Laminar Flow Prediction over a Rotating Disc using Bezier Curves

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Abstract: ***An analysis of fluid dynamic characteristics of the interaction of a laminar flow with a circular disc rotating at uniform angular velocity, is presented. The study emphasizes on a novel method of using parametric Bezier curves for the representation of the solution to the flow problem, which due to the assumptions involved simplifies into exact solutions of three-dimensional flow, i.e., three dimensional Navier Stokes equations near the rotating disk. A MATLAB code has been written to implement the method, whose results are compared with the numerical solution given by the conventional Boundary value problem solver. It is predicted that due to the precision of the results obtained, the novel method is an appropriate means to solve the systems of nonlinear equations. It can have comprehensive application in complex mathematical and engineering problems.***

Keywords: Bezier Curves, Boundary Value problem, Navier-Stokes Equation

1. Introduction

1.1. Background

The wind moving, the water flowing, and other movements of fluids in nature, they are governed by a set of equations called as Navier Stokes Equations. They are considered to be so significant to the dynamics of motion

Navier-Stokes equations can be considered to be have begun on 17th century, most precisely 1687. On this year, Issac Newton published his most important work named 'Mathematical Principles of Natural Philosophy' which had a huge impact on physics and most particularly mechanics, which is the branch of physics that studies the movement of bodies. This work established the foundations of mechanics as we know today as Newtons laws of motion: The first law, which states that every object in a state of uniform motion will remain in that motion unless acted upon by external force. The second law, the sum of forces acting on a body equals mass times the acceleration ($F = m.a$). Lastly, the very popular third law states that for every action there is an equal and opposite reaction. Apart from mechanics, Newton made several other contributions to physics and mathematics. Working independently, he and Gottfried Leibniz developed a set of mathematical tools known today as "infinitesimal calculus" or simply "Calculus".

With the development of calculus, many problems were solved in the frame of ideal fluid or inviscid fluid, which is known as a fluid without viscosity. In 1738, Bernoulli proved that the gradient or pressure is proportional to the acceleration of the fluid. Later, Euler derived the famous differential equations known as 'Eulers Equations', which closely resemble Navier-Stokes equations. However, the action of viscosity was not considered in these equations, providing unrealistic results. In 1758, D'Alembert proved that the drag on the body of any shape moving through a fluid with no viscosity zero which is known as D'alembert's paradox. This result was clearly in contradiction found from the real world experiments; hence a mathematical fluid mechanics & engineering hydrodynamics were established as two discrete branches.

In the 19th century, research in fluid mechanics was focussed in trying to add a friction term in Eulers equations in order to obtain the realistic results. In 1822, Claude Louis Navier by incorporated an extra term to the Eulers equations, in order to represent friction and hence derived Navier-Stokes equations for the first time. Various other scientists such as Cauchy in 1828, Poisson in 1829, and Saint-Venant in 1843 had done the same in the respective years. However, George Stokes in 1845 is credited to have made the first mathematical derivation of the Navier-Stokes equations. From the Eulers equations up to today, different scientists wrote Navier-Stokes equations in various forms. It is Ludwig Prantl, in 1934 who wrote the Navier-Stokes equations in the form that is widely used today.

1.2 Motivation

Solutions to Navier-Stokes equations are used in many practical applications. However, theoretical understanding of the equations is incomplete. In particular, solutions of Navier-Stokes equations often included turbulence, which remains one of the greatest unsolved problems in physics. Even more basic properties of the solutions to Navier-Stokes equations have never been proven so far.

Mathematicians have not yet proved that solutions always exist. Or, and if they do exist, they have bounded energy, which means that they don't blow up. This mathematical problem is widely known as "Navier-Stokes existence and Smoothness" problem. In 2000, Clay Mathematics made this problem one of the seven Millennium prize offering a 1-million-dollar prize to the first person providing a solution. There are, however, some partial results regarding the mathematical behaviour for these equations.

The mathematician John Leray proved, in 1934, the existence of the so called "weak solutions" of the Navier-Stokes Equations. More recently, another mathematician named Terence Tao published in 2016* a finite time blow up result from an average version of the three-dimensional Navier-Stokes equations. Navier-Stokes equations are very useful because they describe the physics of many phenomena of scientific and engineering interest. They can be used to model the ocean currents, weather conditions, water flow in a pipe, design of aircraft and cars, study of blood flow, design of hydro-power stations, design of aircraft and cars, study of blood flow, design of hydro-power stations, analysis of pollution and many other things. It can be said that applications of Navier-Stokes equations are almost everywhere in our life today.

1.3 Aim & Objective of the Research

Until the present day no general analytic methods have become available for the integration of the Navier-Stokes equations. Furthermore, solutions which are valid for all the values the problem of finding the exact solutions of the Navier-Stokes Equations presents insurmountable mathematical difficulties. This is, primarily, a consequence of their being non-linear, so that the application of the principle of super-position, which serves so well in the case of frictionless potential motions, is excluded. Nevertheless, it is possible to find exact solutions in certain particular cases, mostly when the quadratic convective terms vanish in a natural way. This paper devotes our attention to the discussion of an exact solution to the Navier-Stokes equations and its representations using the Bezier curves. Incidentally, it will be shown that in the case of small viscosity, the exact solution has a boundary-layer structure which means that the influence of viscosity is confined to a thin layer near the wall.

2. Navier Stokes Equations

2.1 Classical Mechanics

Hence, Classical mechanics, the father of physics and perhaps of scientific thought, was initially developed in the 1600s by the famous natural philosophers of the 17th century such as Isaac Newton building on the data and observations of astronomers including Tycho Brahe, Galileo, and Johannes Kepler. Classical mechanics mainly sticks to the mathematical explanation of the motion of physical bodies, binding together the notions of force, momentum, velocity, and energy to define the behaviour of macroscopic objects. Classical mechanics holds accurately for scales from 1 picometer (10−12 meters) to 1030 meters. Due to its steady success, classical mechanics has been broadly studied by both physicists and mathematicians alike. Despite the need to consider quantum mechanics for small-scale motion and special relativity for high-velocity travel, classical mechanics have always been considered mostly complete and have solved set of theories over time.

2.2 Fluid Dynamics and the Navier-Stokes Equations

The Navier-Stokes equations, developed by Claude-Louis Navier and George Gabriel Stokes in 1822, are equations which can be used to determine the velocity vector field that applies to a fluid, given some initial conditions. They arise from the application of Newton's second law in combination with a fluid stress (due to viscosity) and a pressure term. For almost all real situations, they result in a system of nonlinear partial differential equations; however, with certain simplifications (such as 1-dimensional motion) they can sometimes be reduced to linear differential equations. Usually, however, they remain nonlinear, which makes them difficult or impossible to solve; this is what causes the turbulence and unpredictability in their results.

2.3 Derivation of the Navier-Stokes Equations

Navier-stokes equations could be derived from the basic laws applied to the properties of fluids, i.e., conservation and continuity equations. The process involves the derivation of continuity equation, and hence applies the equation to conservation of mass and momentum, and finally chain the conservation equations with a physical understanding of what a fluid is.

2.3.1Basic assumptions

The Navier–Stokes equations are based on the assumption

- The fluid, in consideration, is a continuum, within the scale of interest
- All the properties including pressure, velocity, density, temperature and others are differentiable,

The equations are derived from the basic principles of conservation of mass, momentum, and energy,

- Hence, a finite arbitrary volume, called a control volume is considered, over which these principles can be applied
- Finite volume is denoted by Ω, and its bounding surface by ∂Ω.
- Control volume can either remain fixed in space or can move along with the fluid.

2.3.2 Material Derivative

There are two ways to measure the changes in the properties of a moving fluid. One involves carrying out the measurement on a fixed point in space as particles of fluid move through it and the other by following fluid particle along its streamline. The former means of measurement which involves the calculation of the derivative of a field with respect to a fixed position in space is known as the Eulerian derivative while the later that involves calculation of the derivative following a moving parcel is known as the convective or material derivative.

The material derivative $\frac{D}{Dt}$could be defined as:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{v}.\nabla$$

where is $\boldsymbol{v}$ the velocity of the fluid.

- $\frac{\partial}{\partial t}$ denotes ordinary Eulerian derivative (i.e. the derivative on a fixed frame of reference, denoting the changes at a point with respect to time)
- $\boldsymbol{v}.\nabla$ denotes the changes of a quantity with respect to position (advection).

2.3.3 Conservation laws

Navier-Stokes equation could be considered as a special case of the continuity equation. The main and the associated equations such as mass continuity, may be derived from the fundamental principles of:

- Mass
- Momentum
- Energy

The equation which describes the change of an intensive property L, is the basic continuity equation. The volume Ω is

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assumed to be of any form; its bounding surface area is referred to as ∂Ω. After density, the continuity equation could later be derived by applying it to mass and momentum.

Reynold's Transport Theorem

The first basic assumption is that of Reynold's Transport Theorem, usually symbolized as follows:

$$\frac{d}{dt}\int_{\Omega} L\, dV = -\int_{\partial\Omega} L\vec{v}\cdot\vec{n}dA - \int_{\Omega} Q\, dV$$

The rate of change of the property L contained inside the volume Ω is denoted on the left hand side. The right hand side is the sum of two terms:

- A flux term, $\int_{\partial\Omega} L\vec{v}\cdot\vec{n}dA$, which indicates how much of the property L is leaving the volume by flowing over the boundary ∂Ω
- A sink term, $\int_{\Omega} Q\, dV$, which describes how much of the property L is leaving the volume due to sinks or sources inside the boundary

For example, if the intensive property we are dealing with is temperature, the equations states that the total change in heat is the sum of the heat flux (heat flowing out of the boundary) and the heat sources or sinks in the medium. If the intensive property we're dealing with is density, then the equation is simply a statement of conservation of mass: the net change in mass is the sum of what is leaving the boundary and what remains within it; no mass is left unaccounted for.

2.3.4 Divergence Theorem

By the Divergence Theorem, expressing the flux term as a volume integral, we get

$$\int_{\partial\Omega} L\vec{v}\cdot\vec{n}dA = \int_{\Omega} \nabla\cdot(L\vec{v})\, dV$$

Hence, the equation could be re-written as,

$$\frac{d}{dt}\int_{\Omega} L\, dV = -\int_{\Omega} (\nabla\cdot(L\vec{v}) + Q)dV$$

2.3.5 Resulting Equation

Applying Leibniz's rule to the integral on the left and then combining all of the integrals:

$$\int_{\Omega} \frac{\partial L}{\partial t}\, dV = -\int_{\Omega} (\nabla\cdot(L\vec{v}) + Q)dV$$

Equivalently,

$$\int_{\Omega} (\frac{dL}{dt}\, dV + \nabla\cdot(L\vec{v}) + Q)\, dV = 0$$

This relation applies to any control volume Ω; the only way the above equality remains true for all control volumes is if the integrand itself is zero. Thus, we arrive at the general form of the continuity equation

$$\frac{dL}{dt}\, dV + \nabla\cdot(L\vec{v}) + Q = 0$$

2.3.6 Conservation of Mass

Relating the continuity equation to density (the property equivalent to mass), we obtain

$$\frac{d\rho}{dt}\, dV + \nabla\cdot(\rho\vec{v}) + Q = 0$$

This is the same as conservation of mass because we are operating with a constant control volume Ω. ($Q = 0$ when there are no sources or sinks of mass),

$$\frac{d\rho}{dt}\, dV + \nabla\cdot(\rho\vec{v}) = 0, \text{ i.e.,}$$

This is the equation of conversation of mass.

For an incompressible fluid, the density is constant. Setting the derivative of density equal to zero and dividing through by a constant ρ, we obtain the simplest form of the equation $\nabla\cdot(\vec{v}) = 0$.

From this generic equation of continuity, three significant concepts may be concluded:

- Conservation of mass
- Conservation of momentum, and
- Conservation of energy.

2.3.7 Conservation of momentum

On applying the conservation relation to momentum, the most fundamental form of the Navier–Stokes equations could be obtained. Writing momentum as$\rho\vec{v}$ gives:

$$\frac{d}{dt}(\rho\vec{v}) + \; dV + \nabla\cdot(\rho\vec{v}\vec{v}) + Q = 0$$

Basic physics dictates that,

$$\vec{F} = m\vec{a}$$

2.3.8 Class of Forces:

Two distinct classes of force that act on the fluid particles within the control volume.

1) Body forces act on all the fluid within the control volume - Weight of the fluid.
2) Surface forces act only on the control surface - They are always expressed in terms of stress (force per unit area).
   a) Pressure force:
      - Always acts perpendicular to the control surface - in the opposite direction to the unit normal
      - Always tends to compress the fluid in the control volume
      - Depends on the flow speed.
   b) Viscous forces
      - Acts at an angle to any particular face
      - Have two components in two dimensional flow acting on each face
        1. ***Direct stress*** acting perpendicularly to the face
        2. ***Shear stress*** acting tangentially to the face.

$$F = \sum \vec{F}_{body} + \sum \vec{F}_{surface} = m\cdot\frac{D\vec{V}}{Dt}$$

Let body force $\vec{F} = \vec{b}$and substituting density for mass, we obtain the following equation

$$\vec{b} = \rho\frac{d}{dt}\vec{v}(x, y, z, t)$$

NOTE: Substitution of density for mass is possible due to assumption of operating with a fixed control volume and infinitesimal fluid parcels.

The body force $\vec{b}$ is is a force that acts throughout the body of fluid (as opposed to, say, a shear force, which acts parallel to a plane).

Applying the chain rule to the derivative of velocity, we get,

$$\vec{b} = \rho(\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v})$$

Substituting the value in parentheses for the material derivative, the final equation is obtained:

$$\rho\frac{D\vec{v}}{Dt}=\vec{b}$$

2.3.9 Equations of Motion

Assuming that the body force on the fluid particles is composed of two components:

a) Fluid stresses
b) External forces.

Cauchy's equation: $\rho\frac{D\vec{v}}{Dt}=\vec{b}$, where $\vec{b}=\nabla\cdot\sigma_{ij}+\vec{f}$

Here, σ_{ij} is the *stress tensor*, and $\vec{f}$represents *external body forces*. Intuitively, the fluid stress/surface forces is represented as the divergence of the stress tensor because the divergence is the extent to which the tensor acts like a sink/ source. In other words, momentum source or sink, also known as force is derived from the divergence of the tensor.For many applications it is sufficient to say that $\vec{f}$ is composed only of gravity, but for now we will leave the equation in its most general form.

The equations that relate σ_{ij} to other variables such as velocity, pressure, and fluid properties are known as constitutive equations. There are different constitutive equations for different kinds of fluids. The equations of motion depend on the stress tensor σ. The tensor can be represented as

$$Stress\ Tensor=\sigma_{ij}=\begin{pmatrix}\sigma_{xx}\,\tau_{xy}\,\tau_{xz}\\ \tau_{yx}\,\sigma_{yy}\,\tau_{yz}\\ \tau_{zx}\,\tau_{zy}\,\sigma_{zz}\end{pmatrix}$$

$$Rate\ of\ stress\ tensor=\begin{pmatrix}\dot{\varepsilon}_{xx}\,\dot{\gamma}_{xy}\,\dot{\gamma}_{xz}\\ \dot{\gamma}_{yx}\,\dot{\varepsilon}_{yy}\,\dot{\gamma}_{yz}\\ \dot{\gamma}_{zx}\,\dot{\gamma}_{zy}\,\dot{\varepsilon}_{zz}\end{pmatrix}, \dot{\varepsilon}\ is\ the\ direct\ strain\ \&\dot{\gamma}\ is\ the\ shear\ strain$$

A tensor is a generalization of the concept of the higher-order quantity; a vector is represented as a first order tensor, a matrix as a second order tensor, a 3D matrix is a third order tensor, and so on.

2.3.10 General Form of Navier-Stokes Equation

The stress tensor σ_{ij} denoted above is often divided into two terms of interest in the general form of the Navier-Stokes equation. The two terms are the volumetric stress tensor, which tends to change the volume of the body, and the stress deviator tensor, which tends to deform the body. The volumetric stress tensor denotes the force which defines the volume of the body (for example, pressure forces). The stress deviator tensor denotes the forces which govern body deformation and movement, and consists of the shear stresses on the fluid.

Thus, σ_{ij} is broken down into

$$\sigma_{ij}=\begin{pmatrix}\sigma_{xx}\,\tau_{xy}\,\tau_{xz}\\ \tau_{yx}\,\sigma_{yy}\,\tau_{yz}\\ \tau_{zx}\,\tau_{zy}\,\sigma_{zz}\end{pmatrix}= =-\begin{pmatrix}p&0&0\\0&p&0\\0&0&p\end{pmatrix}+ +\begin{pmatrix}\sigma_{xx}+p & \tau_{xy}\,\tau_{xz}\\ \tau_{yx}\,\sigma_{yy}+p & \tau_{yz}\\ \tau_{zx}\,\tau_{zy}\,\sigma_{zz}+p & \end{pmatrix}$$

Denoting the stress deviator tensor as T, and substituting

$$\sigma_{ij}=-pI+T$$

Substituting this in the preceding equation, we arrive at the broad form of the Navier-Stokes equation:

$$\rho\frac{D\vec{v}}{Dt}=-\nabla p+\nabla\cdot T+\vec{f}$$

The left hand side of the equation, $\rho\frac{D\vec{v}}{Dt}$ is the force on each fluid particle. The equation conditions that the force consists of three terms:

- $-\nabla p$: A pressure term (volumetric stress tensor) that inhibits motion due to normal stresses. The fluid presses against itself and keeps it from shrinking in volume.
- $\nabla\cdot T$: A stress term (known as the stress deviator tensor) which causes motion due to horizontal friction and shear stresses. The shear stress results in turbulence and viscous flows.
- $\vec{f}$: The force term which is acting on every single fluid particle.

This intuitively explains turbulent flows and some common scenarios.

Although this is the general form of the Navier-Stokes equation, it cannot be applied until it is precisely specified. First of all, an expression must be determined for the stress tensor T, depending on the type of fluid.

Compressible Newtonian fluid

For Newtonian fluid, Stress is directly proportional to Rate of strain, i.e., $\tau\propto\frac{\partial u}{\partial y}$

To apply this to the Navier–Stokes equations, Stokes made three assumptions:

- The stress tensor is a linear function of the strain rates.
- The fluid is isotropic.
- For a fluid at rest, $\nabla\cdot T=0$. (so that hydrostatic pressure results).

Applying these assumptions will lead to:

$$T_{ij}=\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\delta_{ij}\lambda\nabla\cdot\vec{v}$$

δ_{ij} is the Kronecker delta. μ and λ are proportionality constants related with the assumption that stress depends linearly on strain; such thatμ is the first coefficient of viscosity and λ is the second coefficient of viscosity (also known as bulk viscosity). The sign and value ofλ, the bulk viscosity coefficient which produces a viscous effect related with volume change, is very difficult to determine. The term is often negligible even while considering compressible flows; nonetheless, it can sometimes be important even in nearly incompressible flows and is a topic of controversy. When taken non-zero, the most common estimate is $\lambda\approx-⅔\,\mu$.

A straightforward substitution of T_{ij} into the momentum conservation equation will yield the ***Navier–Stokes equations for a compressible Newtonian fluid***:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}+\lambda\nabla\cdot\vec{v}\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right)+\rho g_x$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left(\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right) + \frac{\partial}{\partial y}\left(2\mu\frac{\partial v}{\partial y}+\lambda\nabla\cdot\vec{v}\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right)+\rho g_y$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}\left(\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right) + \frac{\partial}{\partial z}\left(2\mu\frac{\partial w}{\partial z}+\lambda\nabla\cdot\vec{v}\right)+\rho g_z$$

When denoted in the vector form:

$$\rho\left(\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\nabla\vec{v}\right) = -\nabla p+\nabla\cdot\left(\mu\cdot\left(\nabla\vec{v}+(\nabla\vec{v})^T\right)\right) + \nabla(\lambda\nabla\cdot\vec{v})+\rho\vec{g}$$

where the transpose has been used. Gravity has been considered as the body force, i.e. $\vec{f}=\rho\vec{g}$. The related mass continuity equation is:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{v})=0$$

In addition, an equation of state and an equation for the conservation of energy is required. The conservation of energy could be defined by the following equation:

$$\rho\frac{Dh}{Dt}=\frac{Dp}{Dt}+\nabla\cdot(k\nabla T)+\varphi$$

Where, $h$denotes the enthalpy, Tdenotes the temperature, and φdenotes a function symbolizing the dissipation of energy caused due to viscous effects:

$$\varphi=\mu\left(2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+2\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2\right) + \lambda(\nabla\cdot\vec{v})^2$$

The previously obtained system of equations appears to properly model the dynamics of all known gases and most liquids.

3. Synthetic Curves

Analytic curves that are described by conic is insufficient to meet the geometric design requirements of the complex mechanical parts including that of the aerospace components. Airplane fuselages and wings, secondary control surfaces including elevator, and propeller blades are few of the many examples of aerospace components that need free-form, or synthetic, curves. Generally, the need for synthetic curves in design arises from the need to represent the curves as the collection of measured data points.

To express it mathematically, synthetic curves finds a solution to the problem of constructing a smooth curve that has to pass through a definite set of data points. Hence, polynomials are generally used to represent these curves.

In general, various degrees of continuity can be imposed at specified data points to obtain the desired smoothness for the resulting curve. The order of continuity becomes significant when a complex curve has to be constructed by merging several segments pieced together end-to-end. Zero order continuity yields a position-continuous curve, whereas the first and second order continuities imply slope and curvature continuous curves, respectively.

To explain, a cubic polynomial is the polynomial of minimum order that guarantees the generation of zeroth, first and second order curves. Moreover, it is the lowest-degree polynomial that permits modulation within a curve segment and allows the illustration of the non-planar(twisted) three dimensional curves in space. Computer-aided design is not concerned about higher-order polynomials, as they are computationally inconvenient, and are uneconomical of storing curve and surface representations.

Major CAD/CAM systems provide three types of synthetic curves: Cubic Splines, Bezier Curves and B-Spline curves. The cubic spline curve passes through the data points and therefore is an interpolant, whereas Bezier and B-Spline curves do not through(approximate) or interpolate (pass through) the data points.

3.1 Introduction to Bezier Curves

A Bezier curve is described by a set of data points. Bezier curves and surfaces as foretold by P. Bezier of the fresh car firm Regie Renault, who advanced them in 1962 and made use of them in his software system known as UNISURF, which designers used to define the outer panels of various Renault cars. These curves, also known as Bezier curves were also independently developed by P. De Casteljau of the French car company Citroen (about 1959), which used them as a part of Computer Aided Designing(CAD) system. The Bezier UNISURF system was soon published in the literature which is the reason that the curves now bear Bezier's name.

The few among the significant characteristics of the Bezier Curves are:

1) The shape of the curve is controlled by its defining points. Unlike the preliminary form of synthetic curve, i.e., Hermite cubic spline, tangent vectors are not used in the curve development.
2) The order or the degree of the curve is variable and is related to the number of points defining it. 'n+1' points defining a n^{th} degree curve, which permits higher order continuity.

The data points are also known as the control points. They form the vertices of what is called as the characteristic or control polygon, which uniquely defined the curve shape. Only the first and the last control points or vertices of the polygon actually lie on the curve, whereas the curves approximates through the remaining intermediate points. The curve is also tangent to the first and the last polygon segments. In addition, the shape of the curve follows that of the polygon.

Bezier curve is a parametric curve that uses Bernstein polynomial as its basis. A curve of degree n(order n+1) could be defined as:

$r(t) = \sum_{i=0}^{n} b_i \binom{n}{i} t^i (1-t)^{n-i}$, $0 \le t \le 1$, such that $B_{i,n}(\mathrm{t}) = \binom{n}{i} t^i (1-t)^{n-i}$ is the basis function, which together with the co-efficients, b_i, the control points or Bezier points, determine the shape of the curve.

Lines drawn between the consecutive control points of the curve form the control polygon.

3.2 Properties of the Bezier Curve

a) Bezier curves always lie inside the convex hull of their control points,
b) Bezier curves pass through the first and last control points and the slopes of the curves on both ends equal the slopes of the corresponding control polygon(tangent to the control polygon at the end points),
c) a Bezier curve of degree m can be exactly represented by a new Bezier curve of degree m+1 by degree elevation
d) Partition of unity property of the Bernstein polynomial assures the invariance of the shape of the Bezier curve under rotation of its control points.
e) The curve is symmetric with respect to u and (1-u). This means that the sequence of control points defining the curve can be reversed without changing the curve shape.
f) A closed Bezier curve could be constructed if the first and the last control points coincide.

The first two properties are useful in handling the geometry constrains. The third property permits an increase in the degree of the curve by introducing additional control points so that the curve can represent a more complex shape when needed in the shape optimization process.

4. Flow Over a Rotating Disc

4.1 Problem Description

Considering a flat disk which rotates about an axis perpendicular to its plane with a uniform angular velocity, ω, in a fluid which is otherwise at rest. The layer near the disk is carried by it through friction and is thrown outwards owing to the action of centrifugal force. Particles that flow towards the disk in the axial direction that has to be in turn carried and driven out centrifugally, compensates for it. Thus, the case is seen to be one of the fully three-dimensional flow, i.e., there exists velocity components in the radial direction, r, the circumferential direction φ, and the axial direction, z, which is denoted by V_r, V_θ and V_z. An axonometric representation of this flow field is shown below.

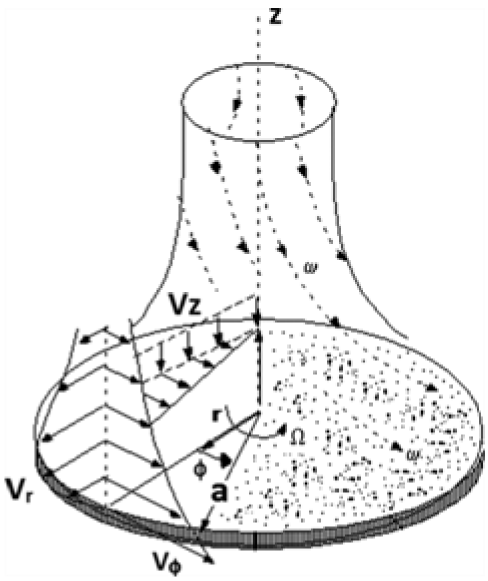

Figure: Model Description of the Laminar flow over the circular rotating disc

At first, the calculation will be performed for the case of an infinite rotating plane. It will then be easy to extend the result to include a disk of finite diameter $D = 2R$, on condition that the edge effect is neglected.

Taking into account of the rotational symmetry as well as the rotation for the problem, the Navier Stokes equations could be written as:

$$\frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{\partial V_z}{\partial z} = 0$$

$$V_r \frac{\partial V_r}{\partial r} - \frac{V_\theta^2}{r} + V_z \frac{\partial V_r}{\partial z} = \frac{-1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 V_r}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{V_r}{r}\right) + \frac{\partial^2 V_r}{\partial z^2}\right]$$

$$V_r \frac{\partial V_\theta}{\partial r} - \frac{V_r V_\theta}{r} + V_z \frac{\partial V_\theta}{\partial z} = +\nu\left[\frac{\partial^2 V_\theta}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) + \frac{\partial^2 V_\theta}{\partial z^2}\right]$$

$$V_r \frac{\partial V_z}{\partial r} + V_z \frac{\partial V_r}{\partial z} = \frac{-1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r}\frac{\partial V_z}{\partial r} + \frac{\partial^2 V_z}{\partial z^2}\right]$$

The no-slip condition at the wall gives the following boundary conditions:

$$z=0: V_r=0;\ V_\theta = r\omega;\ V_z=0$$
$$z=\infty: V_r=0;\ V_\theta = 0$$

4.2 Reduction from the Ordinary differential to Partial Differential Form

To begin by estimating the thickness, δ, of the layer of fluid 'carried' by the disk.

The centrifugal force per unit volume which acts on a fluid particle in the rotating layer at a distance r from the axis is equal to $\rho r \omega^2$. Hence, for a volume of area $dr.ds$ and height, δ, the centrifugal force becomes: $\rho r \omega^2 \delta dr ds$. The same element of fluid is acted upon by a shearing stress τ_w, pointing in the direction in which the fluid is slipping, and forming an angle, say θ, with the circumferential velocity.

The radial component of the shearing stress must now be equal to the centrifugal force, and hence

$$\tau_w sin\theta drds = \rho r\omega^2 \delta drds$$
$$\tau_w sin\theta = \rho r\omega^2 \delta$$

On the other hand, the circumferential component of the shearing stress must be proportional to the velocity gradient of the circumferential velocity at the wall. This condition gives,

$$\tau_w cos\theta \sim \mu r\omega/\delta$$

Eliminating τ_w from these two equations, we obtain

$$\delta^2 \sim \frac{\nu}{\omega} tan\theta$$

If it is assumed that the direction of slip in the flow near the wall is independent of the radius, the thickness of the layer carried by the disk becomes

$$\delta \sim \sqrt{\frac{\nu}{\omega}}$$

Hence, the shearing stress at the wall

$$\tau_w \sim \rho r\omega^2 \delta \sim \rho r\omega\sqrt{\nu\omega}$$

R, denoting the radius of the disk.

In order to integrate the system of equations, it is convenient to introduce a dimensionless distance from the wall

$$Z = z\sqrt{\frac{\omega}{\nu}}$$

Further the following assumptions are made for the velocity components and pressure

$$\boldsymbol{V_r = r\omega F(Z);\ V_\theta = r\omega G(Z);\ V_z = \sqrt{\nu\omega} H(Z); p = p(z) = \rho\nu\omega P(Z)}$$

Inserting these equations into equations ( ), we obtain a system of four simultaneous ordinary differential equations for the functions F, G, H and P. The derivatives are obtained with respect to the independent variable Z, and primes representing derivatives with respect to Z.

$$\boldsymbol{2F + H' = 0}$$
$$\boldsymbol{F^2 + F'H - G^2 - F'' = 0}$$
$$\boldsymbol{2FG + HG' - G'' = 0}$$
$$\boldsymbol{P' + HH' - H'' = 0}$$

Transformed boundary conditions are:

$$\boldsymbol{Z = 0;\ F = 0, G = 1, H = 0, P = 0}$$
$$\boldsymbol{Z = \infty;\ F = 0, G = 0}$$

Hence, to conclude, a disk of radius R is rotating with the angular speed ω in still fluid. The flow could be assumed to be steady, incompressible, and symmetric about the rotating axis(axisymmetric), and has fixed properties. No-slip condition is assumed to be satisfied by the fluid at the disk wall. The centrifugal effects cause the fluid to leave the disk near the disk. The flow above the disk must replace this airflow through a downward spiralling flow. A cylindrical coordinate system (r, θ, z) is used for description. V_r, V_θ and V_z are the velocity components, p is the pressure and ν the dynamic viscosity.

The original Navier-Stokes equations and the corresponding boundary conditions have been transformed from its partial differential form to the ordinary differential form. It is important to note that the last differential equation is not solved with the first three. It can be obtained from the solution of $F, G, and\ H$.

The solution to the problem was first obtained using a power series around $Z = 0$, and an asymptotic series for large values of Z. Numerical solutions are currently accepted as the alternate analytical solutions for this problem today. MATLAB is used to generate the numerical solutions through its boundary value problem solver used for comparison with the Bezier functions. The final value of the residuals, the error in the differential equations over the range of the independent variable, should provide reasonable confidence that the Bezier functions provide an excellent solution.

4.3 Bezier Function Formulation

The functions $F, G,$ and $H$are used to represent the three Bezier functions. This is now a coupled set of non-linear differential equations. The three functions are tied to the same parameter value and the values for the independent variable. These Bezier functions must solve the differential equations and boundary conditions in equations defined above. To avoid additional terminology in defining functions we will identify our three Bezier functions, as $[Z, F], [Z, G],$ and $[Z, H]$. These are determined by three sets of vertices:

$$[\boldsymbol{P_F^1, P_F^2, \ldots. P_F^{m+1}}], [\boldsymbol{P_G^1, P_G^2, \ldots. P_G^{m+1}}], [\boldsymbol{P_H^1, P_H^2, \ldots. P_H^{m+1}}]$$

The optimization problem is: Find the vertices of the Bezier Functions$[Z, F], [Z, G],$ and $[Z, H]$ and their derivatives that To Minimize:

$$\boldsymbol{f = \sum_{i=1}^{n_p} \left\{[2F_i + H'_i]^2 + \left[F_i{}^2 + F'_i H_i - G_i{}^2 - F''_i\right]^2\right\} + \sum_{i=1}^{100} \{[2F_i G_i + H_i G'_i - G''_i]^2\}}$$

Subject to:

$$\boldsymbol{[Z_1, F_1] = [0,0];\ [Z_1, G_1] = [0,1];\ [Z_1, H_1] = [0,0];}$$
$$\boldsymbol{[Z_{m+1}, F_{m+1}] = [0,0]; [Z_{m+1}, G_{m+1}] = [0,0],}$$

Where i represents a point on the curves, and m order of the curves. The number of points n_p on each of the curve is chosen to be 101. The objective function is the residuals of the differential equation, for the three equations, over the domain/trajectory.

5. Results

5.1 Bezier Solution

The solution to the optimisation problem is as follows:

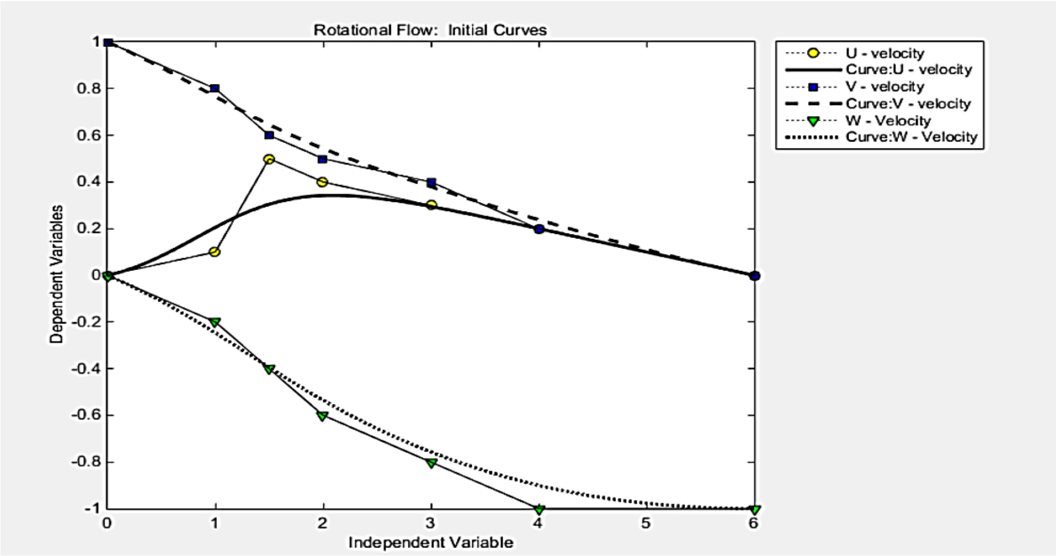

Figure: Initial Curves 1st set of inputs

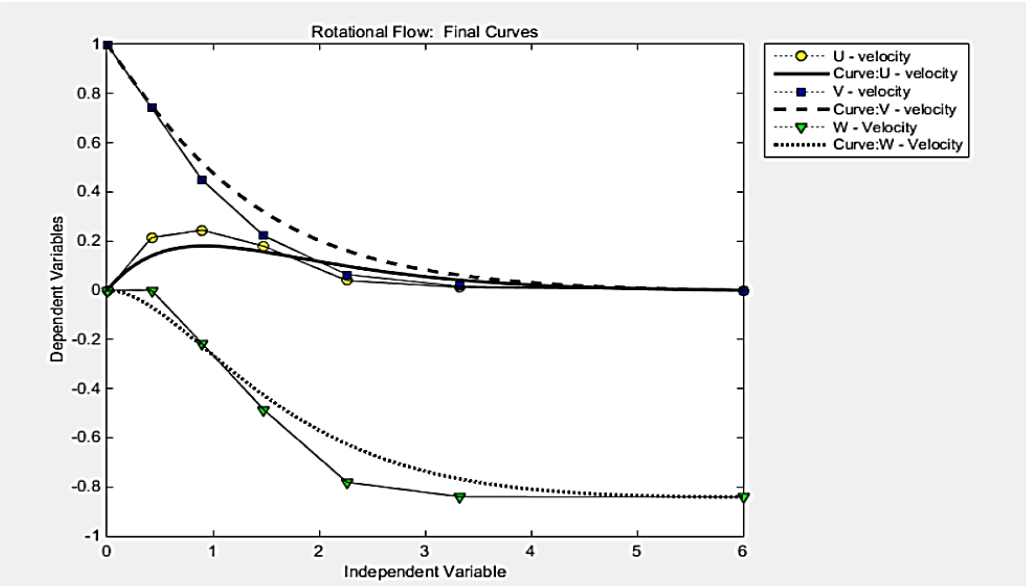

Figure: Final Curves for 1st set of inputs

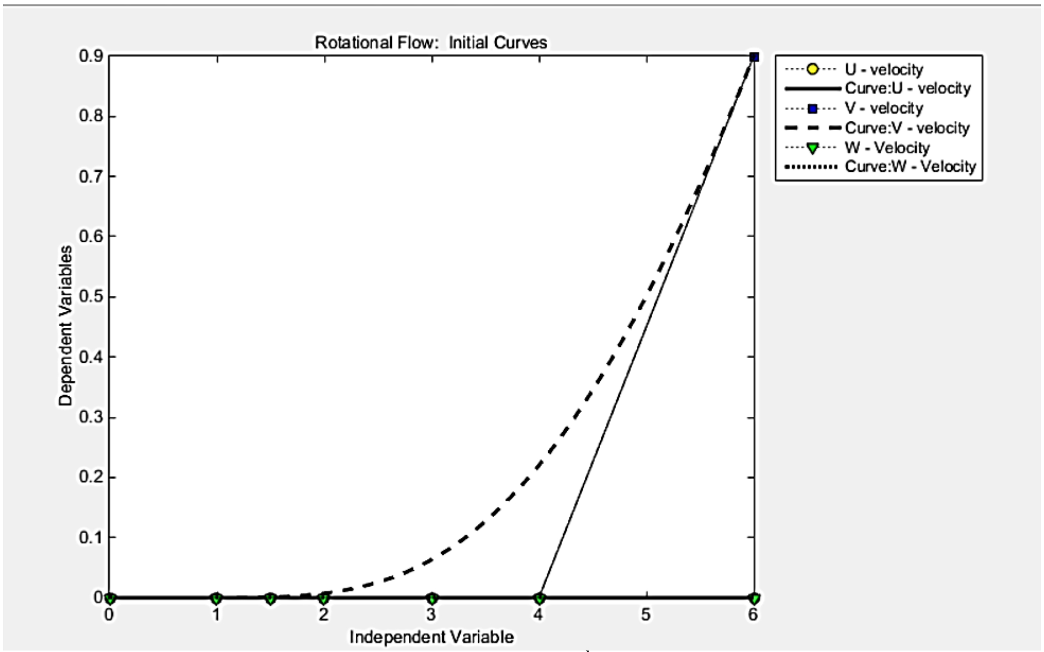

Figure: Initial Curves 2nd set of inputs

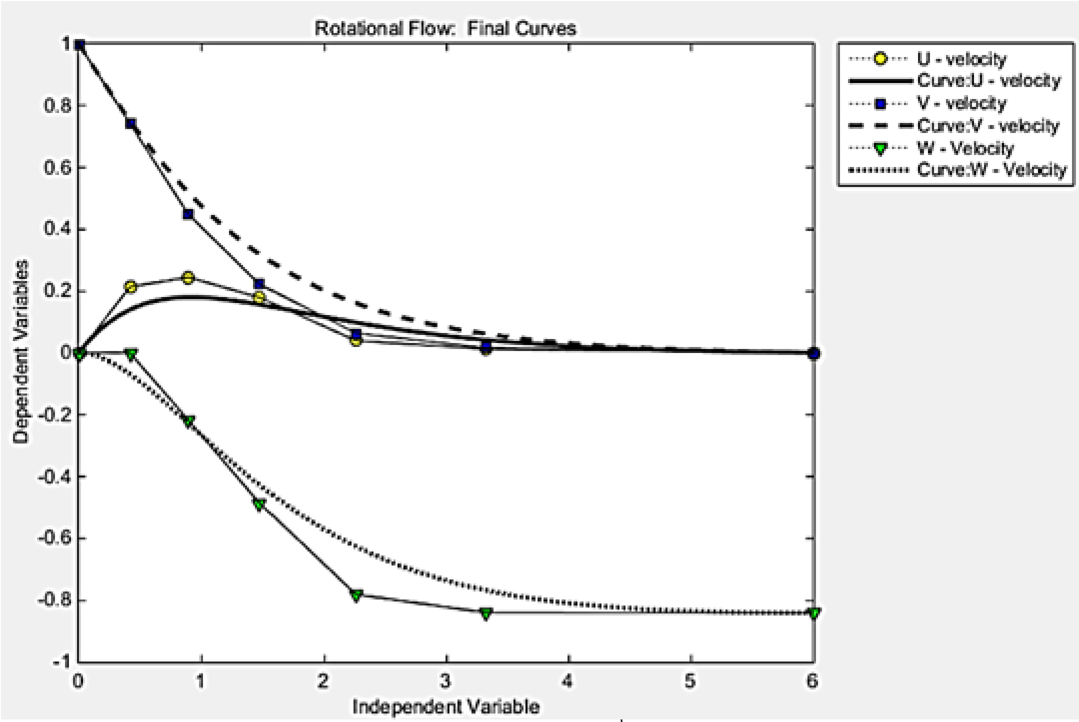

Figure: Final Curves for 2nd set of inputs

The design variables that are the solution to the optimization problem are the vertices, which are also displayed as markers. The Bezier functions that represent the solution to the nonlinear boundary value problem (NLBVP) are the curves that are obtained explicitly from the vertex information and are also shown in the figure.

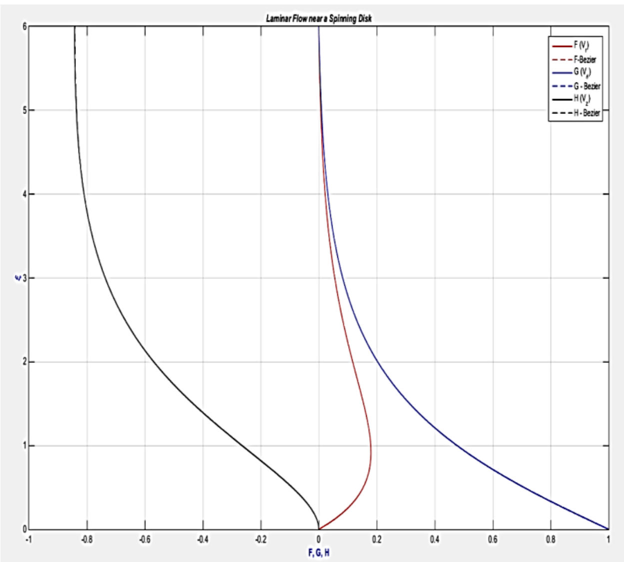

Figure: Comparison of the results obtained from two approaches

Figure also shows the comparison of the Bezier functions with the other numerical solution obtained using the BVP solver in MATLAB. The axes are flipped to display the boundary layer picture.
We find that the two solutions are indistinguishable within the scale of the graph. The residuals of the Bezier functions are exact.

The order of the functions was six for all of the curves. The number of design variables was 28. The convergence criteria for the optimum were set to 1.0e-08. Iterations stopped when the objective function F could not improve further due to meeting the tolerance on the constraints (1.0e-06). The final value of f was 1.21e-06. This is sum of squared error in the three differential equations was over 101 points each. The target value is zero. The number of iterations of the solver was 63. Since the residuals are computed exactly, the low value of the sum of the residuals for the final iteration should provide reasonable confidence that the Bezier functions provide excellent solution.

5.2 Analytical or Explicit Bezier Solution

The explicit solution for the independent variable and the three functions are given here:

$$Z(p) = 0.40468p^6 - 0.10827p^5 + 0.71410p^4 + 0.66665p^3 + 1.1775p^2 + 3.1453p$$

$$F(p) = 0.40468p^6 - 0.10827p^5 + 0.71410p^4 + 0.66665p^3 + 1.1775p^2 + 3.1453p$$

$$G(p) = 0.40468p^6 - 0.10827p^5 + 0.71410p^4 + 0.66665p^3 + 1.1775p^2 + 3.1453p$$

$$H(p) = 0.40468p^6 - 0.10827p^5 + 0.71410p^4 + 0.66665p^3 + 1.1775p^2$$

6. Conclusions

The main thesis of the chapter is to express the representation of the data or the solution to the differential equation out together through a parametric curve or a surface. These curves or surfaces can be required to behave as a function by employing simple side constraints during their creation. These constraints are routine in applied numerical optimization. The solution therefore is the curve that will reduce the squared error over all of the data points or satisfy the differential equations and the boundary conditions.

The Bezier curve, defined through geometric construction by Bezier, can be reproduced using the Bernstein basis. The Bernstein polynomial approximation to a continuous function mimics the gross features of the function remarkably well. Furthermore, as the order of the polynomial is increased, this approximation converges uniformly to the function and its derivatives where they exist. The Bezier curve brings, at the minimum, the equivalent smoothness as the original function it is trying to match.

Several kinds of parametric curves have been investigated to provide solutions to various differential problems. The one that appears often in connection with solving differential equations is the B-Spline. B-Splines are essential elements in computer graphics. Parametric Bezier curves using the Bernstein basis functions are also a special class of the uniform B-Splines using an open-knot vector. B-Splines, however do not possess continuous derivatives over the whole domain, and therefore are not very useful for establishing analytical solutions.

The last known explicit solution was obtained several decades ago and is considered too challenging to be pursued today. Here we have shown how numerical optimization in combination with a parametric function definition can easily establish solutions to the intractable problems of the past and the future. Such an approach can be regular feature in building new mathematical model for physical process and situations.

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