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The Consequence of the Analytic Continuity of Zeta Function Subject to an Additional Term and a Justification of the Location of the Non-Trivial Zeros

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Abstract: ***We review two main results of Riemann Zeta function; the analytic continuity and the first functional equation by the means of Gamma function and Hankel contour. We observe that an additional term is considered in both results. We justify the non-trivial location of Zeta non-trivial zeros subject to an approximation.***

Keywords: Riemann Hypothesis, Analytic Continuity, Functional Equation, Hankel Contour.

1. Introduction

In 1859, Bernhard Riemann published an eight-page paper, in which he estimated "the number of prime numbers less than a given magnitude" using a certain meromorphic function on $\mathbb{C}$. But Riemann did not fully explain his proofs; it took decades for mathematicians to verify his results, and to this day we have not proved some of his estimates on the roots of ζ. Even Riemann did not prove that all the zeros of ζ lie on the line *Re (s)* = 1/2. This conjecture is called the Riemann hypothesis and is considered by many the greatest unsolved problem in mathematics [1, 3, 5, 7]

The difference between the analytic continuity of Gamma and Zeta

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad Re(s) > 1$$

$$\Gamma(s) = \int_0^{\infty} e^{-\tau}\tau^{s-1}\,d\tau. \quad Re(s) > 1$$

Clearly, $\tau = 0$ is the *lower limit* of the integral representation of $\Gamma(s)$. Now, considering the Hankel Contour approach we can certainly provide the analytic continuity for all *s* except *s = 0,-1,-2*...see [2, 3, 4 and 10]

Riemann considered the same approach [1, 2, 3 and 10]

Let $\tau = nt \rightarrow d\tau = n\,dt$ yields

$$\Gamma(s) = n^s \int_0^{\infty} e^{-nt}t^{s-1}\,dt$$

$$\zeta(s)\Gamma(s) = \sum_{n=1}^{\infty}\int_0^{\infty} e^{-nt}t^{s-1}\,dt = \int_0^{\infty}\left(\sum_{n=1}^{\infty} e^{-nt}\right)t^{s-1}dt$$

$$= \int_0^{\infty} \frac{t^{s-1}}{e^t - 1}\,dt, \qquad Re(s) > 0.$$

Now, since we already assumed the limit of the geometric series

$$\sum_{n=1}^{\infty} e^{-nt} = \frac{1}{e^t - 1}, \quad n \rightarrow \infty$$

We may wish to reconsider that

$\tau = nt$ and $n \rightarrow \infty \Rightarrow t \neq 0$. Otherwise we are letting $\tau = \infty \times 0$ which is undefined

In view of that; obviously $t = 0$, is not a *lower limit* of the integral representation of $\zeta(s)$. Consequently;

$$\zeta(s)\Gamma(s) = \int_{t>0}^{\infty} \frac{t^{s-1}}{(e^t - 1)}\,dt, \quad n\,already \rightarrow \infty \quad (1.1)$$

Through the representation above, the one cannot provide any analytic continuity by the means of Hankel contour since there will be always a gap around $t = 0$.

If we assume that

$$\zeta(s)\Gamma(s) = \int_0^{\infty} \frac{t^{s-1}}{(e^t - 1)}\,dt. \qquad (1.2)$$

We are assuming an additional term around $t = 0$

For that, we may expect some accurate conclusions but not sharp. A straight forward approach toward locating the non-trivial zeros can be considered from the following observation:

For a non zero real valued number t if

$$t^{s-1} + t^{-s} = 0. \quad t \neq 0, \text{then}$$

$$s = \frac{1}{2} + i\,\frac{(2m+1)\pi}{lnt}, \quad t \neq 0, lnt \neq 0 \text{ and } m \in \mathbb{Z}$$

That is *if we assume the analytic continuity* from (1.1) and *if we let* $\zeta(s) = \zeta(1-s) = 0$.
Then

$$\zeta(s)\Gamma(s) + \zeta(1-s)\Gamma(1-s) = \int_{t>0}^{\infty} \frac{t^{s-1} + t^{-s}}{(e^t - 1)}\,dt = 0$$

We can let the right hand side equal zero elements wise

$$t^{s-1} + t^{-s} = 0$$

$$s = \frac{1}{2} + i\frac{(2m+1)\pi}{lnt}, \; t \neq 0, lnt \neq 0 \; and \; m \in \mathbb{Z}$$

This implies that

The non-trivial zeros lie on the critical line $Re(s) = \frac{1}{2}$

Next;

We consider the Hankel Contour that is we assume the non sharp version (1.2). We briefly state and prove the analytic continuity and the first functional equation

Riemann Zeta FunctionIntegral Formula and First Functional Equation [1 – 10]

Lemma 1*the Riemann Zeta function is meromorphic everywhere, except at a simple pole s = 1*

Proof

$$\Gamma(s) = \int_0^\infty e^{-\tau}\tau^{s-1}d\tau = n^s \int_0^\infty e^{-nt}t^{s-1}dt, \quad \tau = nt.$$

Multiplying by $\zeta(s)$, implies

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{t^{s-1}}{e^t - 1}\,dt, \quad Re(s) > 0.$$

To extend this formula to $\mathbb{C}\backslash\{1\}$, we integrate $(-t)^s/(e^t - 1$ over a Hankel contour: a path from $+\infty$ inbound along the real line to $\epsilon > 0$, counterclockwise around a circle of radius ϵ at 0, back to ϵ on the real line, and outbound back to $+\infty$ along the real line, around the circle, $t$can be parameterized by $t = \epsilon e^{i\theta}, 0 \leq \theta \leq 2\pi$ and ϵ is a small arbitrary positive constant that we will let tend to 0:

$$\oint_C \frac{(-t)^{s-1}}{e^t - 1}dt = \int_{\rho_1} \frac{(-t)^{s-1}}{e^t - 1}dt + \int_{\rho} \frac{(-t)^{s-1}}{e^t - 1}dt + \int_{\rho_2} \frac{(-t)^{s-1}}{e^t - 1}dt$$

$$= \int_R^\epsilon \frac{(te^{-\pi i})^{s-1}}{e^1 - 1}dt + i\epsilon\int_0^{2\pi} \frac{\left(\epsilon e^{-\pi i}e^{i\theta}\right)^{s-1}}{e^{\epsilon e^{i\theta}} - 1}e^{i\theta}d\theta + e^{2\pi i}\int_\epsilon^R \frac{(te^{-\pi i}e^{2\pi i})^{s-1}}{e^{te^{2\pi i}} - 1}dt$$

$$= -e^{-\pi i(s-1)}\int_\epsilon^R \frac{t^{s-1}}{e^t - 1}dt + i\epsilon\int_0^{2\pi} \frac{\left(\epsilon e^{-\pi i}e^{i\theta}\right)^{s-1}}{e^{\epsilon e^{i\theta}} - 1}e^{i\theta}d\theta + e^{(2\pi i - \pi i)(s-1)}\int_\epsilon^R \frac{t^{s-1}}{e^t - 1}dt$$

$$= \left(e^{\pi i(s-1)} - e^{-\pi i(s-1)}\right)\int_\epsilon^R \frac{t^{s-1}}{e^t - 1}dt + i\epsilon\int_0^{2\pi} \frac{\left(\epsilon e^{-\pi i}e^{i\theta}\right)^{s-1}}{e^{\epsilon e^{i\theta}} - 1}e^{i\theta}d\theta$$

$$= -2i\sin(\pi s)\int_\epsilon^R \frac{t^{s-1}}{e^t - 1}dt + i\epsilon\int_0^{2\pi} \frac{\left(\epsilon e^{-\pi i}e^{i\theta}\right)^{s-1}}{e^{\epsilon e^{i\theta}} - 1}e^{i\theta}d\theta$$

$$\lim_{\substack{\epsilon\to 0\\ R\to\infty}}\left[-2i\sin(\pi s)\int_\epsilon^R \frac{t^{s-1}}{e^t - 1}dt + i\epsilon\int_0^{2\pi} \frac{\left(\epsilon e^{-\pi i}e^{i\theta}\right)^{s-1}}{e^{\epsilon e^{i\theta}} - 1}e^{i\theta}d\theta\right], \; Re(s) > 0$$

$$= 2i\sin(\pi s)\int_0^\infty \frac{t^{s-1}}{e^t - 1}dt.$$

Finally;

$$\oint_C \frac{(-t)^{s-1}}{e^t - 1}dt = 2i\sin(\pi s)\Gamma(s)\zeta(s) \qquad (1.3)$$

Lemma 2

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\,\zeta(1-s)$$

Proof

Here we consider a modified Hankel contour: consisting of two circles centered at the origin and a radius segment along the positive reals. The outer circle has radius $(2n+1)\pi$ and the inner circle has radius $\epsilon < \pi$. The outer circle is traversed clockwiseand the inner one counterclockwise. The radial segment is traversed in both directions. Then by employing the residue theorem

$$\oint_\gamma \frac{(-t)^{s-1}}{e^t - 1}dt = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\sum_{n=1}^\infty n^{s-1}$$

Plugging in equation (1.1) we then prove the functional equation

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\,\zeta(1-s)$$

Remark

From ***Lemma 1***, the integral along the real axis in both directions does not depend on ϵ. Similarly; the integral along the modified Hankel contour in ***Lemma 2*** does not depend on the path. The only significant note is the integral around the small circle vanishes subject to $\epsilon = 0$.

Claim

$\epsilon \nrightarrow 0$. In other words ϵ will remain non-zero no matter how small it is.

Proof

Certainly; there exist a connection between t and ϵ, since t can be parametrized around the small circle by $t = \epsilon e^{i\theta}$, this implies $|t| = \epsilon \neq 0$ (since $\tau = nt, n \to \infty$).

According to our claim, we will keep track on ϵ along the steps of the proofs of ***Lemma 1*** and ***Lemma 2***. We will let the integral around the circle.

$$t = \epsilon e^{i\theta} \Rightarrow i\epsilon\int_{\theta>0}^{2\pi} \frac{\left(\epsilon e^{-\pi i}e^{i\theta}\right)^{s-1}}{e^{\epsilon e^{i\theta}} - 1}e^{i\theta}d\theta$$

$$i\epsilon\int_{\theta>0}^{2\pi} \frac{\left(\epsilon e^{-\pi i}e^{i\theta}\right)^{s-1}}{e^{\epsilon e^{i\theta}} - 1}e^{i\theta}d\theta \to f(s,\epsilon)\epsilon^{s-1} \to \epsilon^{s-1}$$

For shorthand and since the function $f(s,\epsilon)$ will not directly contribute in our approach, we can simply omit it. Now, we consider the slight modification on ***Lemma 1*** and ***Lemma 2***, due to the additional term ϵ^{s-1}, the result in (1.3) will reduce to

$$\oint_C \frac{(-t)^{s-1}}{e^t - 1}\, dt = \epsilon^{s-1} + 2i \sin(\pi s)\Gamma(s)\zeta(s),$$

Similarly, the functional equation in Lemma 2 will be also modified and viewed as*non functional equations*. Since we assumed the additional terms of ϵ we have to add restrictions for accuracy.

$$\zeta(s) = \epsilon^{s-1} + 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\,\zeta(1-s), \quad Re(s) > 1, \qquad (1.4)$$

$$\zeta(1-s) = \epsilon^{-s} + 2^{1-s}\pi^{-s}\sin\left(\frac{\pi(1-s)}{2}\right)\Gamma(s)\,\zeta(s), \quad Re(s) < 0 \qquad (1.5)$$

Theorem *assuming the analytic continuity of* $\zeta(s)$ *and*

$$\text{If } \zeta(s) = \zeta(1-s) = 0, \quad \text{then } Re(s) = \frac{1}{2}$$

Proof

For simplicity we write

$$A(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\,\zeta(1-s)$$

$$A(1-s) = 2^{1-s}\pi^{-s}\sin\left(\frac{\pi(1-s)}{2}\right)\Gamma(s)\,\zeta(s)$$

Let $s = x + iy$ satisfying $\zeta(s) = \zeta(1-s) = 0$,then

$$\zeta(s) - A(s)\zeta(1-s) = \epsilon^{s-1}$$

$$\zeta(1-s) - A(1-s)\zeta(s) = \epsilon^{-s}$$

Adding the last two expressions yield

$$\zeta(s) - A(s)\zeta(1-s) + \zeta(1-s) - A(1-s)\zeta(s) = \epsilon^{s-1} + \epsilon^{-s}$$

Now, if the left hand side of the expression above equals zero implies the right hand side also equals zero, solving for $s \in \mathbb{C}\backslash\{1\}$:

$$\epsilon^{s-1} + \epsilon^{-s} = 0$$

$$y = \frac{i\, ln\epsilon\,(2x-1) + (2m+1)\pi}{2ln\epsilon}, \qquad \epsilon \neq 0,$$

$$ln\epsilon \neq 0 \text{ and } m \in \mathbb{Z}$$

Since y is real valued then the term multiplied by i will vanish that is $x = \frac{1}{2}$, or if we write

$$s = x + i\left[\frac{i\, ln\epsilon\,(2x-1) + (2m+1)\pi}{2\,ln\epsilon}\right]$$

That is

$$s = x + i\left[\frac{(2m+1)\pi}{2\,ln\epsilon}\right]$$

The real part of s will reduce to $x = \frac{1}{2}$

Consequences

For ϵ sufficiently small; we can consider the following equations:

$$\zeta(s) = \epsilon^{s-1} + 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\,\zeta(1-s), \quad Re(s) > 1, \qquad (1.6)$$

$$\zeta(1-s) = \epsilon^{-s} + 2^{1-s}\pi^{-s}\sin\left(\frac{\pi(1-s)}{2}\right)\Gamma(s)\,\zeta(s), \quad Re(s) < 0, \qquad (1.7)$$

$$\zeta(s) = \epsilon^{s} + 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\,\zeta(1-s), \quad 0 < Re(s) < 1 \qquad (1.8)$$

$$\zeta(1-s) = \epsilon^{1-s} + 2^{1-s}\pi^{-s}\sin\left(\frac{\pi(1-s)}{2}\right)\Gamma(s)\,\zeta(s), \quad 0 < Re(s) < 1. \qquad (1.9)$$

$$\zeta(s) + \zeta(1-s) = 0, \text{at } s = \frac{1}{2} + iy,$$

$$y = \frac{(2m+1)\pi}{2lnt}, \text{for some non zero real number } t, m \in \mathbb{Z}$$

$$\zeta(-2n) \sim 0, \qquad n \in \mathbb{Z}.$$

Now, letting $\epsilon \to 0$ is an approximation of $\zeta(s)$ over $\mathbb{C} \setminus \{1\}$. Clearly; the approximation does not divert the purpose of Riemann, since the non-trivial zeros will definitely lie on the critical line. Moreover; computing Zeta Zero through The Euler-Maclaurin summation formula, The Riemann Siegel formula and The Odzyklo-Schölange algorithm remain accurate, and every imaginary part of nontrivial zero can be viewed as $y = \frac{(2m+1)\pi}{2lnt}$, for some non zero real number $t, m \in \mathbb{Z}$.

2. Conclusion

We justified the location of the non-trivial zeta zeros subject to assuming the analytical continuity and subject to considering $\zeta(s) = \zeta(1-s) = 0$ simultaneously. However this does not change the fact that the analytical continuity is *not sharp* nor all of its consequences: the functional equation, the trivial zeros and the non trivial zeros. The only sharp result is the location of non-trivial zeros on the critical line subject to assuming an approximated analytic continuity.

But since the one can manipulate the error, Riemann Zeta function remains an accurate approach to investigate the main objective of Riemann: the distribution of prime numbers.

3. Conflicts of Interest

This manuscript has not been submitted to, nor is under review at, another journal or other publishing venue.

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