Fixed Point Theorem for Expansive Mappings in G - Metric Space

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Abstract: ***In this paper we obtain a result on fixed points for expansivetype mappings in G – metric spaces.Our result includes several fixed point result in ordinary metric spaces as special cases.***

Keywords: G- metric spaces, fixed Point, expansive mappings, etc

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1. Introduction

In 1963, Gahler [4] introduced the concept of 2 –metric spaces. In 2 –metric space d(x, y, z) is to be taken as the area of triangle with vertices x, y and z in R^2. In 1992, Dhage [3] introduced the concept of D-metric space. The central concept of D-metric space is different from 2 –metric spaces. Geometrically a D- metric D9x, y, z) represents the perimeter of the triangle with vertices x, y and z in R^2. However, Recently in 2005,Mustafa et. Al.[11]introduced the concept of generalized metric space called G –metric space which claims most concerning the fundamental topological structure of D –metric space that are incorrect.

To eastablish our main result we need the following definitions :

2. Preliminaries

Definition 2.1([11]) : Let X be a nonempty set and G : X ×X×X → R^+, be a function satisfying the following properties :

(G_1) G(x, y, z) = 0, if x = y = z,

(G_2) 0 < G(x, x, y) for all x, y ∈ X with x ≠ y,

(G_3) G(x, x, y) ≤ G(x, y, z) for all x, y, z ∈ X with z ≠ y,

(G_4) G(x, y, z) = G(x, z, y) = G(y, z, x) = . . . (symmetry in all three variables)

(G_5) G(x, y, z) ≤ G(x, a, a) + G(a, y, z), for all x, y, z, a∈ X (rectangle inequality)

Then the function G is called a generalized metric or, more specifically a G – metric on X and the pair (X, G) is called a G –metric space. (throughout in this paper we denote R^+ the set of all positive real numbers and N the set of all natural numbers)

Definition 2.2([11]) : Let (X, G) be a G –metric space and {x_n} be a sequence in X. A point x ∈X is said to be the limit of the sequence {x_n} if $\lim_{n,m\to\infty} G(x_n, x_m, x) = 0$ and the sequence {x_n} is said to be G-convergent to x ∈X.

Proposition 1 ([11]) : Let (X, G) be a G-metric space. Then the following are equivalent:

(i) {x_n} is G-convergent to x.

(ii) $G(x_n, x_n, x) \to 0$, as n → ∞

(iii) $G(x_n, x, x) \to 0$, as n → ∞

(iv) $G(x_n, x_m, x) \to 0$, as n, m → ∞

Definition 2.3([11]) : Let (X, G) be a G –metric space. Asequence {x_n} is called a G-Cauchy sequence if given ε> 0, there is N ∈N, such that $G(x_n, x_m, x_l)$ <ε for all n. m, l ≥ N.

Proposition 2([11]) : Let (X, G) be a G-metric space. Then the following are equivalent:

(i) The sequence {x_n} is a G-Cauchy.

(ii) For every ε> 0, there exists an N ∈N, such that $G(x_n, x_m, x_m)$ <ε for all n. m ≥ N.

Definition 2.4([11]): Let (X, G) and (X', G')be two G – metric space and let f : (X, G) → (X', G') be a function. Then f is said to be G – continuous at a point a ∈ Xif and only if given ε> 0 there exists a δ > 0 such that for x, y ∈ X and G(a, x, y) < δ implies that G'(f(a), f(x), f(y)) <ε. A function f is G-continuous on X if and only if it is G –continuous at all a ∈ X

Proposition 3([11]) : Let (X, G) and (X', G')be two G – metric space then a function f : (X, G) → (X', G') is G-continuous at x ∈ X if and only if it is G-sequentially continuous at x ; that is, whenever {x_n} is G-convergent to x we have {f(x_n)} is G- convergent to f(x).

Proposition 4([11]) : Let (X, G)be a G –metric space. Then the function G(x, y, z) is jointly continuous in all three of its variables.

Definition 2.5([11]): A G-metric space (X, G) is said to be G-complete (or complete G-metric) if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Definition 2.6 ([11]): A G-metric space (X, G) is called symmetric G-metric space if
G(x, y, y) = G(y, x, x) for all x, y ∈ X.

Definition 2.7 ([11]): Let (X, G) be a G-metric space and T be a self mapping on X. Then f is expansive mapping if there exists a constant a > 1 such that for all x, y, z ∈ X, we have

G(Tx, fy, fy) ≥ aG(x, y, z).

3. Main Results

Theorem 3.1 : Let (X G) bea complete G-metric space and let T : X ⟶ X be a surjective mapping. If there exists non-negative realsa_1, a_2, . . . a_7 with $a_1 + a_3 + a_5 > 0$,$a_2 < 1$ and
$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 > 1$, such that
(3.1.1) $G^2(Tx, Ty, Tz)$
$\geq a_1 G^2(x,y,z) + a_2 G^2(x,Tx,Tz) + a_3 G^2(y,Ty,z)$
$+ a_4 G(x,Tx,Tz).G(x,y,z) + a_5 G(y,Ty,z).G(x,y,z)$
$+ a_6 G(x,Tx,Tz).G(y,Ty,z) a_7 G(Tx,Ty,Tz).G(x,y,z)$
for all x, y, z ∈ X with x ≠ y ≠ z then T has a unique fixed point.

Proof
Let $x_0 \in X$, since T is surjective there exists $x_1 \in T^{-1}(x_0)$. Continuing in this way, we get a sequence $\{x_n\}$, where $x_n \in T^{-1}(x_{n-1})$.

If, $x_n = x_{n-1}$ for some n, then we get x_n as a fixed point of T.

Hence, without loss of generality we may assume that $x_n \neq x_{n-1}$ for every n ∈ N.

From (3.1.1) we have,
$G^2(x_{n-1}, x_{n-1}, x_n) = G^2(Tx_n, Tx_n, Tx_{n+1})$
$\geq a_1 G^2(x_n,x_n,x_{n+1}) + a_2 G^2(x_n,Tx_n,Tx_{n+1}) + a_3 G^2(x_n,Tx_n,x_{n+1})$
$+ a_4 G(x_n,Tx_n,Tx_{n+1}).G(x_n,x_n,x_{n+1})$
$+ a_5 G(x_n,Tx_n,x_{n+1}).G(x_n,x_n,x_{n+1})$
$+ a_6 G(x_n,Tx_n,Tx_{n+1}).G(x_n,Tx_n,x_{n+1}) + a_7 G(Tx_n,Tx_n,Tx_{n+1}).G(x_n,x_n,x_{n+1})$
$= a_1 G^2(x_n,x_n,x_{n+1}) + a_2 G^2(x_n,x_{n-1},x_n) + a_3 G^2(x_n,x_{n-1},x_{n+1})$
$+ a_4 G(x_n,x_{n-1},x_n).G(x_n,x_n,x_{n+1})$
$+ a_5 G(x_n,x_{n-1},x_{n+1}).G(x_n,x_n,x_{n+1})$
$+ a_6 G(x_n,x_{n-1},x_n).G(x_n,x_{n-1},x_{n+1}) + a_7 G(x_{n-1},x_{n-1},x_n).G(x_n,x_n,x_{n+1})$
Thus,
$(a_1 + a_2 + a_3)G^2(x_n, x_{n+1}, x_n) + (a_4 + a_6 + a_7)G(x_{n-1}, x_n, x_{n-1})G(x_n, x_{n+1}, x_n)$
$-(1 - a_2)G^2(x_{n-1}, x_n, x_n) \leq 0$
Or,
(3.1.2) $(a_1 + a_2 + a_3)t^2 + (a_4 + a_6 + a_7)t - (1 - a_2) \leq 0$, where
(3.1.3) $t = \frac{G(x_n,x_{n+1},x_n)}{G(x_{n-1},x_n,x_{n-1})}$
Let g : [0, ∞) ⟶ R be the function
(3.1.4) $g(t) = (a_1 + a_2 + a_3)t^2 + (a_4 + a_6 + a_7)t - (1 - a_2)$

Then by the hypothesis,
$g(0) = a_2 - 1 < 0$ and $g(1) = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 - 1 > 0$.

Let k∈ (0, 1) be the root of the equation g(t) = 0.

Then, g(t) ≤ 0 for t ≤ k and therefore
$G(x_n, x_{n+1}, x_n) \leq kG(x_{n-1}, x_n, x_{n-1})$
$\leq k^2 G(x_{n-2}, x_{n-1}, x_{n-2})$
≤ … … …
$\leq k^n G(x_0, x_1, x_0)$

Then for all n, m ∈ N, m > n, we have
$G(x_{n-1}, x_n, x_n) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \ldots + G(x_{m-1}, x_m, x_m)$
$\leq [k^n + k^{n+1} + k^{n+2} + \ldots + k^{m-1}]G(x_0, x_1, x_1)$
$\leq \frac{k^n}{1-k} G(x_0, x_1, x_1)$

So, $G(x_m, x_n, x_n) \to 0$ as n, m ⟶ ∞ and hence $\{x_n\}$ is a G-Cauchy sequence.
Hence, by the G- completeness of (X, G) there exists u ∈ X such that $\{x_n\}$ is G-converges to u.
Let $y \in T^{-1}(u)$, for n such that $x_n \neq$ u, we have
$G^2(x_n, u, u) = G^2(Tx_{n+1}, Ty, Ty)$
$\geq a_1 G^2(x_{n+1},y,y) + a_2 G^2(x_{n+1},Tx_{n+1},Ty) + a_3 G^2(y,Ty,y)$
$+ a_4 G(x_{n+1},Tx_{n+1},Ty).G(x_{n+1},y,y)$
$+ a_5 G(y,Ty,y).G(x_{n+1},y,y)$
$+ a_6 G(x_{n+1},Tx_{n+1},Ty).G(y,Ty,y) a_7 G(Tx_{n+1},Ty,Ty).G(x_{n+1},y,y)$
$= a_1 G^2(x_{n+1},y,y) + a_2 G^2(x_{n+1},x_n,Ty) + a_3 G^2(y,Ty,y)$
$+ a_4 G(x_{n+1},x_n,Ty).G(x_{n+1},y,y)$
$+ a_5 G(y,Ty,y).G(x_{n+1},y,y)$
$+ a_6 G(x_{n+1},x_n,Ty).G(y,Ty,y) a_7 G(x_n,Ty,Ty).G(x_{n+1},y,y)$

On taking limit n ⟶ ∞, we get
$0 = G^2(u, u, u) \geq a_1 G^2(u,y,y) + a_2 G^2(u,u,u) + a_3 G^2(y,u,y)$
$+ a_4 G(u,u,u).G(u,y,y) + a_5 G(y,u,y).G(u,y,y)$
$+ a_6 G(u,u,u).G(y,u,y) a_7 G(u,u,u).G(u,y,y)$
$= (a_1 + a_3 + a_5)G^2(y,u,y)$
Since, $a_1 + a_3 + a_5 > 0$, therefore u = y = Tu.

Thus, u is a fixed point of T.

To prove the uniqueness, let v be an another fixed point of T , i.e. Tv = v,

Then from (3.1.1), we have
$G^2(u, u, v) = G^2(Tu, Tu, Tv)$
$\geq a_1 G^2(u,u,v) + a_2 G^2(u,Tu,Tv) + a_3 G^2(u,Tu,v)$
$+ a_4 G(u,Tu,Tv).G(u,u,v) + a_5 G(u,Tu,v).G(u,u,v)$
$+ a_6 G(u,Tu,Tv).G(u,Tu,v) + a_7 G(Tu,Tu,Tv).G(u,u,v)$
$= a_1 G^2(u,u,v) + a_2 G^2(u,u,v) + a_3 G^2(u,u,v)$
$+ a_4 G(u,u,v).G(u,u,v) + a_5 G(u,u,v).G(u,u,v)$
$+ a_6 G(u,u,v).G(u,u,v) + a_7 G(u,u,v).G(u,u,v)$
$G^2(u, u, v) \geq (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7)\, G(u,u,v)$
Which is a contradiction, since $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 > 1$.
Thus, u = v.
Hence, T has a unique fixed point.

References

[1] Agarwal, Ravi P. and ErdalKarapinar; Remarks on coupled fixed point theorems in G- metric spaces, Fixed Point Theory and Applications, Vol. 1(2),1-33,(2013).

[2] DingwelZheng; Fixed point theorems for gegeralized θ – Φ contraction in G-metric spaces, Journal of Function Spaces, Article ID 1418725, Vol. (2018),1- 8.
[3] Dhage, B.C. ;Generalized metric spaces and mapping with fixed point, Bull. Calcutta Math.Soc., Vol.84, 329-336,(1992).
[4] Gahler, S.; 2 -Metricheraume and ihretopologischestruc.,Math.Nach, 26, 115-148, (1964).
[5] Rhoades, B.E.; A comparision of various definitions of contractive mappings, Trans. Amer.Math.Soc., 257-290, (1976).
[6] Rhoades, B.E., Generalized contractions, Bull.Calcutta Math. Soc., 71, 323-330 (1979).
[7] Saaditi R., Vaezpour P. and Rhoades B.E.; Fixed point theorem in generalized partially ordered G-metric spaces and applications, Abst. Appl. Anal., Article Id 126205, (2011).
[8] Shang Zhi Wang, Bo YuLi, Min Gao and Iseki, K.; Some fixed point theorems on expansion mappings, Math. Japonica, 29(4),631 – 636, (1984).
[9] Wong, S.Z., L.L., B.Y., Gao, Z.M. and Iseki, K.; Some fixed point theorems on expansion mappings, Math. Japonica, 29(4), 631 – 636, (1984).
[10] Zead Mustafa; A new structure for generalized metric spaces with applications to fixed point theory, Ph.D.Thesis, The University of Newcastle, Australia,2005.
[11] Zead Mustafa and Braileysims; A new approach to Generalized metric spaces, Journal of Nonlinear and Convex Analysis, 7(2), 289 – 297, (2006).
[12] Zead Mustafa, Shantanawi W. and Bataineh, M.; Existence of fixed point results in G- metric spaces, Int. J. of math. And Math. Sci., vol. 2009, Article Id 283028, doi : 10.1155/2009/283028.