International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2019): 7.583

Technical Inequalities on the Arcs of a Circle

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Abstract

Let $\Delta := \{e^{i\theta}: \theta \in [\alpha, 2\pi - \alpha]\}$ be an arc of the circle and let $R(z) = (z - e^{i\alpha})(z - e^{-i\alpha}) = z^2 - 2z\cos\alpha + 1$. Then we prove

a. $\left|\vartheta\left(e^{i\theta}\right)\right| \leq \frac{6}{n}$
b. $|\vartheta(z) - \vartheta(a)| \leq 14|z - a|$, for $a, z \in \Delta$.
c. $\frac{1}{2} \leq \frac{\vartheta(z)}{\vartheta(a)} \leq \frac{3}{2}$, for $a, z \in \Delta$ such that $|z - a| \leq \frac{1}{28}\vartheta(a)$.

Here,

$$\vartheta(z) = \vartheta_n(z) = \frac{1}{n}\left[\frac{|R(z)| + \left(\frac{\pi - \alpha}{n}\right)^2}{|z - 1|^2 + \left(\frac{1}{n}\right)^2}\right]^{1/2} = \frac{1}{n}\left[\frac{\left|\sin\left(\frac{\theta - \alpha}{2}\right)\sin\left(\frac{\theta + \alpha}{2}\right)\right| + \left(\frac{\pi - \alpha}{n}\right)^2}{4\sin\left(\frac{\theta}{2}\right)^2 + \left(\frac{1}{n}\right)^2}\right]^{1/2}$$

Introduction and Results

The classical Markov-Bernstein Inequality for trigonometric polynomials

$$S_n(\theta) := \sum_{j=0}^{n}\left(c_j \cos j\theta + d_j \sin j\theta\right)$$

of degree $\leq n$ is

$$\|S'_n\|_{L_\infty[0,2\pi]} \leq n\|S\|_{L_\infty[0,2\pi]}$$

Over the past years, there are several developments and improvements in this area. In the 1950's V.S.Videnskii generalized the L_∞ inequality to the case where the interval over which the norm is taken shorter than the period [1]. D. S. Lubinsky and K Kobindarajah [6] and [7] worked on to find L_p analogue over the arcs of a circle. In this paper, we establish the following inequalities improving earlier ineqlaities which will be helpful to researcher for their research in this area.

Theorem

Let $\Delta := \{e^{i\theta}: \theta \in [\alpha, 2\pi - \alpha]\}$ be an arc of the circle and let $R(z) = \left(z - e^{i\alpha}\right)\left(z - e^{-i\alpha}\right) = z^2 - 2z\cos\alpha + 1$. Let

$$\vartheta(z)=\vartheta_n(z)=\frac{1}{n}\left[\frac{|R(z)|+\left(\frac{\pi-\alpha}{n}\right)^2}{|z-1|^2+\left(\frac{1}{n}\right)^2}\right]^{1/2}=\frac{1}{n}\left[\frac{\left|\sin\left(\frac{\theta-\alpha}{2}\right)\sin\left(\frac{\theta+\alpha}{2}\right)\right|+\left(\frac{\pi-\alpha}{n}\right)^2}{4\sin\left(\frac{\theta}{2}\right)^2+\left(\frac{1}{n}\right)^2}\right]^{1/2}$$

Then we prove the following inequalities

a. $\left|\vartheta\left(e^{i\theta}\right)\right|\leq\frac{6}{n}$

b. $|\vartheta(z)-\vartheta(a)|\leq\sqrt{2}\pi^2|z-a|$, for$a,z\in\Delta$.

c. $\frac{1}{2}\leq\frac{\vartheta(z)}{\vartheta(a)}\leq\frac{3}{2}$ for $a,z\in\Delta$ such that $|z-a|\leq\frac{1}{2\sqrt{2}\pi^2}\vartheta(a)$

where,

$$\vartheta(z)=\vartheta_n(z)=\frac{1}{n}\left[\frac{|R(z)|+\left(\frac{\pi-\alpha}{n}\right)^2}{|z-1|^2+\left(\frac{1}{n}\right)^2}\right]^{1/2}=\frac{1}{n}\left[\frac{\left|\sin\left(\frac{\theta-\alpha}{2}\right)\sin\left(\frac{\theta+\alpha}{2}\right)\right|+\left(\frac{\pi-\alpha}{n}\right)^2}{4\sin\left(\frac{\theta}{2}\right)^2+\left(\frac{1}{n}\right)^2}\right]^{1/2}$$

Proof

$$\begin{aligned}R\left(e^{i\theta}\right)&=\left(e^{i\theta}-e^{i\alpha}\right)\left(e^{i\theta}-e^{-i\alpha}\right)\\&=-4e^{i\theta}\sin\left(\frac{\theta-\alpha}{2}\right)\sin\left(\frac{\theta+\alpha}{2}\right)\\&=-4e^{i\theta}\left(\cos^2\frac{\alpha}{2}-\cos^2\frac{\theta}{2}\right)\\&=-4e^{i\theta}\left(\sin^2\frac{\theta}{2}-\sin^2\frac{\alpha}{2}\right)\quad(1)\end{aligned}$$

From (1), we derive,

$$\left|R\left(e^{i\theta}\right)\right|\leq 4\left(\sin\frac{\theta}{2}\right)^2\qquad(2)$$

$$\left|R\left(e^{i\theta}\right)\right|\leq 4\left(\cos\frac{\alpha}{2}\right)^2\qquad(3)$$

$$\left|R\left(e^{i\theta}\right)\right|\leq 4\left|\sin\frac{\theta}{2}\right|\cos\frac{\alpha}{2}\qquad(4)$$

valid for $\theta\in[\alpha,2\pi-\alpha]$.

Paper ID: SR21117192200 DOI: 10.21275/SR21117192200

For our convenient, we shall write

$$f(\theta) \coloneqq \left|R\left(e^{i\theta}\right)\right| + \left(\frac{\pi-\alpha}{n}\right)^2;$$

$$g(\theta) \coloneqq 4\left(\sin\frac{\theta}{2}\right)^2 + \left(\frac{1}{n}\right)^2$$

Thus $\vartheta\left(e^{i\theta}\right)$ can be written as

$$\vartheta\left(e^{i\theta}\right) = \frac{1}{n}\left(\frac{f(\theta)}{g(\theta)}\right)^{1/2}.$$

Proof of (a) of the theorem:

From the equation (2)

$$f(\theta) \le 4\left(\sin\frac{\theta}{2}\right)^2 + \left(\frac{\pi}{n}\right)^2 \le \pi^2 g(\theta)$$

Therefore,

$$\vartheta\left(e^{i\theta}\right) = \frac{1}{n}\left(\frac{f(\theta)}{g(\theta)}\right)^{1/2} \le \frac{\pi}{n}$$

Also, from the inequality

$$\frac{\pi-\alpha}{\pi} \le \cos\frac{\alpha}{2} = \sin\left(\frac{\pi-\alpha}{2}\right) \le \frac{\pi-\alpha}{2} \qquad (5)$$

and from (3), we get

$$\vartheta\left(e^{i\theta}\right) \le \frac{(4+\pi^2)^{1/2}}{n}\frac{\cos\frac{\alpha}{2}}{\left|\sin\frac{\theta}{2}\right|} \le \frac{4}{n}\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}$$

Then the two bounds on ϑ give

$$\frac{\vartheta\left(e^{i\theta}\right)}{\cos\frac{\alpha}{2}} \le \frac{4}{n}\min\left\{\frac{1}{\cos\frac{\alpha}{2}}, \frac{1}{\sin\frac{\alpha}{2}}\right\} \le \frac{6}{n}$$

This gives,

$$\left|\vartheta\left(e^{i\theta}\right)\right| \le \frac{6}{n}\left|\cos\frac{\alpha}{2}\right| \le \frac{6}{n} \qquad \blacksquare$$

Proof of (b) of the theorem:

We write $z = e^{i\theta}; a = e^{is}$. We shall assume, as we may, that

$$\left|\sin\frac{s}{2}\right| \geq \left|\sin\frac{\theta}{2}\right| \qquad (6)$$

or equivalently, that s is closer to π than θ. Note from the definition of f, g and from (1) that

$$f(\theta) = g(\theta) + c,$$

where,

$$c = -4\left(\sin\frac{\alpha}{2}\right)^2 + \frac{(\pi-\alpha)^2 - 1}{n^2}$$

Then

$$\vartheta\left(e^{i\theta}\right) = \frac{1}{n}\left(1 + \frac{c}{g(\theta)}\right)^{1/2},$$

so,

$$n\left[\vartheta\left(e^{i\theta}\right) - \vartheta\left(e^{is}\right)\right] = \frac{\left(1 + \frac{c}{g(\theta)}\right) - \left(1 + \frac{c}{g(s)}\right)}{\left(1 + \frac{c}{g(\theta)}\right)^{\frac{1}{2}} + \left(1 + \frac{c}{g(s)}\right)^{\frac{1}{2}}}$$

$$= \frac{c[g(s) - g(\theta)]}{g(\theta)g(s)\left[\left(1 + \frac{c}{g(\theta)}\right)^{\frac{1}{2}} + \left(1 + \frac{c}{g(s)}\right)^{\frac{1}{2}}\right]}$$

Here,

$$|g(s) - g(\theta| = 4\left|\sin\left(\frac{s-\theta}{2}\right)\sin\left(\frac{s+\theta}{2}\right)\right|$$

$$= 2\left|e^{is} - e^{i\theta}\right|\left|\sin\frac{s}{2}\cos\frac{\theta}{2} + \cos\frac{s}{2}\sin\frac{\theta}{2}\right|$$

$$\leq 4\left|e^{is} - e^{i\theta}\right|\min\left\{\sin\frac{s}{2}, \cos\frac{\alpha}{2}\right\}$$

(We have used the fact $s, \theta \in [\alpha, 2\pi - \alpha]$ and (6)). Also

$$|c| \leq 4\left(\sin\frac{\alpha}{2}\right)^2 + \left(\frac{\pi}{n}\right)^2$$

$$\leq 4\left(\sin\frac{\theta}{2}\right)^2 + \left(\frac{\pi}{n}\right)^2$$

$$\leq \pi^2 g(\theta).$$

Then

$$n\left|\frac{\vartheta\left(e^{i\theta}\right)-\vartheta\left(e^{is}\right)}{e^{i\theta}-e^{is}}\right| \le \frac{4\pi^2\min\left\{\sin\frac{s}{2},\cos\frac{\alpha}{2}\right\}}{g(s)\left(1+\frac{c}{g(s)}\right)^{\frac{1}{2}}}$$

$$\le \frac{4\pi^2\min\left\{\sin\frac{s}{2},\cos\frac{\alpha}{2}\right\}}{(f(s)g(s))^{1/2}}$$

If $\alpha > \frac{\pi}{2}$ then using

$$(f(s))^{\frac{1}{2}} \ge \frac{\pi-\alpha}{n} \ge \frac{\pi}{2n} \quad \text{(by (5))}$$

and

$$(g(s))^{\frac{1}{2}} \ge 2\left|\sin\frac{s}{2}\right| \ge 2\sin\frac{\pi}{4}$$

we deduce that

$$\left|\frac{\vartheta\left(e^{i\theta}\right)-\vartheta\left(e^{is}\right)}{e^{i\theta}-e^{is}}\right| \le \frac{\pi^2}{\sin\frac{\pi}{4}} = \sqrt{2}\pi^2$$

Now, if $\alpha \le \frac{\pi}{2}$, we use

$$(f(s))^{\frac{1}{2}} \ge \frac{\pi-\alpha}{n} \ge \frac{2\cos\frac{\alpha}{2}}{n};$$

$$(g(s))^{\frac{1}{2}} \ge 2\left|\sin\frac{s}{2}\right|$$

to deduce

$$\left|\frac{\vartheta\left(e^{i\theta}\right)-\vartheta\left(e^{is}\right)}{e^{i\theta}-e^{is}}\right| \le 4\pi \le \sqrt{2}\pi^2$$

Therefore, in both cases, we have established that

$$\left|\frac{\vartheta\left(e^{i\theta}\right)-\vartheta\left(e^{is}\right)}{e^{i\theta}-e^{is}}\right| \le \sqrt{2}\pi^2$$

Thus,

$$|\vartheta(z)-\vartheta(a)| \le \sqrt{2}\pi^2|z-a| \qquad \blacksquare$$

Proof of (c) of the theorem:

From (b)

$$\begin{aligned} |\vartheta(z)-\vartheta(a)| &\le \sqrt{2}\pi^2|z-a| \\ &\le \sqrt{2}\pi^2 \frac{1}{2\sqrt{2}\pi^2}\vartheta(a) \\ &= \frac{1}{2}\vartheta(a) \end{aligned}$$

Thus

$$-\frac{1}{2}\vartheta(a) \le \vartheta(z)-\vartheta(a) \le \frac{1}{2}\vartheta(a)$$

This implies

$$\frac{1}{2}\vartheta(a) \le \vartheta(z) \le \frac{3}{2}\vartheta(a)$$

and therefore

$$\frac{1}{2} \le \frac{\vartheta(z)}{\vartheta(a)} \le \frac{3}{2} \qquad \blacksquare$$

I would like to thank to Professor D S Lubinsky for supervising my Ph.D studies and his valuable ideas in my research.

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