Some Fixed Point Theorems of Integral Function in Menger Spaces with Property (E.A.)

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Abstract: In this paper, we obtain some fixed point theorems using integral type inequality in Menger space employing the property (E.A). Our results improve and generalize several known fixed point theorems existing in the literature.

Keywords: Menger space, Integral Function, Weakly compatible mappings, Property (E.A)

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1. Introduction

In the year 1942 Menger [21] introduced the notion of a probabilistic metric space (PM-space) which was, in fact, a generalization of metric space. The idea behind this is to associate a distribution function with a pair of points, say (p,q), denoted by Fp,q(t) where t > 0 and interpret this function as the probability that distance between p and q is less than t, whereas in the metric space, the distance function is a single positive number. Sehgal [37] initiated the study of fixed points in probabilistic metric spaces. The study of these spaces was expanded rapidly with the pioneering works of Schweizer and Sklar [7]. Jungck [13] introduced the notion of compatible mappings and utilized the same to prove existence theorems on common fixed points. However, the study of common fixed points of non-compatible mappings was initiated by Pant [29]. Recently, Aamri and Moutawakil [1] and Liu et al. [34] respectively defined the property (E.A) and the common property (E.A) and proved interesting common fixed point theorems in metric spaces. Most recently, Kubiaczyk and Sharma [15] adopted the property (E.A) in PM-spaces and used it to prove results on common fixed points. Recently, Imdad et al. [26] adopted the common property (E.A) in PM-spaces and proved some coincidence and common fixed point results in Menger spaces.

1) Preliminaries: Before going to our main result we require some more definitions and Lemma,

Definition 1.1 [8]: Let X be a non empty set and L denote the set of all distribution functions. A probabilistic metric space is an ordered pair (X,F) where F : X * X→ L. We shall denote the distribution function by F (p, q) or Fp,q : p, q ∈ X and F(p, q, x) will represent the value of F (p, q) at x ∈ R. The function F p, q is assumed to satisfy the following conditions:

1. F p, q (t) = 1, ∀ t > 0 if and if p = q
2. F p, q (0) = 0 for every p, q ∈ X
3. F p, q (t) = F q, p (t) for every p, q ∈ X
4. If F p, q (t) = 1 and F q, r (s) = 1 it follows that F q r (t + s) = 1 ∀ p, q, r ∈ X and t, s ≥ 0.

In metric space (X,d), the metric d induces a mapping F : X * X→ L such that F p, q (t) = H(t-d(p, q)) for all p, q ∈ X and t ∈ R , where H is the distribution function defined as

H(x) = \begin{cases} 0, & \text{if } x ≤ 0 \\ 1, & \text{if } x > 0 \end{cases}

Definition 1.2 [8]: A mapping A : [0, 1] * [0, 1]→ [0, 1] is called t-norm if the following conditions are satisfied:

1. A (a, 1) = a for all a ∈ [0, 1], A (0,0) = 0,
2. A (a, b) = A (b, a)
3. A (c, d) ≤ A (a, b) for c ≥ a, d ≥ b, and
4. A (A (c, d),c) = A (a,A (b, c)) for all a, b, c ∈ [0, 1]

Example 1 [8]: The following are the four basic t-norms:

(i) The minimum t-norm: T_m(a,b) = \min{a,b}.
(ii) The product t-norm: T_p(a,b) = a.b
(iii) The Lukasiewicz t-norm: T_l(a, b) = max{a + b - 1, 0}.

In respect of above mentioned t-norms, we have the following ordering:

T_m < T_p < T_l < T_l

Definition 1.3 [21]: A Menger probabilistic space is a triplet (X,F,∆) where (X, F) is a PM-space and ∆ is a t-norm with the following condition

F_p,r (t+s) ≥ ∆ (F_p,r (t), F_p,r (s)) for all p, q, r ∈ X and t, s ≥ 0.

The above inequality is called Menger’s triangle inequality.

Definition 1.4 [28]: A sequence \{x_n\} in (X, F, ∆) is said to be a convergent to a point x ∈ X if for every ε > 0 and λ > 0, there exists an integer N=N (ε ,λ) such that F_{x_n, x} (ε )→1- λ ∀ n ≥ N (ε ,λ).

Definition 1.5 [28]: A sequence \{x_n\} in (X, F, ∆) is said to be a Cauchy sequence if for every ε > 0 and λ > 0 , there exists an integer N=N (ε ,λ) such that F_{x_n, x_m} (ε )→1- λ ∀ n, m ≥ N (ε ,λ).

Definition 1.6 [28]: A Menger Space (X, F, ∆) with the continuous t-norm is said to be complete if every Cauchy sequence in X converges to a point in X.
Definition 1.7 [24]: Let (X, F, ∆) be a Menger PM Space. A pair (f, g) of self mapping on X is said to be weakly commuting if and only if \( F_{f,g_x}(t) \geq F_{f,x}(t) \) for each \( x \in X \) and \( t > 0 \).

Definition 1.8 [31]: Let (X, F, ∆) be a Menger PM Space . A pair (f, g) of self mapping on X is said to be compatible if and only if \( F_{f,g_x}(t) \rightarrow 1 \) for all \( t > 0 \) whenever \( \{x_n\} \) in X such that \( f x_n \rightarrow z \) for some \( z \in X \) as \( n \rightarrow \infty \).

Clearly, a weakly commuting pair is compatible but every compatible pair need not be weakly commuting.

Definition 1.10 [19]: Let (X, F, ∆) be a Menger PM Space . A pair (f, g) of self mapping on X is said to be non–compatible if and only if there exist at least one sequence \( \{x_n\} \) in X such that \( \lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = z \), for some \( z \in X \), implies that \( \lim_{n \to \infty} F_{f,g_x}(t_0) \) (for some \( t_0 > 0 \)) is either less than 1 or non-existent.

Definition 1.11 [15] : Let (X, F, ∆) be a Menger PM Space . A pair (f, g) of self mapping on X is said to satisfy the property (E,A) if there exist a sequence \( \{x_n\} \) in X such that \( \lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = z \), for some \( z \in X \).

Clearly, a pair of compatible mappings as well as non-Comatible mappings satisfies the property (E,A).

Inspired by Liu et al. [39], Imdad et al. [26] defined the following:

Definition 1.12 [34]: Two pairs (f, g) and (p, q) of self mappings of a Menger PM space (X, F, ∆) are said to satisfy the common property (E.A) if there exist two sequences \( \{x_n\}, \{y_n\} \) in X and some \( t \) in X such that \( \lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = \lim_{n \to \infty} p x_n = \lim_{n \to \infty} q x_n = z \).

Definition 1.13. [24] Two finite families of self mappings \( \{A_i\} \) and \( \{B_i\} \) are said to be pair wise commuting if:

(i) \( A_i A_j = A_j A_i \), \( i, j \in \{1, 2, ..., m\} \),

(ii) \( B_i B_j = B_j B_i \), \( i, j \in \{1, 2, ..., n\} \),

(iii) \( A_i B_j = B_j A_i \), \( i \in \{1, 2, ..., m\}, \; j \in \{1, 2, ..., n\} \).

2. Main Result

The following lemma is useful for the proof of succeeding theorems.

Lemma 2.1 [14]: Let (X, F, ∆) be a Menger space. If there exists some k ∈ (0, 1) such that for all p, q, X and all \( x > 0 \),

\[
\int_0^{F_{p,q}(kt)} \phi(u) du \geq \int_0^{F_{p,q}(t)} \phi(u) du
\]

- - - (2.1.1)

Where \( \phi : [0, \infty) \to [0, \infty) \) is a non-negative summable Lebesque integral function such that \( \int_0^\infty \phi(u) du > 0 \) for each \( \epsilon \in [0, 1) \), then p = q.

Proof: From (2.1.1)

\[
\int_0^{F_{p,q}(t)} \phi(u) du \geq \int_0^{F_{p,q}(k^{-1}t)} \phi(u) du
\]

one can inductively write (for \( m \in \mathbb{N} \))

\[
\int_0^{F_{p,q}(t)} \phi(u) du \geq \int_0^{F_{p,q}(k^{-1}t)} \phi(u) du \geq \cdots \geq \int_0^{F_{p,q}(k^{-m}t)} \phi(u) du
\]

\[
\geq \cdots \geq \int_0^{F_{p,q}(0)} \phi(u) du = \int_0^\infty \phi(u) du
\]

Therefore

\[
\int_0^{F_{p,q}(t)} \phi(u) du - \int_0^{F_{p,q}(t)} \phi(u) du \leq 0
\]

And hence,

\[
\int_0^{F_{p,q}(t)} \phi(u) du \left( \int_0^{F_{p,q}(t)} \phi(u) du - \int_0^{F_{p,q}(t)} \phi(u) du \right) \geq 0
\]

Or,

\[
\int_0^{F_{p,q}(t)} \phi(u) du \leq 0.
\]

which amounts to say that \( F_p q(t) \geq 1 \) for all \( t \geq 0 \), thus, we get \( p = q \).

Remark : By setting \( \phi(t) = 1 \) (for each \( t \geq 0 \)) in (2.1.1) of Lemma 2.1, we have

\[
\int_0^{F_{p,q}(kt)} \phi(u) du = F_{p,q}(kt) \geq F_{p,q}(t) = \int_0^{F_{p,q}(t)} \phi(u) du,
\]

which shows that Lemma 1 is a generalization of the Lemma 2 (contained in [34]).

In what follows, ∆ is a continuous t-norm (in the product topology).

Lemma 2.2: Let (X,F, ∆) be a complete Menger Space and let f, g, p and q be self mapping of X satisfying the conditions:

(i) Pairs \( \{p, f\} \) and \( \{q, g\} \) satisfies the property E.A.

(ii) \( B(y_u) \) converges for every sequence \( \{y_n\} \) in X whenever \( T(y_u) \) converges,

(iii) for any \( x, y \in X \) and for each \( t > 0 \),

\[
\int_0^{F_{p,q}(kt)} \phi(u) du \geq \int_0^{m(x,y)} \phi(u) du
\]

- - - (2.2.1)

Where \( \phi : [0, \infty) \to [0, \infty) \) is a non-negative summable Lebesque integral function such that \( \int_0^\infty \phi(u) du > 0 \) for each \( u \in [0, 1) \), where 0 < k < 1 and

\[
m(x,y) = \min\{ F_{f,x}(t), F_{f,x,y}(t), F_{g,y}(t), F_{f,x,y}(t), F_{f,x,y}(t), F_{f,x,y}(t), F_{f,x,y}(t), F_{f,x,y}(t), F_{f,x,y}(t), F_{f,x,y}(t), F_{f,x,y}(t), F_{f,x,y}(t), F_{f,x,y}(t) \}
\]

(2)

\[
p(x) \in G(X) \text{ (or } q(X) \in f(X)\text{).}
\]

Then the pair \( (p,f) \) and \( (q,g) \) share the common property (E,A).

Proof : Suppose that the pair \( (p,f) \) enjoys the property (E,A) then there exist a sequence \( \{x_n\} \) in X such that

\[
\lim_{n \to \infty} p x_n = \lim_{n \to \infty} f x_n = u, \text{ for some } u \in X.
\]

Since \( p(x) \in G(X) \), for each \( x_n \) there exists \( y_n \in X \).

Such that \( p x_n = q y_n \), and hence

\[
\lim_{n \to \infty} q y_n = \lim_{n \to \infty} p x_n = u
\]

Thus in all, we have \( p x_n \to u, \; f x_n \to u \) and \( g y_n \to u \).

Now we assert that \( q y_n \to u \).

To accomplish this, using (2.2.1) with \( x = x_n \) and \( y = y_n \), we get

\[
\int_0^{F_{p,q}(kt)} \phi(u) du \geq \int_0^{m(x_n,y_n)} \phi(u) du
\]

Where,

\[
m(x_n,y_n) = \min\{ F_{f,x,y}(t), \}
\]

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917
Therefore, on using (2.2) with \( x = y_u \), \( y = y_u \), we get

\[
F_{fz,pz_x}(t), F_{gy,u,y_u}(t), F_{fx,n,y_u}(t), F_{gy,pz_x}(t),
F_{fz,x,y_u}(t), F_{gy,pz_u}(t),
F_{fz,y_u}(t), F_{gy,pz_u}(t) = \frac{F_{fz,y_u}(t)}{F_{gy,pz_u}(t)} + \frac{F_{fz,y_u}(t)}{F_{gy,pz_u}(t)}
\]

\[
F_{fz,y_u}(t) + F_{gy,pz_u}(t)
\]

\[
\frac{\phi(u)}{2} \geq \int_0^\Phi(u)du \geq \int_0^\Phi(u)du
\]

Let, \( \lim_{n \to \infty} q(y_u) = v \)

Also, let \( t > 0 \) be such that \( F_{u,t}(.), \) is continuous in \( t \) and \( kt \).

Then, on making \( n \to \infty \) in the above inequality, we get

\[
\int_0^\Phi(u)du \geq \int_0^\Phi(u)du
\]

On taking \( n \to \infty \), reduces to

\[
\int_0^\Phi(u)du \geq \int_0^\Phi(u)du
\]

Now on appealing Lemma 2.1, we get \( pz = u \) and hence \( pz = fz \). Therefore, \( z \) is a coincidence point of the pair \( (p, f) \).

Since \( g(x) \) is a closed subset of \( X \), therefore \( \lim_{n \to \infty} g(y_u) = u \in g(X) \) and hence we can find a point \( w \in X \) such that \( gw = u \)

Now we show that \( qw = gw \).

To accomplish this, on using (2.2.1) with \( x = x_u, y = w, \) we have

\[
F_{fz,u,w}(t), F_{fz,u,w}(t), F_{fz,u,w}(t), F_{fz,u,w}(t), F_{fz,u,w}(t) = \frac{F_{fz,u,w}(t)}{F_{fz,u,w}(t)} + \frac{F_{fz,u,w}(t)}{F_{fz,u,w}(t)}
\]

\[
F_{fz,u,w}(t) + F_{fz,u,w}(t)
\]

\[
\frac{\phi(w)}{2} \geq \int_0^\Phi(w)du \geq \int_0^\Phi(w)du
\]

On employing Lemma 2.1, we get \( qw = u \) and \( gw = qw \)

Therefore, \( w \) is a coincidence point of the pair \( (q, g) \).

Since the pair \( (p, f) \) is weakly compatible and \( pz = fz \), therefore \( pu = pfz = fpz = fu \).

Again, on using (2.2.1) with \( x = u, y = w, \) we have

\[
F_{fz,u,w}(t), F_{fz,u,w}(t), F_{fz,u,w}(t), F_{fz,u,w}(t) = \frac{F_{fz,u,w}(t)}{F_{fz,u,w}(t)} + \frac{F_{fz,u,w}(t)}{F_{fz,u,w}(t)}
\]

\[
F_{fz,u,w}(t) + F_{fz,u,w}(t)
\]

\[
\frac{\phi(u)}{2} \geq \int_0^\Phi(u)du \geq \int_0^\Phi(u)du
\]

Therefore, \( w \) is a coincidence point of the pair \( (q, g) \).

Theorem 2.3: Let \( f, g, p \) and \( q \) be self mappings of a Menger space \((X, F, A)\) which satisfy the inequality (2.2.1) together with the conditions:

(i) the pairs \((p, f)\) and \((q, g)\) share the common property \((E.A)\),

(ii) \( f(X) \) and \( g(X) \) are closed subsets of \( X \).

Then the pairs \((p, f)\) and \((q, g)\) have a point of coincidence each. Moreover, \( f, g, p \) and \( q \) have a unique common fixed point provided both the pairs \((p, f)\) and \((q, g)\) are weakly compatible.

Proof. Since the pairs \((p, f)\) and \((q, g)\) share the common property \((E.A)\), there exist two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that

\[
\lim_{n \to \infty} px_n = \lim_{n \to \infty} qy_n = \lim_{n \to \infty} qy_n = u, \text{ for some } u \in X.
\]

Since \( f(X) \) is a closed subset of \( X \), hence \( \lim_{n \to \infty} f x_n = u \in f(X) \).

Therefore, there exists a point \( z \in X \) such that \( fz = u \).

Now, we assert that \( pz = fz \).

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\end{align*}

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