International Journal of Science and Research (IJSR)
ISSN: 2319-7064
ResearchGate Impact Factor (2018): 0.28 | SJIF (2019): 7.583

Covariant Formulation for Second Law of Motion

Dr. Dev Raj Mishra

Department of Physics, R. H. Government Post Graduate College, Kashipur, U. S. Nagar, Uttarakhand-244713, India

Abstract: ***Theory of Relativity has created the need to revisit the fundamental laws of Physics. The inclusion of time as a forth co-ordinate of Minkowski space and the quest for the corresponding fourth components of Physical Quantities suggest new modifications to the laws of Physics. This study presents modification to Newton's Second law of motion leading to a compact covariant formulation which successfully explains the occurrence of various pseudo forces in nature.***

Keywords: Covariance, Four vectors, Minkowski Space time, Special theory of Relativity

1. Introduction

Newton's laws are successful in explaining the physical observations which we see around in our non-relativistic everyday life. However theories of relativity both the special theory and the general theory have forced us to change the way we see the laws of Physics [1]. Newton's Laws are no exceptions to this changed scenario. It becomes important as they are still being used to explain the physical phenomenon. General Theory of Relativity has modified the law of gravity by adding a small correction term as a consequence of the solution to the Einstein's field equations [2]. As we know Newton's Law of Inertia tells us "Everybody continues in its state of rest or of uniform motion in a straight line unless it is compelled by some external force to change that state". There is no need to rethink over this except that the observer is making observations in an inertial frame of reference [3]. Second law tells us "The rate of change of momentum is proportional to the applied force and takes place in its direction". This law needs a covariant formulation, which we are discussing below. Third law of action and reaction seems alright and we shall not discuss it here.

2. Results and Discussion

Let us assume a particle P at a point defined by co-ordinates (x, y, z)is moving with a momentum**p**. Let a force **F** acts on the particle and causes a change in its momentum.Newton's Second law of motion, as stated above, tells us that the rate of change of momentum is proportional to the applied force and takes place in its direction. Mathematically we write it as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{1}$$

As we know, the units of force are so defined to make the constant of proportionality as unity. The components of force **F** and momentum **p** are given by the familiar expressions $\boldsymbol{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}$where F_x, F_y, F_z are components of the force, and $\mathbf{p} = p_x\hat{\mathbf{i}} + p_y\hat{\mathbf{j}} + p_z\hat{\mathbf{k}}$ wherep_x, p_y, p_z are components of the momentum. Therefore the x-component of the force is given by

$$F_x = \frac{\partial p_x}{\partial t} \tag{2}$$

Of course, this force can be a field, and therefore possessing a unique value at each point. Extending the scenario to Minkowski four dimensional space-time concepts, let E be the event of this force being applied to the particle P at time, say t. Then the co-ordinates of event E are x, y, z and t. The time co-ordinate is taken as 'ict' where c is the velocity of Light. Let the particle has a potential energy V due to this field. Then, we know force is a gradient of the potential energy or we can write

$$\mathbf{F} = -\nabla V \tag{3}$$

where $\nabla = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}$ is the gradient operator.

The x-component of force from Equation (3) is then given by

$$F_x = -\frac{\partial V}{\partial x} \tag{4}$$

Comparing the equations (2) and (4) we get

$$-\frac{\partial V}{\partial x} = \frac{\partial p_x}{\partial t} \tag{5}$$

This result however is not covariant in the sense **p** is a three vector. The corresponding four vector is $\vec{p} = \left(\frac{iE}{c}, p_x, p_y, p_z\right)$, E being the total energy of the particle and c being the velocity of light. The four coordinates are given by$x = (ict, x, y, z)$. The only modification we can do to make above equation covariant is replacing V by E

$$-\frac{\partial E}{\partial x} = \frac{\partial p_x}{\partial t} \tag{6}$$

Dividing both sides by ic, We get

$$-\left(\frac{1}{ic}\right)\frac{\partial E}{\partial x} = \frac{\partial p_x}{\partial (ict)}$$

This is same as

$$\frac{\partial\left(\frac{iE}{c}\right)}{\partial x} = \frac{\partial p_x}{\partial (ict)} \tag{7}$$

Replacing coordinates $x = (ict, x, y, z)$ by $x^\mu = (x^0, x^1, x^2, x^3)$ and $\vec{p} = \left(\frac{iE}{c}, p_x, p_y, p_z\right)$ by $p^\mu = (p^0, p^1, p^2, p^3)$, we can write above equation as

$$\frac{\partial p^0}{\partial x^1} = \frac{\partial p^1}{\partial x^0}$$

Rearranging, we have

$$\frac{\partial p^1}{\partial x^0} - \frac{\partial p^0}{\partial x^1} = 0$$

or

$$\frac{\partial p^\nu}{\partial x^\mu} - \frac{\partial p^\mu}{\partial x^\nu} = 0 \tag{8}$$

for $\mu = 0$ and $\nu = 1$.

Further using tensor notations ∂_μ for $\frac{\partial}{\partial x^\mu}$etc., we can write Equation (8) as

$$\partial_\mu p^\nu - \partial_\nu p^\mu = 0 \qquad (9)$$

for $\mu = 0$ and $\nu = 1$.

Since we are using co-ordinates are$x = (ict, x, y, z)$,$\vec{p} = \left(\frac{iE}{c}, p_x, p_y, p_z\right)$. We have metric $g_{\mu\nu} = g^{\mu\nu} = (1,1,1,1) = \delta^\mu_\nu$. Therefore Equation (9) can be equivalently written as

$$\partial^\mu p^\nu - \partial^\nu p^\mu = 0 \qquad (10)$$

And

$$\partial_\mu p_\nu - \partial_\nu p_\mu = 0 \qquad (11)$$

for $\mu = 0$ and $\nu = 1$.

We see that Equation (9) is $\mu\nu$ component of an anti-symmetric tensor $(\partial \wedge p)$with $\mu = 0$ and $\nu = 1$. Let us generalize this result assuming it true for allμ and ν, we can write the covariant form substitute for Newton's Second Law as

$$\boldsymbol{\partial \wedge p = 0} \qquad (12)$$

The tensor $\boldsymbol{\partial \wedge p}$ has six components given by $(\boldsymbol{\partial \wedge p})_{\boldsymbol{\mu\nu}} = \partial_\mu p_\nu - \partial_\nu p_\mu$for $\boldsymbol{\mu}$, $\boldsymbol{\nu} = \mathbf{0, 1, 2, 3.}$ Therefore assigning different values to$\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, we get six relations between four components of Energy-Momentum four-vector.

$$-\frac{\partial E}{\partial x} = \frac{\partial p_x}{\partial t} \qquad (i)$$
$$-\frac{\partial E}{\partial y} = \frac{\partial p_y}{\partial t} \qquad (ii)$$
$$-\frac{\partial E}{\partial z} = \frac{\partial p_z}{\partial t} \qquad (iii)$$
$$\frac{\partial p_y}{\partial x} = \frac{\partial p_x}{\partial y} \qquad (iv)$$
$$\frac{\partial p_z}{\partial x} = \frac{\partial p_x}{\partial z} \qquad (v)$$
$$\frac{\partial p_z}{\partial y} = \frac{\partial p_y}{\partial z} \qquad (vi)$$

(13)

The first three relations are between spatial variation of energy and the temporal rate of change of three-momentum. The last three relations just correlate the spatial variations of the components of the three- momentum. **These relations prove the fact that whenever there is a spatial variation in Energy; a force appears in the direction of the decrease in Energy.** This force we usually call as pseudo-force. The examples of these forces are upward force in a falling lift or frame of reference, centrifugal force on a particle doing circular motion etc.

Following is the discussion on illustrative examples of these results.

3. Illustrative Examples

The illustrative examples chosen here have kinetic energy variations which are the prominent energy variations in these cases but Tensor Equation (12) is equally and in fact, valid for the total energy variations.

3.1 Pseudo force in a freely falling frame

Let us consider an object of mass m in a frame of reference falling vertically downward along z-axis with acceleration$-\boldsymbol{a_z}$ (negative sign to show that the acceleration is along negative z axis) as shown in Figure. 1. We get from Equation (13(iii))

$$\frac{\partial E}{\partial z} + \frac{\partial p_z}{\partial t} = 0 \qquad (14)$$

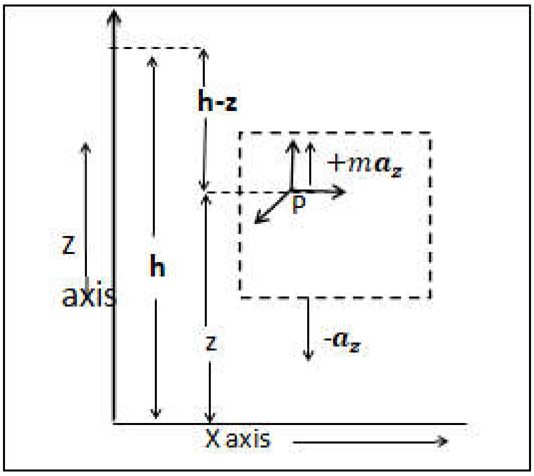

Figure 1: Frame at particle P in a container moving downwards with acceleration $\boldsymbol{a_z}$. The particle experiences a force m$\boldsymbol{a_z}$ upwards.

Let the frame falls from a height h. We have, change in energy at height z is equal to gain in the kinetic energy$\frac{1}{2}m\dot{z}^2$. But from elementary kinematics

$$0 - \dot{z}^2 = -2a_z(h - z).$$

Therefore the increase in the kinetic energy is$ma_z(h - z)$. Thus we get from Equation (14)

$$\frac{\partial p_z}{\partial t} = -\frac{\partial E}{\partial z} = -\frac{\partial(ma_z(h - z))}{\partial z} = ma_z \qquad (15)$$

Therefore the object experiences a force ma_z along positive z direction.

3.2 Centrifugal Force

Consider a particle of mass m moving in a circle of radius r with an angular velocityω as shown in Figure.2. The velocity of the particle is $v = \omega r$. The kinetic energy, which is the only variable part of the total energy of the particle under non-relativistic conditions, is given by

$$E = \frac{1}{2}mv^2$$
$$\frac{\partial E}{\partial r} = mv\frac{\partial v}{\partial r} \qquad (16)$$

The change in velocity is $\Delta v = \omega\Delta r$ and is along $-\Delta r$ i.e. points towards the center of the circle. Therefore $\Delta v = -\omega\Delta r$. For small variations, this becomes

$$\frac{\partial v}{\partial r} = \lim_{\Delta r \to 0}\frac{\Delta v}{\Delta r} = -\omega$$

Therefore from Equation (17), we have

$$\frac{\partial E}{\partial r} = -mv\omega \qquad (17)$$

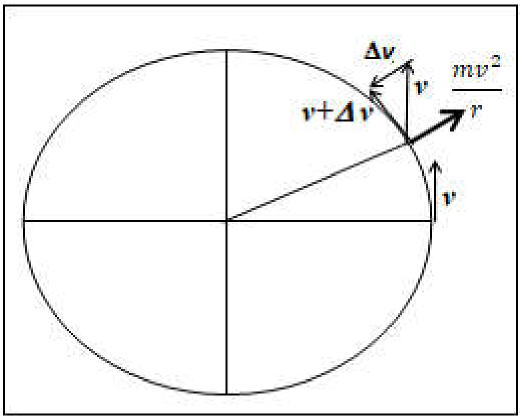

Figure 2: Particle rotating in circular path with velocity v experiences an outward force of $\frac{mv^2}{r}$

We get from Equation (13 (i))

$$\frac{\partial E}{\partial x} + \frac{\partial p_x}{\partial t} = 0 \qquad (18)$$

Let the particle is at an infinitely small displacement from the x-axis, we can safely replace x by r, and we get

$$\frac{\partial E}{\partial r} + \frac{\partial p_r}{\partial t} = 0$$

Therefore from Equation (17)

$$\frac{\partial p_r}{\partial t} = -\frac{\partial E}{\partial r} = mv\omega \qquad (19)$$

Therefore

$$\frac{\partial p_r}{\partial t} = \frac{mv^2}{r} \; as \; \omega = \frac{v}{r} \qquad (20)$$

which is obviously an outward force acting along positive **r** direction, and is known as centrifugal force.

Conversion Negative sign shows energy decreases over distance. Also momentum of the photon is mc and this process completes in time interval of $\frac{1}{\nu}$. Therefore

$$\frac{\partial p_x}{\partial t} = \frac{mc}{\frac{1}{\nu}} = mc\nu \qquad (22)$$

We get from (13(i))

$$\frac{\partial E}{\partial x} = -\frac{\partial p_x}{\partial t}$$

Therefore we have from Equations (21) and (22)

$$-\frac{h\nu}{\lambda} = -mc\nu$$

or

$$h\nu = mc\nu\lambda = mc^2 \text{as } c = \nu\lambda \qquad (23)$$

which is true by the Einstein Mass-Energy relation.

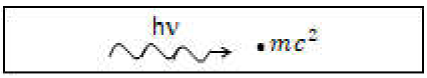

Figure 3: Photon of Energy hν converts to mass m according to relation $h\nu = mc^2$.

These were few examples to illustrate the validity of $\partial \wedge p = 0$ as covariant form substitute for Newton's Second law. The illustrative examples chosen above have kinetic energy variations which are the prominent energy variations in these cases but Tensor Equation (**Error! Reference source not found.**) is equally and in fact, valid for the total energy variations.

4. Conclusion

Substitution of Energy-Momentum four-vector leads to a covariant form of Newton's Second Law of motion which successfully explains the various pseudo-forces observed in nature such as in a falling frame of reference, in angular motion and also explains a hypothetical process of material particle production from a photon.

5. Acknowledgement

Author will like to thank Head, Department of Physics, Kumaun University, Nainital, Uttarakhand, India and Principal, R.H. Govt PG College, Kashipur (U.S. Nagar), India for their generous support in pursuing these studies.

References

[1] A. Einstein "A Brief Outline of the Development of the Theory of Relativity" Nature 106, pp. 782-784, 1921.
[2] R. Adler, M. Bazin, & M. Schiffer, Introduction to General Relativity (Second Edition) McGraw-Hill, New York, 1975.
[3] Bernard F. Schutz, A First Course in General Relativity (Second Edition), Cambridge University Press, New York, 2009.
[4] Weblink: www.cambridge.org/9780521887052.

Author Profile

[5] M. Bazin, & M. Schiffer, Introduction to General Relativity (Second Edition) McGraw-Hill, New York, 1975.
[6] Bernard F. Schutz, A First Course in General Relativity (Second Edition), Cambridge University Press, New York, 2009.
[7] Weblink: www.cambridge.org/9780521887052.

Author Profile

Dr Dev Raj Mishra Gold Medalist at Graduation did M.Sc. from M.D. University, Rohtak (Haryana) India. Did his Ph.D. in High Temperature Superconductivity from Cryogenics Group in National Physical Laboratory and University of Delhi, Delhi. Joined as Lecturer in the Department of Higher Education, Government of Uttarakhand in Jan. 2000. Presently working as Professor in Physics at Department of Physics, R.H. P.G. College, Kashipur, U. S. Nagar, India.