-1}=1}^{r_{p-1}^*}\dots\sum_{r_1=0}^{r_1^*}\left[\prod_{t=1}^{m}(k-r_pl-r_{p-1}l-\dots-r_1l)_l^{(pt)}\right]$$

$$= \sum_{i_1=0}^{x_2}\sum_{i_2=0}^{x_3}\dots\sum_{i_{q-1}=0}^{x_q}\sum_{r=1}^{x_m-\sum i_s}\begin{pmatrix}x_2\\ i_1\end{pmatrix}\begin{pmatrix}x_3\\ i_2\end{pmatrix}\dots\begin{pmatrix}x_q\\ i_{m-1}\end{pmatrix}(-1)^{i_1}(-2)^{i_2}\dots$$

$$\times(-(m-1))^{i_{m-1}}S_r^{x_m-\sum i_s}(l)^{x_q-(r+n)}\frac{k_l^{(r+n)}}{\prod_{i=1}^{n}(r+i)} + a_{(p-1)j}\left(\frac{k_l^{(p-1)}}{(p-1)!l^{(p-1)}}\right)$$

$$+ a_{(p-2)j}\left(\frac{k_l^{(p-2)}}{(p-2)!l^{(p-2)}}\right) + \dots + a_{1j}\left(\frac{k_l^{(1)}}{l}\right) + a_{0j} \qquad (13)$$

where $\sum i_s = i_1 + i_2 + \dots, i_{q-1}$ and the constants $c_{(n-1)j}, c_{(n-2)j}, \dots, c_0 j$ are given by solving the n equations obtained by putting $k = (p_m + a)l + j$ for

$a = n-1, n, n+1, \ldots, 2n-2$

Proof:

From the binomial Theorem and (9), we find

$$k^{x_1}(k-l)^{x_2}\ldots,(k-(m-1)l)^{x^m}$$
$$= \sum_{i_1=0}^{x_2}\sum_{i_2=0}^{x_3}\cdots\sum_{i_{q-1}=0}^{x_q}\sum_{r=1}^{x_m-\sum i_s}\begin{pmatrix}x_2\\ \\ i_1\end{pmatrix}\begin{pmatrix}x_3\\ \\ i_2\end{pmatrix}\cdots\begin{pmatrix}x_q\\ \\ i_{m-1}\end{pmatrix}$$
$$\times(-1)^{i_1}(-2)^{i_2}\ldots(-(m-1))^{i_m-1}S_r^{x_m-\sum i_s}(l)^{x_q-r} - k_l^{(r)}. \tag{14}$$

Now, applying the inverse operator of the nth kind and making the substitution $k = (P_m + a)l + j$ for $a = n-1, n, n+1, \ldots, 2n-2$ in (15), we obtain the required result.

The following corollary shows the formula for the sum of partial sums for products of the p_i^{th} powers of m - consecutive terms $k^{x_1}(k-l)^{x_2}\ldots,(k-(m-1)l)^{x^m}$ of the arithmetic progression $k, k-l, \ldots, j$, where $j = k - \left[\frac{k}{l}\right] l$.

Corollary 3.5

If $P_m, \sum i_s, S_r^n, k$ and l are as in Theorem 3.4, then

$$\sum_{r_2=2}^{\left[\frac{k}{l}\right]}\sum_{r_1=0}^{\left[\frac{k}{l}\right]-r_2}\left[\prod_{t=1}^{m}(k-r_2l-r_1l-(t-1)l)^{pt}\right]$$
$$= \sum_{i_1=0}^{x_2}\sum_{i_2=0}^{x_3}\cdots\sum_{i_{q-1}=0}^{x_q}\sum_{r=1}^{x_m-\sum i_s}\begin{pmatrix}x_2\\ \\ i_1\end{pmatrix}\begin{pmatrix}x_3\\ \\ i_2\end{pmatrix}\cdots\begin{pmatrix}x_q\\ \\ i_{m-1}\end{pmatrix}$$

$$\times(-1)^{i_1}(-2)^{i_2}\dots(-(m-1))^{i_{m-1}}S_r^{x_m-\sum i_s}(l)^{x_q-(r+2)}\frac{k_l^{(r+2)}}{(r+1)(r+2)}+\frac{a_{1j}k}{l}+a_{0j}, \tag{15}$$

where a_{0j}, a_{1j}, and a_{2j} are constants obtained by solving the two simultaneous equations obtained by substituting $k=(p_m+a)l+j$

Proof:

The proof follows by putting $n=2$ in theorem 3.4

The following corollary putting $n=3$ in theorem 3.4

Corollary 3.6

If $P_m, \sum i_s, S_r^n, kandl$ are as in theorem 3.4,then

$$\sum_{r_3=2}^{\left[\frac{k}{l}\right]}\sum_{r_2=1}^{\left[\frac{k}{l}\right]-r_3}\sum_{r_1=0}^{\left[\frac{k}{l}\right]-r_3-r_2}\left[\prod_{t=1}^{m}(k-r_2l-r_1l-(t-1)l)^{pt}\right]$$

$$=\sum_{i_1=0}^{x_2}\sum_{i_2=0}^{x_3}\dots\sum_{i_{q-1}=0}^{x_q}\sum_{r=1}^{x_m-\sum i_s}\begin{pmatrix}x_2\\ i_1\end{pmatrix}\begin{pmatrix}x_3\\ i_2\end{pmatrix}\dots\begin{pmatrix}x_q\\ i_{m-1}\end{pmatrix}$$

$$\times(-1)^{i_1}(-2)^{i_2}\dots(-(m-1))^{i_{m-1}}S_r^{x_m-\sum i_s}(l)^{x_q-(r+2)}\frac{k_l^{(r+3)}}{(r+1)(r+2)(r+3)}$$

$$+\frac{a_{2j}k_l^2}{2!l^2}+\frac{a_{1j}k}{l}+a_{0j}, \tag{16}$$

where c_{0j}, a_{1j}, and a_{2j} are constants obtained by solving the two simultaneous equations obtained by substituting $k=(p_m+a)l+j$

4 Applications

In this section, we present some examples to illustrate the main results. The following example is an illustration of corollary 3.5.

Example 4.1

The sum of partial sums of products of first, second and third powers of three consecutive terms $(k(k-l)^2(k-2l)^3)$ of A.P. $k, k-l, \ldots, j$ where $j = k - \left[\frac{k}{l}\right] l$ is given by

$$\begin{aligned}
&\sum_{t=2}^{\left[\frac{k}{l}\right]} \sum_{s=0}^{\left[\frac{k}{l}\right]-t} (k-tl-sl)(k-l-tl-sl)^2(k-2l-tl-sl)^3 \\
&\quad = \frac{1}{56l^2}\left[k_l^{(8)} - (3l+j)_l^{(8)}\right] + \frac{1}{6l}\left[k_l^{(7)} - (3l+j)_l^{(7)}\right] \\
&\quad + \frac{1}{3}\left[k_l^{(6)} - (3l+j)_l^{(6)}\right] + \frac{l}{10}\left[k_l^{(5)} - (3l+j)_l^{(5)}\right] \\
&\quad + \sum_{t=2}^{3}\sum_{s=0}^{3-t}(3l+j-tl-sl)(3l+j-l-tl-sl)^2(3l+j-2l-tl-sl)^3 \\
&\quad \times \left(\frac{k-(3l+j)}{l}\right)\left\{\sum_{s=1}^{5}(3l+j-sl)(2l+j-sl)^2(l+j-sl)^3\right\} \\
&\quad - \left\{\frac{1}{56l^2}\left[(4l+j)_l^{(8)} - (3l+j)_l^{(8)}\right] + \frac{1}{6l}\left[(4l+j)_l^{(7)} - (3l+j)_l^{(7)}\right]\right\} \\
&\quad + \left\{\frac{1}{3}\left[(4l+j)_l^{(6)} - (3l+j)_l^{(6)}\right] + \frac{l}{10}\left[(4l+j)_l^{(5)} - (3l+j)_l^{(5)}\right]\right\}
\end{aligned}$$

Solution:

By taking $x_1 = 1, x_2 = 2, x_3 = 3, n = 2, m = 3$ in corollary 3.5, we find

$$\sum_{t=2}^{\left[\frac{k}{l}\right]} \sum_{s=0}^{\left[\frac{k}{l}\right]-t} (k-tl-sl)(k-l-tl-sl)^2(k-2l-tl-sl)^3$$

$$= \frac{k_l^{(8)}}{56l^2} + \frac{k_l^{(7)}}{6l} + \frac{k_l^{(6)}}{3} + \frac{lk_l^{(5)}}{10} a_{1j} \frac{k}{l} + a_{0j} \tag{17}$$

Putting $k = (3l + j)$ and $k = (4l + j)$ in (17), we get

$$\sum_{t=2}^{3} \sum_{s=0}^{3-t} (3l + j - tl - sl)(3l + j - l - tl - sl)^2 (3l + j - 2l - tl - sl)^3$$

$$= \frac{(3l + j)_l^{(8)}}{56l^2} + \frac{(3l + j)_l^{(7)}}{6l} + \frac{(3l + j)_l^{(6)}}{3} + \frac{l(3l + j)_l^{(5)}}{10} a_{1j} \frac{3l + j}{l} + a_{0j} \tag{18}$$

$$\sum_{t=2}^{4} \sum_{s=0}^{4-t} (4l + j - tl - sl)(4l + j - l - tl - sl)^2 (4l + j - 2l - tl - sl)^3$$

$$= \frac{(4l + j)_l^{(8)}}{56l^2} + \frac{(4l + j)_l^{(7)}}{6l} + \frac{(4l + j)_l^{(6)}}{3} + \frac{l(4l + j)_l^{(5)}}{10} a_{1j} \frac{4l + j}{l} + a_{0j} \tag{19}$$

Hence, a_{0j} and a_{1j} are obtained by solving (18) and (19). Now the proof follows by substituting the values of a_{0j} and a_{1j} in (17).

$$a_{0j} = 0,\ a_{1j} = 0$$

In particular, taking $k = 20$ and $l = 4$, we obtain

$$\sum_{t=2}^{5} \sum_{s=0}^{5-t} (20 - tl - sl)(20 - l - tl - sl)^2 (20 - 2l - tl - sl)^3$$

$$= [(12)(8)^2(4)^3] = 49152$$

$$= \frac{20_l^{(8)}}{56l^2} + \frac{20_l^{(7)}}{6l} + \frac{20_l^{(6)}}{3} + \frac{20l_l^{(5)}}{10} a_{1j} \frac{20}{l} + a_{0j}$$

$$= \frac{4}{10} [20(16)(12)(8)(4)] = 49152$$

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