Pareto Optimal Solution Analysis of Multi Objective Quadratic Programming Problem

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Abstract: ***Multi Objective optimization is nowadays a word of order in engineering projects and many more sectors. A Pareto outcome is an action that harms no one and helps at least one. The aim of this paper is to define a solution concept of Pareto optimality for a Multi Objective Quadratic Programming Problem (MOQPP) and design two methods to extract Pareto optimal solution of MOQPP. In this paper, the methods of norm ideal point and membership function are used to solve the MOQPP which are effective in getting Pareto optimal solution.***

Keyword: MOQPP, Pareto optimal solution, Norm ideal point method, Membership function method

1. Introduction

Life is about making decisions and the choice of the optimal solutions is not an exclusive subject of scientists, engineers and economists. Decision making is present in day to day life. Edgeworth (1881) was pioneer to define an optimum for multi criteria economic decision making problem at King's College, London. In 1896, Pareto, at the University of Lausanne, Switzerland, formulated his two main theories, Circulation of the Elites and the Pareto optimum: " The optimum allocations of the resources of a society is not attained so long as it is possible to make at least one individual better off in his own estimation while keeping others as well off as before in their own estimation." Many researchers have been dedicated to develop methods to solve this kind of problems. Interestingly, solution for problems with multiple objectives, also called multi criteria optimization or vector optimization are treated as Pareto optimal solutions or Pareto front.

In multi-objective programming problem, it is difficult to find an optimal solution to achieve the extreme value of every objective function, so that the decision maker is exploring for the compromise solution. Based on this idea, the concepts of Pareto optimal solution and weakly Pareto optimal solution are introduced into multi- objective programming problem [1]. The main method of solving multi-objective programming problem is converting multi-objective programming problem to single objective programming problem and we can get Pareto optimal solution or weakly Pareto optimal solution. K. Suga, S. Kato, and K. Hiyama discussed structure of Pareto optimal solution, presented the analysis process and showed the method they proposed is effective at finding an acceptable solution for multi-objective optimization problems [2]. B.A. Ghaznavi-ghosoni and E. Khorram analyzed the relationships between ∈- efficient points of multi-objective optimization problem and ϵ -optimal solutions of the related scalarized problem and obtained necessary and sufficient conditions for approximating efficient points of a general multi- objective optimization problem via approximate solutions of the scalarized problem [3]. M. D. Monfared, A. Mohades and J. Rezaei proposed a new method for ranking the solutions of an evolutionary algorithm's population, and the proposed algorithm was very suitable for the convex multi-objective optimization problems [4].

Y. Liu, Z. Peng and Y. Tan analyzed the relations among absolutely optimal solutions, effective solutions and weakly effective solutions of multi-objective programming problem [5]. Caramin M. and Dell'olmo describe scalarization techniques, ∈- constraints methods, Goal problem, multi level programming to solve Multiobjective optimization problem (MOOP) [6]. There are a lot of methods of converting multi-objective programming problem to single objective programming problem. In norm ideal point method, for the given weights, the optimal solution of the corresponding single objective programming problem is Pareto optimal solution of multi-objective programming problem. In membership function method, for the given weights, the optimal solution of the corresponding single objective programming problem is M-Pareto optimal solution of multi-objective programming problem, which is similar to Pareto optimal solution [7]. O. Britto, F. Bennis and S. Caro bring new approach to solve MOOP providing a rapid solution for Pareto set if the objective function involved are quadratic [8]. G.Zhang and H. Zuo then analyze the Pareto solution for convex MOLPP [9].

In our present study, we have extended the work by introducing the methods of Norm-Ideal point and Membership function for solving constrained MOOP involving quadratic function and discuss the effectiveness of the solution. Illustrative examples are used to highlight the potentiality. All Pareto optimal solutions and M- Pareto optimal solutions can be got through norm ideal point method and membership function method for convex Multi objective programming problem and Pareto optimal solution is equal to M- Pareto optimal solution. For any Pareto optimal solution there exist weights such that Pareto optimal solution or M- Pareto optimal solution of Multi objective programming problem is the optimal solution of the corresponding single objective programming problem.

2. Multi-Objective Optimization

Multi-objective optimization is an area of multiple criteria decision making that is concerned with mathematical optimization problems involving more than one objective function to be optimized simultaneously.

Mathematically, Multi objective decision making problems can be expressed as:

$Max\ /Min[f_1(x), f_2(x), \dots, f_k(x)]$
s/t $\quad x \in X = \{x |\ g_h(x): \{\geq, =, \leq 0,\ \ h=1, 2, \dots, m\}$
Where, $\quad f_j(x) = Objective\ for\ maximization,\ j \in J$
$\quad f_i(x) = Objective\ for\ minimization,\ i \in I$

The problem consists of n decision variables, m constraints and k objectives. $f_j(x), f_i(x)\ and\ g_h(x)\ \forall\ i, j, h$ might be linear or nonlinear.

2.1 Multi-Objective Linear Programming Problem:

Mathematically, the Multi-Objective Linear Programming Problem (MOLPP) can be defined as:

$$Max\ f_i = C_i x + \alpha_{i,} \qquad i = 1, \dots, r$$
$$Min\ f_i = C_i x + \alpha_{i,} \qquad i = r+1, \dots, s$$
$$\text{Subject to} \quad AX \begin{bmatrix} \leq \\ = \\ \geq \end{bmatrix} Bx \geq 0$$

where x is an n-dimensional vector of decision variables c is n-dimensional vector of constants, B is m-dimensional vector of constants, r is the number of objective function to be maximized, s the number of objective function to maximized plus minimized, (s-r) is the number of objective that is to be minimized, A is a $(m \times n)$ matrix of coefficients all vectors are assumed to be column vectors unless transposed, $\alpha_i (i = 1, \dots s)$ are scalar constants, $C_i x + \alpha_{i,} \quad i = 1, \dots, s$ are linear factors for all feasible solutions.

2.2 Multi-Objective Quadratic Programming Problem

Mathematically the multi objective quadratic programming problem (MOQPP) can be stated as:

$$Max\ \ F_r = \frac{1}{2} x^T P_r\ x + C_r^T x$$
$$Min\ \ F_s = \frac{1}{2} x^T P_s\ x + C_s^T x$$
$$\text{Subject to} \qquad Ax \begin{bmatrix} \leq \\ = \\ \geq \end{bmatrix} b \qquad\qquad x \geq 0$$

Where r is the number of objective function to be maximized, s is the number of objective function to be maximized and minimized and (s-r) is the number of objective function to be minimized. Here P is a $(n \times n)$ symmetric matrix of coefficients, x is an n-dimensional vector of decision variables, C is the n-dimensional vector of constants, b is m-dimensional vector of constants. A is $(m \times n)$ matrix of coefficients. All vectors are assumed to be column vectors unless transposed.

2.3 Convex Multi-objective Optimization Problem

A convex multi-objective optimization problem can be stated as follows:

$$Minimize\ \ [f_1(x), f_2(x) \dots . f_m(x)]$$
Subject to $\qquad g_j(x) \leq 0; \quad j = 1, 2, \dots p$

where x is an n -dimensional vector of decision variables, $f_1(x), f_2(x), \dots, f_m(x)$ are convex functions defined on X, and $X = \{x \mid g_j(x) \leq 0;\ j = 1, 2, \dots, p\}$ is convex set. The given Problem is called convex multi- objective programming problem.

3. Pareto Optimal Solution

A vector $x^* \in W$ is said to be Pareto optimal for a multi-objective problem if all other vectors $x \in W$ have a higher value for at least one of the objective function f_i, with $i = 1, \dots, n$, or have the same value for all the objective functions. We have the following definitions:

- A point x^* is said to be a **weak Pareto optimum** or a weak efficient solution for the multi-objective problem if and only if there is no $x \in W$ such that $f_i(x) < f_i(x^*)\ for\ all\ i \in \{1, \dots, n\}$.
- A point x^* is said to be a **strict Pareto optimum** or a strict efficient solution for the multi-objective problem if and only if there is no $x \in W$ such that $f_i(x) \leq f_i(x^*)\ for\ all\ i \in \{1, \dots, n\}$ with at least one strict inequality.
- Construct a function for convex multi-objective programming problem(MOPP)

$$\alpha_i(f_i(x)) = \frac{f_i(x) - f_i^1}{f_i^* - f_i^1}, \quad i = 1, 2, \dots m; \quad f_i^* = \min_{x \in X} f_i(x), \qquad f_i^1 = \max_{x \in X} f_i(x)$$

$x^* \in X$ is said to be a **M-Pareto optimal solution** of convex MOPP, if and only if there does not exist another $x \in X$ such that $\alpha_i\ (\ f_i(x)) \geq\ \alpha_i\ (\ f_i\ (x^*)),\ i = 1, 2, \cdots, m$, with strict inequality holding for at least one i.

- $x^* \in X$ is said to be a **Weakly M-Pareto optimal solution** of convex MOPP, if and only if there does not exist another $x \in X$ such that

$$\alpha_i\ (f_i\ (x)) > \alpha_i\ (\ (f_i(x^*)),\ i = 1, 2, \cdots, m$$

The image of all the efficient solutions is called Pareto front or Pareto curve or surface. The shape of the Pareto surface indicates the nature of the trade-off between the different objective functions. An example of a Pareto curve is showed in Fig. 3.1, where all the points between $(f_2(\hat{x}), f_1(\hat{x}))$ and $(f_2(\tilde{x}), f_1(\tilde{x}))$ define the Pareto front. These points are called non-inferior or non-dominated points.

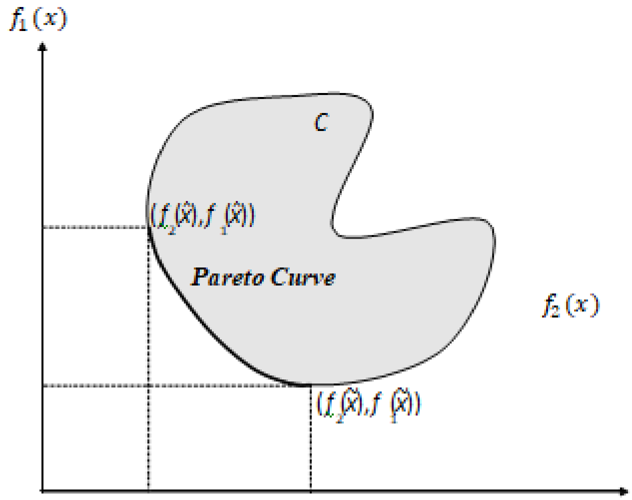

Figure 3.1: Example of a Pareto curve

An example of weak and strict Pareto optima is shown in Fig. 3.2: points p_1 and p_5 are weak Pareto optima; points p_2,

p_3 and p_4 are strict Pareto optima.

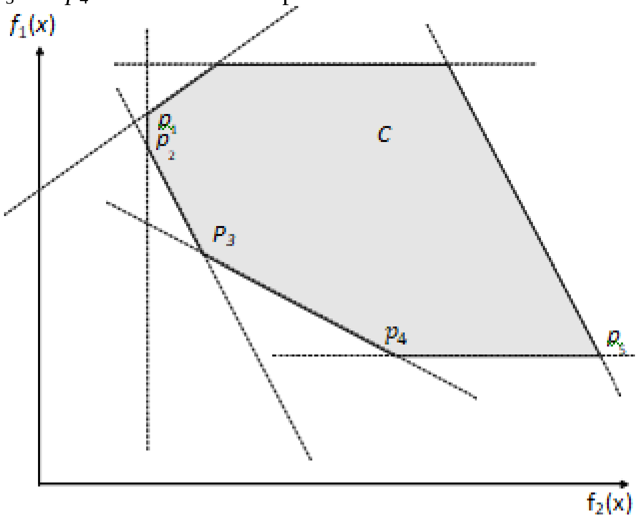

Fig. 3.2 Example of weak and strict Pareto optima

4. Techniques to solve MOOP

Pareto curves cannot be computed efficiently in many cases. Even if it is theoretically possible to find all these points exactly, they are often of exponential size; a straightforward reduction from the knapsack problem shows that they are hard to compute. Thus, approximation methods for them are frequently used. However, approximation does not represent a secondary choice for the decision maker. Indeed, there are many real-life problems for which it is quite hard for the decision maker to have all the information to correctly and/or completely formulate them.

Approximating methods can have different goals: representing the solution set when the latter is numerically available (for convex multi-objective problems). Approximating the solution set when some but not all the Pareto curve is numerically available, approximating the solution set when the whole efficient set is not numerically available (for discrete multi-objective problems).

There are several techniques to solve multi-objective optimization problem:

- The Scalarization Technique
- ε − constraints Method
- Goal Programming
- Multi-Level Programming
- The Norm-Ideal Point Method
- The Membership-Function Method.

Here, we discuss about the method of Norm-Ideal point and Membership function.

4.1 The Norm-Ideal Point Method:

For the convex MOPP, firstly give ideal value $\bar{f_i}$ for every objective function $f_i(x)$, which satisfies $\bar{f_i} \le \min_{x\in X} f_i(x), i = 1, 2, \ldots, m$, $\bar{f}=(\bar{f}1, \bar{f}2, \ldots, \bar{f}m)$ is called Ideal point, after then introduce the norm ||.||, finally get the feasible solution which is having the nearest distance with the given ideal point $\bar{f}$ in the norm.

Use the absolute value norm to structure the corresponding single objective programming $(S_{\bar{f}})$:

$$\min_{x\in X} \sum_{i=1}^{m} w_i |f_i(x) - \bar{f_i}| \;\; where\; w = (w_1, w_2, \ldots, w_m)^T \in R_+^m \backslash \{0\}$$

Because $\bar{f_i} \le \min_{x\in X} f_i(x), i = 1, 2, \ldots, m$; $(S_{\bar{f}})$ can be simplified to $\min_{x\in X} \sum_{i=1}^{m} w_i (f_i(x) - \bar{f_i})$.

For the given ideal point $\bar{f}$and weights $w \in R_+^m \backslash \{0\}$, the optimal solution of $S_{\bar{f}}$ is weakly Pareto optimal solution [11].

There states a theorem that: If $x^* \in X$ is weakly M-Pareto optimal solution of convex MOPP then there exist $w \in R_+^m \backslash \{0\}$ such that x^* is the optimal solution of the corresponding single objective programming problem.

Attention should be paid that Pareto optimal solution must be weakly Pareto optimal solution, which implies that if all weakly Pareto optimal solutions can be obtained, then all Pareto optimal solutions can be obtained. This also shows that theoretically all Pareto optimal solutions can be obtained through changing weights.

4.2 Membership Function Method:

Firstly structure membership function $\alpha_i\ (f_i\ (x))$ for every objective function $f_i(x)$, then use $\alpha_i\ (f_i\ (x))$ as the new objective functions to structure the new multi objective programming problem and then turn the new multi objective programming problem to single objective programming problem through some appropriate methods, finally solve the single objective programming problem to get the optimal solution, which is also the M-Pareto optimal solution of the original multi objective programming problem.

Now, Structure the membership function for every objective function as follows:

$$\alpha_i(f_i(x)) = \frac{f_i(x) - f_i^1}{f_i^* - f_i^1}, \;\; i = 1, 2, \ldots m; \;\; f_i^* = \min_{x\in X} f_i(x), \quad f_i^1 = \max_{x\in X} f_i(x)$$

Without loss of generality $f_i^* < f_i^1, i = 1, 2, \ldots, m$

There states two necessary theorems:

Theorem 1: x^* is M-Pareto optimal solution of problem convex MOPP, if and only if x^* is Pareto optimal solution of problem convex MOPP.

Theorem 2: If x^* is weakly M-Pareto optimal solution of problem convex MOPP, then there exists $w \in R^m \setminus \{0\}$ such that x^* is the optimal solution of the corresponding problem S_α.

The membership function of objective function is set by using simple linear function. According to the properties of the composition of convex function, if the membership function of $f_i\ (x)$ is non increasing and concave about $f_i(x)$, then the above calculation still holds.

5. Example

Consider the following Multi Objective Quadratic Programming problem with linear constraints:

$$Max\ \ Z_1 = 4x_1 + 2x_2 - x_1^2 - x_2^2 + 5$$
$$Max\ \ Z_2 = 2x_1 + x_2 - x_1^2$$
$$Min\ \ Z_3 = 6 - 6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2$$
$$Min\ \ Z_1 = 2x_1 + 3x_2 - 2x_1^2$$
$$s/t \qquad x_1 + 4x_2 \le 9$$
$$x_1 + x_2 \le 3$$
$$3x_1 + 2x_2 \le 8$$
$$x_1, x_2 \ge 0$$

Convert the system into convex MOQPP:

$$Min\ \ Z_1 = -4x_1 - 2x_2 + x_1^2 + x_2^2 - 5$$
$$Min\ \ Z_2 = -2x_1 - x_2 + x_1^2$$
$$Min\ \ Z_3 = 6 - 6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2 \qquad (5.1)$$
$$Min\ \ Z_4 = 2x_1 + 3x_2 - 2x_1^2$$
$$s/t \qquad x_1 + 4x_2 \le 9$$
$$x_1 + x_2 \le 3$$
$$3x_1 + 2x_2 \le 8$$
$$x_1, x_2 \ge 0$$

Now we will find Pareto optimal solution of this system by using Norm-Ideal Point method and Membership Function Method:

Using Norm-Ideal Point method

For convenience, the feasible region of the given problem is:

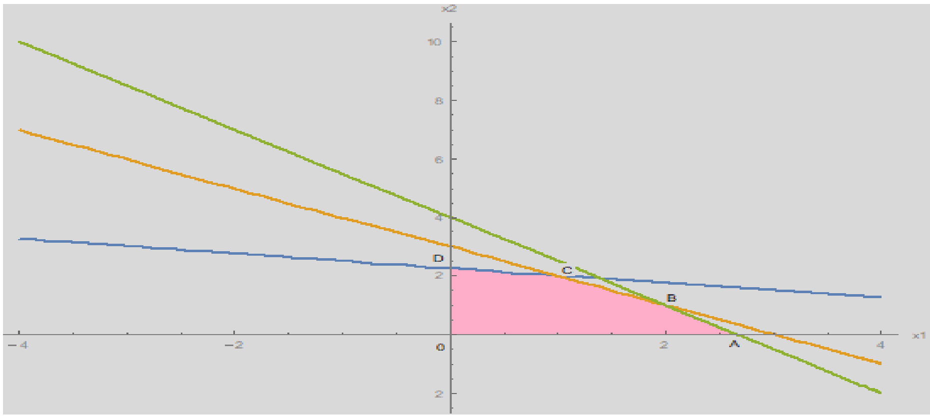

Figure 5.1: The graph of feasible region

The feasible region is 0ABCD. The vertices of feasible region formed by constraints are $\mathbf{0(0,0)}, \boldsymbol{A}(\mathbf{2.6,0})$, $\boldsymbol{B}(\mathbf{2,1})$, $\boldsymbol{C}(\mathbf{1,2})$, $\boldsymbol{D}(\mathbf{0,2.25})$ and the function values of the objective functions in vertices are given in the following table I:

Table I: function values at vertices

	Z_1	Z_2	Z_3	Z_4
$0(0,0)$	-5	0	6	0
$A(2.6,0)$	-8.64	1.56	3.92	-8.32
$B(2,1)$	-10	-1	0	-1
$C(1,2)$	-8	-3	6	6
$D(0,2.25)$	-4.4	-2.25	16.12	6.75

It can easily say that, $Min\{Z_1\} = -10, Min\{Z_2\} = -3, Min\{Z_3\} = 0, Min\{Z_4\} = -8.32$.
All Pareto optimal solutions of problem (5.1) is : $A(2.6,0)$, $B(2,1)$, $C(1,2)$

The Ideal point method is used to solve the given MOQPP and illustrate that there exists weights $\boldsymbol{w}$ such that each Pareto optimal solution of (5.1) is the optimal solution of corresponding single objective function. In the Ideal point method, we take the Ideal point $\bar{p} = (\overline{p_1}, \overline{p_2}, \overline{p_3}, \overline{p_4})$ where

$$\overline{p_1} = \min_{x\in X}\{-4x_1 - 2x_2 + x_1^2 + x_2^2 - 5\} = -10, \qquad \overline{p_2} = \min_{x\in X}\{-2x_1 - x_2 + x_1^2\} = -3,$$
$$\overline{p_3} = \min_{x\in X}\{6 - 6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2\} = 0, \qquad \overline{p_4} = \min_{x\in X}\{2x_1 + 3x_2 - 2x_1^2\} = -8.32$$

So the MOQPP (5.1) can be turned into the following single objective programming problem:

$$Min\ \ w_1[z_1 + 10] + w_2[z_2 + 3] + w_3[z_3 - 0] + w_4[z_4 + 8.32]$$
$$\Rightarrow\ \ Min\ \ w_1[-4x_1 - 2x_2 + x_1^2 + x_2^2 - 5 + 10] + w_2[-2x_1 - x_2 + x_1^2 + 3] + w_3[6 - 6x_1 + 2x_1^2 - 2x_1x_2 +$$
2x22−0+w4[2x1+3x2−2x12+8.32]
$$s/t \qquad x_1 + 4x_2 \le 9$$
$$x_1 + x_2 \le 3$$
$$3x_1 + 2x_2 \le 8$$
$$x_1, x_2 \ge 0 \qquad (5.2)$$

The objective function can be written as:

$$x_1[-4w_1 - 2w_2 - 6w_3 + 2w_4] + x_2[-2w_1 - w_2 + 3w_4] + x_1^2[w_1 + w_2 + 2w_3 - 2w_4] + x_2^2[w_1 + 2w_3] - 2w_3x_1x_2 + [5w_1 + 3w_2 + 6w_3 + 8.32w_4]$$

In order to take Minimum in point $\boldsymbol{B}(\mathbf{2,1})$ according to the characteristic of linear programming, the slope of objective function only need to satisfy

$$-4w_1 - 2w_2 - 6w_3 + 2w_4 > 0\ \ \&\ \ \frac{4w_1 + 2w_2 + 6w_3 - 2w_4}{-2w_1 - w_2 + 3w_4} < -1 \qquad (5.3)$$

Or

$$-4w_1 - 2w_2 - 6w_3 + 2w_4 > 0\ \ \&\ \ \frac{4w_1 + 2w_2 + 6w_3 - 2w_4}{-2w_1 - w_2 + 3w_4} < -3/2 \qquad (5.4)$$

Taking, $w_1 = \frac{1}{4}, w_2 = \frac{1}{4}, w_3 = \frac{1}{4}, w_4 = \frac{1}{4}$

The multi objective programming problem (5.2) can be converted into the following single objective programming problem:

$$Min\ -\frac{5}{2}x_1 + \frac{1}{2}x_1^2 + \frac{3}{4}x_2^2 - \frac{1}{2}x_1x_2 + \frac{279}{50}$$
$$s/t \qquad x_1 + 4x_2 \le 9$$

$$x_1 + x_2 \le 3$$
$$3x_1 + 2x_2 \le 8$$
$$x_1, x_2 \ge 0 \qquad (5.5)$$

After solving we get, $x_1 = 2.29$ & $x_2 = 0.9$

The optimal solution of single objective programming problem (5.5) is $(2.29, 0.9)$ which is also the Pareto optimal solution (approximately) of problem (5.2). So there exist weights $w_1 = \frac{1}{4}, w_2 = \frac{1}{4}, w_3 = \frac{1}{4}, w_4 = \frac{1}{4}$ such that $B(2,1)$ is the optimal solution of the corresponding single objection programming problem (5.5).

In order to take Minimum in point $\boldsymbol{C(1,2)}$ according to the characteristic of linear programming, the slope of objective function only need to satisfy

$$-4w_1 - 2w_2 - 6w_3 + 2w_4 > 0 \ \& \ \frac{4w_1+2w_2+6w_3-2w_4}{-2w_1-w_2+3w_4} < -1 \qquad (5.6)$$

Or $$-4w_1 - 2w_2 - 6w_3 + 2w_4 > 0 \ \& \ \frac{4w_1+2w_2+6w_3-2w_4}{-2w_1-w_2+3w_4} < -1/4 \qquad (5.7)$$

Taking, $w_1 = \frac{2}{5}, w_2 = \frac{1}{5}, w_3 = \frac{1}{5}, w_4 = \frac{1}{5}$

The multi objective programming problem (5.2) can be converted into the following single objective programming problem:

$$Min\ -2.8x_1 - 0.4x_2 + 0.6x_1^2 + 0.8x_2^2 - 0.4x_1x_2 + 5.46$$

$$s/t \quad x_1 + 4x_2 \le 9$$
$$x_1 + x_2 \le 3$$
$$3x_1 + 2x_2 \le 8$$
$$x_1, x_2 \ge 0 \qquad (5.8)$$

After solving we get, $x_1 = 2.23$ & $x_2 = 0.91$

The optimal solution of single objective programming problem (5.8) is $(2.23, 0.91)$ which is also the Pareto optimal solution (approximately) of problem (5.2). So there exist weights $w_1 = \frac{2}{5}, w_2 = \frac{1}{5}, w_3 = \frac{1}{5}, w_4 = \frac{1}{5}$ such that $B(2,1)$ is the optimal solution of the corresponding single objection programming problem (5.8).

So for both cases, the Pareto optimal solution is (2, 1).

For all Pareto optimal solutions of multi objective programming problem, there exists weights such that the Pareto optimal solution is the optimal solution of the corresponding single objective programming problem. From our previous paper [10], the optimal solution has been found.

<u>Using Membership Function method:</u>
For convenience, the feasible region of the given problem is:

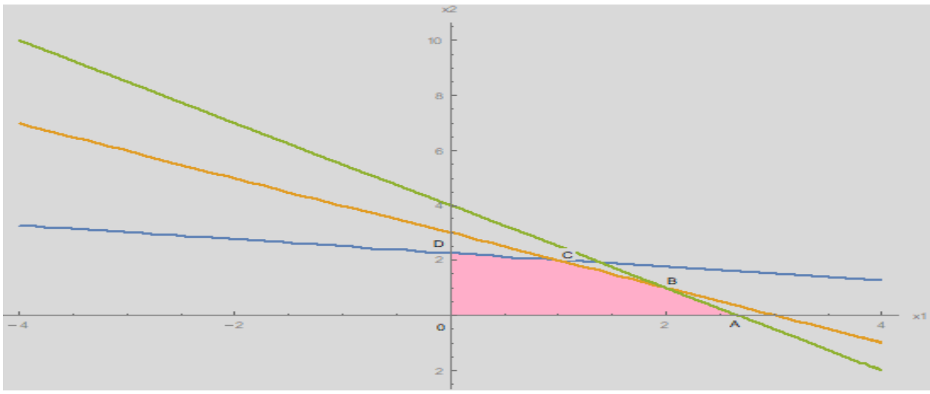

Figure 5.2: The graph of feasible region

For convenience denote that, $f_1 = -4x_1 - 2x_2 + x_1^2 + x_2^2 - 5$, $f_2 = -2x_1 - x_2 + x_1^2$

$$f_3 = 6 - 6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2, \quad f_4 = 2x_1 + 3x_2 - 2x_1^2$$

The feasible region is 0ABCD. The vertices of feasible region formed by constraints are $\mathbf{0(0,0)}$, $\boldsymbol{A(2.6,0)}$, $\boldsymbol{B(2,1)}$, $\boldsymbol{C(1,2)}$, $\boldsymbol{D(0,2.25)}$ and the function values of the objective functions in vertices are given in the following table II:

Table II: Function values at vertices

	f_1	f_2	f_3	f_4
$0(0,0)$	-5	0	6	0
$A(2.6,0)$	-8.64	1.56	3.92	-8.32
$B(2,1)$	-10	-1	0	-1
$C(1,2)$	-8	-3	6	6
$D(0,2.25)$	-4.4	-2.25	16.12	6.75

Due to the particularity of linear programming, it is obvious that,

$$Min\ f_1 = -10, \quad Max\ f_1 = -4.4$$
$$Min\ f_2 = -3, \quad Max\ f_2 = 1.56$$
$$Min\ f_3 = 0, \quad Max\ f_3 = 16.12$$
$$Min\ f_4 = -8.32, \quad Max\ f_4 = 6.75$$

For each objective function, membership function can be structured as follows:

$$\alpha_1(f_1) = \frac{-f_1 - 4.4}{5.6}, \alpha_2(f_2) = \frac{-f_2 + 1.56}{4.56}, \alpha_3(f_3) = \frac{-f_3 + 16.12}{16.12}, \alpha_4(f_4) = \frac{-f_4 + 6.75}{15.07}$$

The membership function values in vertices are given in the following table III:

Table III: Membership function values at vertices

	$\alpha_1(f_1)$	$\alpha_2(f_2)$	$\alpha_3(f_3)$	$\alpha_4(f_4)$
$0(0,0)$	3/28	13/38	253/403	675/1507
$A(2.6,0)$	53/70	0	305/403	1
$B(2,1)$	1	32/57	1	775/1507
$C(1,2)$	9/14	1	253/403	75/1507
$D(0,2.25)$	0	127/152	0	0

From the definition of M-Pareto optimal solution, The M-Pareto optimal solution of problem (5.1) are: $\mathbf{0(0,0)}, \boldsymbol{A(2.6,0)}, \boldsymbol{B(2,1)}, \boldsymbol{C(1,2)}, \boldsymbol{D(0,2.25)}$.

Next, membership function method is used to solve the multi objective programming problem (5.1) and illustrates that there exist weights $\boldsymbol{w}$ such that the M-Pareto optimal solution of problem (5.1) is the optimal solution of the corresponding single objective programming problem.

According to model (S_α), the multi objective programming problem (5.1) is converted to the single objective programming problem:

$$\begin{aligned} Max \quad & w_1\alpha_1(f_1) + w_2\alpha_2(f_2) + w_3\alpha_3(f_3) + w_4\alpha_4(f_4) \\ s/t \quad & x_1 + 4x_2 \leq 9 \\ & x_1 + x_2 \leq 3 \\ & 3x_1 + 2x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{aligned} \tag{5.9}$$

Due to the way of structuring membership function of objective functions f_1, f_2 and f_3, the above method can be written as:

$$Max \quad w_1\left(\frac{-f_1-4.4}{5.6}\right) + w_2\left(\frac{-f_2+1.56}{4.56}\right) + w_3\left(\frac{-f_3+16.12}{16.12}\right) + w_4\left(\frac{-f_4+6.75}{15.07}\right)$$

$$\begin{aligned} s/t \quad & x_1 + 4x_2 \leq 9 \\ & x_1 + x_2 \leq 3 \\ & 3x_1 + 2x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{aligned} \tag{5.10}$$

This can be written as,

$$Max \quad x_1\left[\frac{4}{5.6}w_1 + \frac{2}{4.5}w_2 + \frac{6}{16.12}w_3 - \frac{2}{15.07}w_4\right] + x_2\left[\frac{2}{5.6}w_1 + \frac{1}{4.56}w_2 - \frac{3}{15.07}w_4\right] + x_1^2\left[-\frac{1}{5.6}w_1 - \frac{1}{4.56}w_2 -\right.$$

216.12w3+215.07w4+x22
−15.6w1−216.12w3+216.12x1x2w3+.65.6w1+1.564.5
w2+w3+6.7515.07w4

$$\begin{aligned} s/t \quad & x_1 + 4x_2 \leq 9 \\ & x_1 + x_2 \leq 3 \\ & 3x_1 + 2x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{aligned} \tag{5.11}$$

In order to take maximum in point $B(2,1)$, the slope of objective function only need to satisfy

$$\frac{4}{5.6}w_1 + \frac{2}{4.5}w_2 + \frac{6}{16.12}w_3 - \frac{2}{15.07}w_4 > 0 \quad and$$

$$-\frac{\frac{4}{5.6}w_1+\frac{2}{4.5}w_2+\frac{6}{16.12}w_3-\frac{2}{15.07}w_4}{\frac{2}{5.6}w_1+\frac{1}{4.56}w_2-\frac{3}{15.07}w_4} < -1$$

Or

$$\frac{4}{5.6}w_1 + \frac{2}{4.5}w_2 + \frac{6}{16.12}w_3 - \frac{2}{15.07}w_4 > 0 \quad and$$

$$-\frac{\frac{4}{5.6}w_1+\frac{2}{4.5}w_2+\frac{6}{16.12}w_3-\frac{2}{15.07}w_4}{\frac{2}{5.6}w_1+\frac{1}{4.56}w_2-\frac{3}{15.07}w_4} < -\frac{3}{2}$$

If we take, $w_1 = \frac{1}{4}, w_2 = \frac{1}{2}, w_3 = \frac{1}{6}, w_4 = 2/7$, the multi objective programming problem (5.1) can be converted into single objective programming problem (5.11) and putting values we get,

$$Max \quad 0.422\, x_1 + 0.142\, x_2 - 0.137\, x_1^2 - 0.065\, x_2^2 + 0.02\, x_1x_2 + 0.492$$

$$\begin{aligned} s/t \quad & x_1 + 4x_2 \leq 9 \\ & x_1 + x_2 \leq 3 \\ & 3x_1 + 2x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{aligned} \tag{5.12}$$

After solving we get, $x_1 = 1.9$ and $x_2 = 1$.

The optimal solution of (5.12) is $x_1 = 1.9 \cong 2$ and $x_2 = 1$, which is also the M-Pareto optimal solution of problem (5.1). So there exist weights $w_1 = \frac{1}{4}, w_2 = \frac{1}{2}, w_3 = \frac{1}{6}, w_4 = 2/7$, such that $B(2,1)$ is the optimal solution of the corresponding single objective programming problem (5.12). Similarly, we can show that for point $C(1,2)$.

Therefore, for all M-Pareto optimal solution of multi objective programming problem (5.1), there exist weights, such that M-Pareto optimal solution is the optimal solution of the corresponding single objective programming problem.

6. Discussion

In this paper, an attempt is made to build a theoretical framework for MOQPP by defining the solution concept of Pareto optimality. We carried out two methods to solve a multi objective optimization problem and found approximately same results. MOQPP is given to illustrate that for any Pareto optimal solution there exist weights such that Pareto optimal solution is the optimal solution of the corresponding single objective programming problem. Since weights are not unique, all Pareto optimal solution and M-Pareto optimal solution can be obtained through taking out different weights.

7. Conclusion

This paper performed structural analysis of Pareto optimal solution and M- Pareto optimal solution for convex multi objective quadratic programming problem. Generally, all Pareto optimal solution and M- Pareto optimal solution cannot be obtained through changing weights except convex multi objective optimization problem. Two methods discussed here are very effective to obtain Pareto optimal solution for convex MOQPP. These methods not only have important theoretical remarks but also have a lot of practical advantages.

Acknowledgement

This work is financially supported by UGC [Code: 3632104] and selected by ORE, BUTEX.

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