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Completely Regular Semiring

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Abstract: ***Semiring theory is one of the most developing branches of Mathematics with wide application in many disciplines such as Computer Science. Coding theory, Topology space and many researchers studies different structure of semirings like complemented semirings, Boolean like semirings etc. In this paper, we discuss some properties of Completely regular semiring, In this paper, we have many proved criteria on the different structure of semirings, Anti-inverse (S,+) and quasi-separative (S,.) is proved through completely regular semiring application, completely regular semiring a + x = a is proved, as Distributive, (S,+) is a Boolean ring. On the condition of completely regular semiring anti-inverse is proved as monosemiring and some other relations are proved. We determine the additive and multiplicative structures of a completely regular semiring by assuming different properties on the additive (multiplicative) structures.***

Keywords: Semirings, Completely regular semiring, Monosemiring, Simple semiring Anti-inverse, Boolean Semiring, Quasi-separative semiring, Distributive law

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1. Introduction

The notion of semiring was first introduced by Vandiver [13], in 1934. Vandiver introduced an algebraic system, which consists of non empty set S with two binary operations addition (+) and multiplication (.).The system (S, +,.) satisfies both distributive laws but does not satisfy cancellation law of addition. The system he constructed was ring like but not exactly rings. Vandiver [13] called this system a 'Semiring'. The first steps towards completely regular semirings were done in J. Zeleznekow [11] where a good overview about additively regular semirings has been given. In the last decade, there have been different approaches to introduce a concept of completely regular semirings so that these inherit the structural regularity from their additive reducts. F. Pastijn and Y.Q. Guo[8] analyzed semirings which are composed of disjoint rings. A more generalized approach was chosen by Sen, Maity and Shum in [7] when they claimed that for each element α of a completely regular semiring there has to exist an element x such that a = a + x + a, a + x = x + a and a(a + x) = a + x. We shall follow the terminology proposed by P.M. Drazin[10]. M.K. Sen, S. K. Maity[5] studied the completely regular semiring. The research of separative semigroup was being began from the famous paper of Hoeitt and Zuclevman. Drazin [3] introduced the term 'quasi-separativity' and studied connection between it and other semigroup properties. They proved some results on commutative separative semigroup. Venkateswarlu. [4] studied "Boolean Like Semirings", Bedrich, Pondelicek [12] proved the least separative, separative congruence on a weakly commutative semigroup. Maity & Ghosh [1] studied on the class of idempotent semirings and also we follow the terminology proposed by Shobhalatha and Bhat [3] in an anti-inverse semirings. Heinz Mitsch [14] defined partial order relation on a semigroup.

2. Preliminaries

Definition 2.1: A semiring is a non empty set S on which operations of addition "+" and multiplication "." have been defined such that the following conditions are satisfied:
(i) (S,+) is a semigroup
(ii) (S,.) is a semigroup
(iii) Multiplication distributes over addition from either side.

Examples of semiring
i) The set of natural numbers under the usual addition, multiplication.
ii) Every distributive lattice (L,V,^), (iii) Any ring (R,+,.)

Definition 2.2: An element 'a' of a semiring (S,+,.) completely regular if there exists an element xϵS, satisfying the following conditions (i) a = a + x + a, (ii) a + x = x + a, (iii) a(a + x) = a + x

Definition 2.3: A completely regular semigroup S, if for each element 'a' in S there is an element 'b' in S such that aba = b and bab = a The elements a and b are then called anti-inverse. e.g.

*	a	b	C
a	a	a	c
b	a	b	c
c	c	c	a

aaa =a, bbb = b since a and b are their own anti-inverses. ccc = ac = c, cac = cc = a, aca = ac = c, hence a and c are anti-inverses.

Definition 2.4: A semiring S is called simple of a + 1 = 1 = 1 + a for and a∈S.

Definition 2.5: A semiring S is called quasi-separative if for any a, b∈S, $a^2 = ab = b^2$ implies a = b

Definition 2.6: A system (S,+,.) a Boolean semiring if and only if the following properties hold:

i) (S,+) is an abelian group.
ii) (S,.) is a semigroup.
iii) a(b + c) = ab + ac and
iv) abc = bac, for all a, b, c in S.

3. Some theorems on properties of completely regular semirings x

Theorem 3.1 Let (S, +) be a completely regular semiring if a + x = a

Proof: Let (S, +) be a completely regular semiring. Consider, a + x = a + x + a + x => a + x + x + a => a + x + a => a

Conversly, if a + x = a then (S,+) be a completely regular semiring

i) Consider, a+x=a=> a + x + a = a + a => a + x + a = a
ii) Consider, a+x= a=>x+ a + x = x + a => a + x = x + a
iii) a+ x = a => a(a + x) = a.a => a(a + x) = a => a(a + x) = a + x + a => a(a + x) = a + a + x => a(a + x) = a + x

Theorem 3.2: Let (S,+,.) be a completely regular semiring, then (S,+) is an Anti-inverse

Proof: We first suppose that S is completely regular semiring, Then for any a∈S, satisfying the condition a + x + a = a, and a + x = x + a, We need to prove that a + x + a = x and x + a + x = a, By the definition of completely regular semiring, we have a + x + a = a implies a+a+ x = a suppose that (S,+) is band, a+x =a and we can also implies that a+x+x=a, by the definition of completely regular semiring, we have x + a + x = a

Similarly we have prove a + x + a = x, by the definition of completely regular semiring we have a + x + a = a implies (a + x + a) + a = a which implies that a + a = a, then add x∈S on both side, such that a+a+x=a+x implies a + x = a + x + x

From completely regular semiring condition a + x = x + a + x, suppose that (S,+) is band a + a + x = x + a + x which also implies that a + x + a = x, Therefore (S,+) is Anti-inverse.

Lemma: Let (S,+,.) be a completely regular semiring, a + b + a = a => b + a + a = a => b + a = a => b + a + 1 = a + 1 => b + a + 1 + 1 = a + 1 + 1 => b + 1 + a + 1 = 1 + a + 1 => b + 1 = 1

Similarly, a (a + b) = a + b => a.a + a.b + 1 = a + b +1 => a.a + a.b + 1 + 1 = a + b + 1 + 1 => a + a.b + 1 + 1 = a + 1 + b + 1 => a(1 + b)+ 1 + 1 = a + 1 => a + 1 + 1 = a + 1 => 1 + a + 1 = a + 1 => 1 = a + 1

Therefore (S,+) is simple semiring.

Theorem3.3: Let (S,+,.) be a completely regular semiri ng then (S,.) is quasi-separative.

Proof: Let (S,+,.) be a Completely regular semiring

Consider, a^2 = ab, post adding by ‘b’ whichimplies as a^2 + b = ab + b by the second definition of completel y regular semiring, a (a + b + a) + b = a(b + a + b) + b this can be written as a(a+b) + a.a+ b= a(b+a)+a.b+ b, if (S,.) is a band and third condition of completely regular implies that a + b + a + b = b + a + ab + b implies a+ b = b + a(1 + b) + b from definition completely regular and b the simple semiring implies that a + b + a + b= b + a +b since S is commutative a + a + b + b= b + a + bimp lies a + a + b =b + a + b (S is a band) then implies a + b + a = b + a + b therefore a = b. Similarly, we can show this if, ab = b^2 then a = b,

Consider ab = b^2 preadding by ‘a’ which implies as a + a.b = a + b.b from completely regular semiring a + a(b + a + b) = a + b (b + a + b) implies that a + a(b + a) +a.b = a + b(b + a)+ b.b then implies that a + b+ a + a.b = a +b + a + b (from completely regular definition) implies a + b + a + ab = a + b + a + b this Implies a + b + a(1 + b) = a + b from definition completely regular and by the simple semiring implies that a + b +a = a + b + a + b since S is commutative a + b+ a = a + a + b + b implies a +b +a= a + b + b (S is a band) then implies a + b+ a = b + a + b. Therefore a = b.

Theorem3.4: Let (S,+,.) be an completely regular semiring then ‘S’ is Distributive.

Proof: Given that (S,+,.) be an completely regular semiring. Thus aba = a for all a, b in S. To prove that S is distributive. It is enough to show that ‘+’ is distributive over ‘.’ (a.b) + c = (a + c).(b + c) for all a, b, c in S. Consider, (a + c).(b + c) = (a + c).b + (a + c).c implies that (a + c). (b + c) = (a.b)+ (c.b) + (a.c) + (c.c) which implies (a + c).(b + c) = (a.b) + (c.b) + (a.c) + c then (a + c).(b + c) = (a.b) + (c.b) + (aca)(cac) + c implies (a + c).(b + c) = (a.b) + (c.b) + ac(ac)ac + c implies (a + c).(b + c) = (a.b) + (c.b) + ac(ca)ac + c then implies (a + c).(b + c) = (a.b) + (c.b) + (acc) (aa)c + c implies (a + c). (b + c) =(a.b) + (c.b) + ac(cac) + c implies (a + c).(b + c) = (a.b) + (c.b) + acc + c (from condition of completely regular) (a + c).(b + c) = (a.b) + (c.b) + cac + c implies (a + c).(b + c) = (a.b) + (c.b) + c + c implies that (a + c). (b + c) = (a.b) + (cbc)(bcb) + c + c then (a + c).(b + c) = (a.b) + (cbc)(bcb) + c + c then (a + c).(b + c) = (a.b) + c.c (b.b) c. b + c + c implies (a + c).(b + c) = (a.b) + c.c (bcb) + c + c implies (a + c).(b + c) = (a.b) + c.c.b + c + c implies (a + c). (b + c) = (a.b) + c.b.c + c + c then (a + c).(b + c) = (a.b)+ c + c + c therefore (a + c).(b +c) = (a.b) + c, similarly we can prove all distributive condition, this implies c + (a.b) = (c + a). (c + b) for all a, b, c in S, Hence S is distributive.

Theorem 3.5: Let (S,+,.) be a completely regular semiring, and iff (S,+) Anti-inverse is (S,+) is a Monosemiring

Proof: Let (S,+,.) be a completely regular semiring
Let (S,+) is Anti-inverse.
Consider,a + x + a = x => a(a + x + a) =a.x =>a.a + a.x + a.a = a.x =>a(a + x)

+ a.a = a.x => a + x + a.a = a.x => a(1 + a) + x = a.x => a+x = ax
Therefore (S,+) is Monosemiring
Conversely,
Consider a + x = ax => a(a + x)= ax => a.a + a.x = a.x => a(a + x + a) + a.x = a.x => a(a + x + a + x)= a.x => a + x +a + x = x => a + x + a = x Therefore (S,+) is Anti-inverse.

Theorem 3.6: Let (S,+,.) be a completely regular semiring, then (S,+) is simple semi ring iff (S,.) is idempotent

Proof: Let (S,+,.) be a completely regular semiring,
If (S,+) is simple semiring then a + 1 = 1 pre multipl y by 'a' then, a(a + 1) = a.1
implies a.a + a.1= a implies thata.a + a = athen by sec ond condition of completely
regular which can written as a.a + a + b + a = a impl ies a.a + (a + b) + a = a from completely regular condition a.a + a(a + b)
+ a = a which implies a.a + a.a + a = a implies that a.a + a + a.a=a therefore from completely regular condition a.a = atherefore(S,.) is idempotent.
Conversely, If (S,.) is idempotent, we can prove this as (S,+) is simple semiring.
Let a.a = a implies by completely regular a(a + b + a) = a implies a(a + b) + a = a then implies that a + b + a = a this impliesa + b + a. a = a implies a + a.a = a then a(1 + a) = a.1 from left cancellation 1 + a = a therefore (S,+) is simple semiring.

Lemma: Let (S,+,.) be a completely regular semiring, then it proved as aba = bab such that aba = (aba)(bab)(aba) = a(bab)(aba)ba = a.b.a.b.a = b.a.a.a.b = bab

Theorem 3.7: Let (S,+,.) be a completely regular semiri ng, in which (S,.) is defined by a.b = aab for all a,b i n S then (S,+,.) is a Boolean semiring.

Proof: Let (S,+,.) be a completely regular semiring, sin ce a.a = a.a.a = a and a.0 = 0.a = 0 for all a in S, F or a,b in S by condition of completely regular we can say that a.b= b.a similarly by using same condition we can prove (S,.) is a commutative semigroup, consider a.b.c = a.(bbc) = a.a.b.b.c = a(aba)bbc = a.a(bab)b.c = a.a.b.b.c = ababc = a.b.c

Therefore (S,.) is a commutative semigroup.

Consider a.(b + c) = a.a(b + c) = aab + aac = a.b + a .c implies a.(b + c) = a.b + a.c (a + b). c = c. (a + b) since (S,.) is commutative, implies a + b).c = c.a + c.b implies (a + b).c = a.c + b.c thus a.(b + c) = (a + b).c
Since a + a = 0 for all a in S, every element of S has additive inverse, a.b.c = (aab).c implies a.b.c = (aab)(aab)c = (aba)a (bab) (aba) (aba) (bab)c = a(baa) b(aba)(baa) (bab) (acc) = b(aaa)baabc = b.a.(aba).b.c = b.a.a.b.c = b(aba)c = b.a.c
Hence (S,+,.) is Boolean Semiring.

Theorem 3.8: An element of a semiring S is quasi completely regular if and only if for each a∈S, there exits an element x∈S such that a = a + x + a,
(1.1.1) a(a + b) = a + b
(1.1.2) a(a + b) = ab(a + b)
(1.1.3) a(a + b) = b
(1.1.4) a(a + b) = a
(1.1.5) a(a + b) = ab

Proof: First, we suppose that a is completely regular. Then by quasi- completely regular, there exist an element x∈S such that
(1.1.1) Consider a(a + b) = a + b implies, by second condition of completely regular a(a + b) = aba + bab implies a(a + b) = ab(a + b)
(1.1.2) a(a + b)= a + b implies, by first condition of completely regular, a(a + b) = a + b+ a + b then implies that a(a + b) = b + a + a + b implies a(a + b) = b + a + b therefore from second condition of completely regular a(a+b)=b
(1.1.3) a(a + b) = a + b implies,by first condition of completely regular a(a + b) = a+b+a+b then implies that a(a + b)= a + a + b + bimplies a(a+b)= a + a + b therefore form second condition of completely regular a(a + b) = a + b + a such that a(a + b) = a
(1.1.4) a(a + b) = a(aba + bab)
= a[ab(a + b)] = ab[a(a + b)] = ab(a + b)
= b[a(a + b)] = b(a + b) therefore, a(a + b) = b(a + b)
(1.1.5) a(a + b) = a(aba + bab)=a[ab(a + b)]
= aab(a + b) = ab[a(a + b)] = ab[b(a + b)]= abb(a + b) = a(bba + bbb) = a(b ab + b) = a(b + b) = ab since S is a band.

Theorem 3.9: Let (S,+,.) be a completely regular semiring, satisfying the identity a + b + 1 = ab for all a, b in S

Proof: Let (S,+,.) be a completely regular semiring to prove that a+b+1 = ab

Consider,
i) a+b+a=a=>a+b+a+b= a+b=>a+b+a+b+1=a+b+1 => ab + ab +1 = a+ b+1 => ab + 1 + ab = a + b + 1 => ab = a + b + 1

ii) Consider, a(a + b) = a + b => a.a + a.b = a + b => a.a + a.b + 1 = a + b + 1
=>a + ab + 1 = a + b + 1 => ab + a + 1= a + b + 1 =>ab + 1 + ab = a + b + 1 + ab
=> ab + 1 + ab = a + b + ab + 1 => ab + 1 + ab = a + b(1 + a) + 1 => ab + 1 + ab = a + b + 1 => ab = a + b + 1

Theorem 3.10: Let (S,+,.) be completely regular semiring, Define a relation '≤' on 'S'such that a ≤ b iff a + b + 1 = ab for all a,b in S, If 'e' be the additive identity, (S,+) is simple semiring then (S,+,.) partially ordered semiring.

Proof: Let (S,+,.) be a completely regular semiring Define a relation '≤' on 'S', such that a ≤ b iff a + b + 1 = ab for all a,b in S

Consider aa = a + a + 1 => a = a + 1 + a => a = a => a ≤ a. Hence '≤' is reflexive.

Let a ≤ b and b ≤ a => a = b

Consider, ab = a + b + 1 => ab = b + a + 1 => ab = b + 1 => ab + b = b + 1 + b=>b(a + e) = b => a + e = b => a=b

Hence '≤' is Antisymmetric

Let a ≤ b, b ≤ c then a ≤ c

Consider, ab = a + b + 1 => ab + c = a + b + 1 + c => a + b + 1 + c = a + 1 + c => a + c + b + 1 = ac => a + c + 1 = ac=> a ≤ c
Hence '≤' is transitive.
If a ≤ b => a + b + 1 = ab
TPT ac ≤ bc
Consider, ab = a + b + 1 => ac.b = ac + bc + c => ac.bc = ac^2 + bc^2 + c.c.1 =>
(ac)(bc) = ac + bc + c.1 => (ac)(bc) = ac + bc + 1
Therfore ac ≤ bc
Similarly, we can prove that ca ≤ cb
If a ≤ b => a + b + 1 = ab,
To prove that a + c ≤ b + c
ab = a + b + 1 =>ab + c = a + b + 1 + c => ab + c + c = a + c + b + c + 1 =>
ab + ac + bc = (a + c) + (b + c) + 1 =>
ab + ac + bc + c^2 = (a + c) + (b + c) + 1 + c^2 =>(a + c)(b + c) = (a + c) + (b + c)
+ 1 Similarly we can prove that c + a ≤ c + b
(S,+, . ,≤) is a partially ordered semiring.

4. Conclusion

In Completely regular semirings,the algebraic structure o f multiplicative semigroup (S,.) determine the additive structure of (S,+) and vice versa. In a completely regular semirings satisfies the condition a+ x = a, and monosemiring, antiinverse, and partially ordered semiring satisfying the identity a + b + 1 = ab for all a, b in S. This work is supported by the Visvesvaraya Technological University, Belgavi, Karnataka_590018 India. My sincere thanks to my guide Dr. A. Rajeswari and Research centre EWIT, Bangalore, Karnataka-560091, India.

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