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An Application of Eigen Value and Eigen Vector to Discrete Dynamical System

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Abstract: ***Eigen value and eigen vectors have a variety of applications in physical and biological sciences. Dynamical system of Markov chain plays very important role in the study of behavior of phenomena in different areas of Physics, Chemistry, Biology, Testing, Queueing Theory, Computer and Information Technology. In this paper, we discuss an application of eigen values and eigen vectors to predator-prey dynamical system.***

Keywords: Eigen value, Eigen vector, Markov chain, Probability vector, Stochastic matrix.

1. Introduction

Eigen values are often introduced in context of linear algebra or matrix theory. Historically, however, they arouse in the study of quadratic forms and differential equations. In the beginning of the 20th century, David Hilbert [2] studied the eigen values of integral operators by viewing the operators as infinite matrices. He was the first person to use German word *eigen* which means "own", to denote eigen values and eigen vectors in 1904. Several mathematicians have made contributions in this field. The first numerical algorithm for computing eigen values and eigen vectors appeared in 1929, when Richard von Mises published the 'power method'.

There are many applications of matrices in both engineering and sciencewhich utilize eigen values and, sometimes, eigen vectors. Control theory, vibration analysis, electrical circuits, advanced dynamics and quantum mechanics [1] are some application areas. The application in biological population dynamics is interesting one when it takes the form of linear difference equation which is the context of present study.

2. Eigen Value and Eigen Vector

2.1 Definition: ([3])

Let V and W be two vector spaces over the same field $\mathbb{K}$(where $\mathbb{K}$ is the set of real or complex numbers). A transformation $T: V \to W$ is said to be a linear transformation if

(i) For all$v_1, v_2 \in V$, $T(v_1 + v_2) = T(v_1) + T(v_2)$
(ii) For all $v \in V$ and $\lambda \in \mathbb{K}$, $T(\lambda v) = \lambda T(v)$.

2.2 Definition: ([3])

Let V and W be two vector spaces over the same field $\mathbb{K}$ and $T: V \to W$ be a linear transformation.A scalar $\lambda \in \mathbb{K}$is said to be an eigenvalue (or a characteristic value) of linear transformation T if there exists a non-zero vector $v \in \mathbb{K}^n$ such that T(v) $= \lambda v$. And, then vector $v \in \mathbb{K}^n$ is called is called an eigen vector (or a characteristic vector) of transformation T corresponding to eigenvalue λ.

2.3 Theorem: ([3])

Let $V = \mathbb{K}^n$ and $W = \mathbb{K}^m$ be two vector spaces over the same field $\mathbb{K}$ (where dim(V) = n and dim(W) = m) and $T: V \to W$ be a linear transformation. Then, for transformation T there exists an$m \times n$ matrix A such that$T(v) = Av$ for each $v \in V$ and vice-versa. And, there exists an isomorphism between the set of all linear transformations $T: V \to W$ and set of all $m \times n$ matrices over $\mathbb{K}$.

2.4 Definition: ([3])

Let A be an$n \times n$ real or complex matrix. A scalar $\lambda \in \mathbb{K}$is said to be aneigen value (or a characteristic value) of matrix A if there exists a non-zero vector $x \in \mathbb{K}^n$ such that $Ax = \lambda x$. And, then vector$x \in \mathbb{K}^n$ is called is called an eigen vector (or a characteristic vector) of matrix A corresponding to eigenvalue λ.

2.5 Definition: ([5])

If λ is an eigen value of a square matrix A, then $\det(A - \lambda I)$ is called characteristic matrix and the polynomial $\det(A - \lambda I=0$ is called characteristic equation for the matrix A.

2.6 Theorem: ([5])

Let A be an$n \times n$ matrix and λ be a scalar. The following statements are equivalent:

(i) λ is an eigen value of A.
(ii) $(A - \lambda I)x = 0$has a non-trivial solution, where I is unit $n \times n$ matrix.
(iii) Null space of $A - \lambda I$, $N(A - \lambda I) \neq 0$.
(iv) $A - \lambda I$is singular.
(v) $\det(A - \lambda I) = 0$.

3. Markov Chain and Discrete Dynamical System

A Markov process is a stochastic process with properties: (i) the set of possible outcomes or states is finite (ii) the probability of the next outcome depends only on the previous outcome (iii) the probabilities are constant over time.

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3.1 Definition: ([4])

A vector with non-negative entries is called a probability vector of sum of its entries is equal to 1. A matrix is called a stochastic matrix if its column are probability vectors.

3.2 Definition: ([4])

If v_t denotes a vector depending upon time t and P is a stochastic matrix then a sequence of probability vectors $v_0, v_1, v_2, v_3, v_4, \ldots$satisfying

$$v_1 = Pv_0, v_2 = Pv_1, v_3 = Pv_2, v_4 = Pv_3, \ldots$$

i.e. $v_{t+1} = Pv_t$ (linear difference equation)
is called a Markov Chain.

Here, v_t is called state vectors and v_0 is called initial state vector. And, matrix P is called transition matrix for Markov Chain.

3.3 Definition: ([4])

If v_t denotes a vector depending upon time t and P is a stochastic matrix then set of equations:

$$v_1 = Pv_0, v_2 = Pv_1, v_3 = Pv_2, v_4 = Pv_3, \ldots$$

i.e. $v_{t+1} = Pv_t$
is called a discrete linear dynamical system.

4. Predator-prey model

Predator-prey cycles are based on a feeding relationship between two species. If prey species rapidly multiplies, the number of predator increases until the predators eventually eat so many prey that prey population dwindles again. Soon afterwards, predator numbers likewise decrease due to starvation. This in turn leads to a rapid increase in the prey population and a new cycle begins. Some examples of predator and prey are lion and zebra, bear and fish, and fox and rabbit. The relationship can be put in mathematical model of linear dynamical system.

4.1 Application

Let F_t denote number of foxes and R_t denote number (in thousand) of rabbits at time t. Let $v_t = \begin{pmatrix} F_t \\ R_t \end{pmatrix}$ denotes the fox and rabbit population at time t (where t is in months) and $P = \begin{pmatrix} 0.5 & 0.4 \\ -p & 1.1 \end{pmatrix}$ be the predator-prey stochastic matrix of these two population. We discuss the behavior of both predator and prey at any time t and estimate their long term behavior. Also, we observe the ratio of population fox and rabbit for a given prediction parameter $p = 0.104$.

Solution

For $p = 0.104$, we have, $P = \begin{pmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{pmatrix}$.

The eigen values are given by characteristic equation,

$$\det(P - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 0.5 - \lambda & 0.4 \\ -0.104 & 1.1 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 1.6\lambda + 0.5916 = 0$$

$$\Rightarrow \lambda = \frac{1.6 \pm \sqrt{(-1.6)^2 - 4 \times 1 \times 0.5916}}{2}$$

$$\Rightarrow \lambda = \frac{1.6 \pm 0.44}{2}$$

$$\Rightarrow \lambda = 1.02, 0.58$$

So, eigen values are: $\lambda = 1.02, 0.58$

Now, we find eigen vector corresponding to $\lambda = 1.02$. For this we must find solution of $Px = \lambda x$.

The corresponding augmented matrix is,

$$[P - \lambda I \quad 0] = [P - 1.02\, I \quad 0]$$

$$= \begin{bmatrix} -0.52 & 0.4 & 0 \\ -0.104 & 0.08 & 0 \end{bmatrix}$$

$$\backsim \begin{bmatrix} -0.52 & 0.4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow -0.52\, x_1 + 0.4\, x_2 = 0 \text{and} x_2 \text{ is free variable.}$$

$$\Rightarrow x_1 = \frac{0.4}{0.52} x_2$$

$$\Rightarrow x_1 = \frac{10}{13} x_2$$

Therefore, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{10}{13} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{10}{13} \\ 1 \end{bmatrix}$.

Thus, eigen vector corresponding to eigen value $\lambda = 1.02$is $v_1 = \begin{bmatrix} 10 \\ 13 \end{bmatrix}$.

Similarly, we can find eigen vector corresponding to eigen value $\lambda = 0.58$is $v_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

The initial state vector is $x_0 = c_1 v_1 + c_2 v_2$.

So, $Px_0 = P(c_1 v_1 + c_2 v_2)$

$$\therefore x_1 = c_1 P v_1 + c_2 P v_2$$

$$\therefore x_1 = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2$$

Again, $Px_1 = P(c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2)$

$$\therefore x_2 = c_1 \lambda_1\, P v_1 + c_2 \lambda_2\, P v_2$$

$$\therefore x_2 = c_1 \lambda_1 \lambda_1 v_1 + c_2 \lambda_2 \lambda_2 v_2$$

$$\therefore x_2 = c_1\, \lambda_1^2\, v_1 + c_2\, \lambda_2^2\, v_2$$

Similarly, $x_k = c_1\, \lambda_1^k\, v_1 + c_2\, \lambda_2^k\, v_2$.

Hence, $x_k = c_1 (1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix} + c_2 (0.58)^k \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

As $k \to \infty$, $(0.58)^k \to 0$. So for any $c_1 > 0$,

$$x_k = c_1 (1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix} \qquad \ldots\text{(i)}$$

Therefore, $x_{k+1} = c_1 (1.02)^{k+1} \begin{bmatrix} 10 \\ 13 \end{bmatrix}$

$$= (1.02) c_1 (1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix}.$$

$$= 1.02\, x_k$$

Thus, $x_{k+1} = 1.02\, x_k$ $\qquad \ldots$(ii)

Last equation (ii) shows that population of both fox and rabbit grow by factor 1.02. Also, the growth rate is 2 % monthly. Equation (i) shows that when $F_t = 10$ then $R_k = 13$ i.e. for every 10 foxes there are 13 thousand rabbits.

We remark that for different prediction parameter p, we get different behavior about number of foxes and rabbits for different levels of time.

5. Conclusion

We see that theory of eigen value and eigen vectors can be used to predict the population of species by observing their past and present behavior. Several such phenomena exist in nature which can thus be studied with help of the theory and

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a wise decision can be taken to make policies to balance ecosystem in nature.

References

[1] Fernandes, S & Ramos, C et al (2018), Discrete Dynamical System: A Brief Survey. *Journal of the Institute of Engineering* **14** (1): 35-51.
[2] Francis, JGF (1961), The QR Transformation, part-I, II. *Computer Journal*, ***4****(3): 265-271, 332-345.*
[3] Kolman, Bernard (2007), Introductory Linear Algebra with Applications. *7th Edition*, *Pearson Education.*
[4] Lay, DC (2014), Linear Algebra its Applications, *3rd Edition, Pearson Education.*
[5] Leon, SJ (2015), Linear Algebra with Applications.*Pearson Education.*

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