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On the Summations of Some Special Sequences

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Abstract: ***This paper is concerned with the summations of the Pell, Lucas, Pell-Lucas, Jacopsthal, Jacopsthal-Lucas, generalized Pell, generalized dual Pell and generalized dual Pell quaternion sequences. Also, summing infinite series of reciprocals of the Pell numbers is calculated.***

Keywords: Pell number, Lucas number, Pell-Lucas number, Jacopsthal number, Jacopsthal-Lucas number, generalized Pell sequences, generalized dual Pell sequences, generalized dual Pell quaternion sequences

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1. Introduction

It is well known that many number sequences have important parts in mathematics. Especially in the fields of combinatorics and number theory [15,16]. Recently, interest has been shown in summing infinite series of reciprocals of some special numbers, for example Fibonacci numbers [5], [6] and [7]. It is not easy, in general, to derive the sum of a series whose terms are reciprocals of Fibonacci, Pell, etc. Numbers such that the subscripts are terms of geometric progressions. It seems even more difficult if the subscripts are in arithmetic progression [13]. Many problems related to the sum of the terms of the Fibonacci series were first proposed in [3]. After that Ken Siler proved the$\sum_{k=1}^{n} F_{ak-b}$summations provided that $a > b$ in [9]. Single-indexed or double-indexed ones could be summed, for example, some summations of the Pell Quaternions and the Pell-Lucas Quaternions were studied in [2], some summations of the generalized Pell sequence were studied in [10], some summations of the generalized dual Pell and some summations of the generalized dual Pell Quaternions were also studied in [14], but no generalizations were made. In this publication, both these generalizations are made for Pell, Lucas, Pell-Lucas, Jacopsthal, Jacopsthal-Lucas, generalized Pell, generalized dual Pell, and generalized dual Pell quaternion sequences and summing infinite series of reciprocals of the Pell numbersare calculated.

2. A Reciprocal Series of Pell(P) Numbers

It is not easy, in general, to derive the sum of a series whose terms are reciprocals of Fibonacci(F) numbers such that the subscripts are terms of geometric progressions. However, in [5] Good shows that$\sum_{n=0}^{\infty}\frac{1}{F_{2^n}}=\frac{7-\sqrt{5}}{2}$, a problem proposed by Millin [12]. This particular series can be summed in several different ways in [6].

Now, let's calculate infinite series of reciprocals of the Pell numbers;

$$\sum_{n=0}^{\infty}\frac{1}{P_{2^n}}\ , \qquad (1)$$

whereP_n is then^{th}term of the Pell sequence 1, 2, 5, 12, 29…. Write out the first few terms of (1),

$$1, 1+\frac{1}{2}=\frac{3}{2}, 1+\frac{1}{2}+\frac{1}{12}=\frac{19}{12},\ 1+\frac{1}{2}+\frac{1}{12}+\frac{1}{408}=\frac{647}{408}, \dots\ .$$

Now,

$$\frac{647}{408}=\frac{3}{2}+\frac{35}{408}=\frac{3}{2}+\frac{P_6}{2.P_8} \qquad (2)$$

which suggests that

$$\frac{1}{P_1}+\frac{1}{P_2}+\frac{1}{P_4}+\cdots+\frac{1}{P_{2^n}}=\frac{3}{2}+\frac{P_{2^n-2}}{2P_{2^n}}\ .$$

$$\frac{3}{2}+\frac{P_{2^n-2}}{2P_{2^n}}.\frac{q_{2^n}}{q_{2^n}}+\frac{1}{P_{2^{n+1}}}=\frac{3}{2}+\frac{P_{2^{n+1}-2}}{2P_{2^{n+1}}}$$

since$P_m.q_m = P_{2m}$, where$q_n$is then^{th}term of the Pell-Lucas sequence 2, 6, 14, 34, 82... Thus, we can prove (2) by mathematical induction. If we compute the limit as$n\to\infty$for (2), then we have the infinite sum of (1), for (see[11])

$$\lim_{n\to\infty}\left(\frac{3}{2}+\frac{P_{2^n-2}}{2P_{2^n}}\right)=\lim_{n\to\infty}\left(\frac{3}{2}+\frac{P_{2^n-2}}{2P_{2^n}}\frac{P_{2^n-1}}{P_{2^n-1}}\right)=\frac{3}{2}+\frac{r^2}{2}$$
$$=s+2r$$

which simplifies to$(3-\sqrt{2})$.

The limits used above can be derived from the well-known Binet's formulas

$$P_n=\frac{r^n-s^n}{r-s}\ , \qquad q_n=r^n+s^n,$$

where$r=1+\sqrt{2},\ \ s=1-\sqrt{2}$are the roots of$x^2-2x-1=0$.

$$\lim_{n\to\infty}\frac{P_n}{P_{n+1}}=\lim_{n\to\infty}\frac{q_n}{q_{n+1}}=-s=\sqrt{2}-1,$$

$$\lim_{n\to\infty}\frac{P_{n+1}}{P_n}=\lim_{n\to\infty}\frac{q_{n+1}}{q_n}=r=1+\sqrt{2}$$

since$r-s=2\sqrt{2}$and$\frac{s}{r}<-1$. In an entirely similar manner, we could show that

$$\lim_{n\to\infty}\frac{P_{2^n-2}}{P_{2^n-1}}=-s,\qquad \lim_{n\to\infty}\frac{P_{2^n-1}}{P_{2^n}}=r^{-1}.$$

2.1 Pell Summations

Some summations of the Pell sequence are shown below;

$$2\sum_{k=1}^{n} P_{2k} = P_{2n+1} - 1,$$
$$2\sum_{k=1}^{n} P_{2k-1} = P_{2n},$$
$$14\sum_{k=1}^{n} P_{3k} = 3P_{3n+2} - P_{3n+1} - 5,$$
$$14\sum_{k=1}^{n} P_{3k-1} = 3P_{3n+1} - P_{3n} - 3,$$
$$14\sum_{k=1}^{n} P_{3k-2} = 3P_{3n} - P_{3n-1} + 1,$$
$$16\sum_{k=1}^{n} P_{4k} = q_{4n+2} - 6,$$
$$16\sum_{k=1}^{n} P_{4k-1} = q_{4n+1} - 2,$$
$$16\sum_{k=1}^{n} P_{4k-2} = q_{4n} - 2,$$
$$16\sum_{k=1}^{n} P_{4k-3} = q_{4n-1} + 2.$$

After that, the question arises how can the general sum$\sum_{k=1}^{n} P_{ak-b}$be achieved? It is almost impossible to find this sum intuitively using these above sums. One is led therefore to adopt a more mathematical approach to solving the general case of all Pell series summations with subscripts in arithmetic progression, namely,

$$\sum_{k=1}^{n} P_{ak-b}$$

where$a$and b are positive integers and$a \geq b$.

We recall that Pell numbers can be given in terms of the roots of the equation $x^2 - 2x - 1 = 0$[8], [11], [1]. If these roots are

$$r = 1+\sqrt{2} \quad \text{and} \quad s = 1-\sqrt{2}$$

then

$$P_n = \frac{r^n - s^n}{r-s} \quad \text{and} \quad q_n = r^n + s^n .$$

In these terms

$$\sum_{k=1}^{n} P_{ak-b} = \frac{1}{2\sqrt{2}}\left(\sum_{k=1}^{n} r^{ak-b} - \sum_{k=1}^{n} s^{ak-b}\right)$$

One can restate the summations on the right-hand side of the equation by using the formula for geometric progressions,

$$\sum_{k=1}^{n} r^{ak-b} = r^{a-b}\left(1 + r^a + r^{2a} + r^{3a} + \cdots + r^{(n-1)a}\right)$$
$$= r^{a-b}\left(\frac{r^{an}-1}{r^a-1}\right).$$

There is an entirely similar formula forthe “s” summation. Substituting into the original formula and combining fractions, one obtains

$$\sum_{k=1}^{n} P_{ak-b} = \frac{1}{2\sqrt{2}}\left(r^{a-b}\left(\frac{r^{an}-1}{r^a-1}\right) - s^{a-b}\left(\frac{s^{an}-1}{s^a-1}\right)\right)$$
$$= \frac{1}{2\sqrt{2}}\left(\frac{s^a r^{an+a-b} - r^a s^{an+a-b} - r^{an+a-b} + s^{an+a-b}}{s^a r^a - s^a - r^a + 1} - \frac{s^a r^{a-b} - r^a s^{a-b} - r^{a-b} + s^{a-b}}{s^a r^a - s^a - r^a + 1}\right).$$

Various simplifications result using the definitions of $P_n$and $q_n$in terms of$r$and$s$together with the relation$rs = -1$, the product of the roots in the equation$x^2 - 2x - 1 = 0$being the constant term-1. For example,

$$-s^a r^{a-b} + r^a s^{a-b} = -r^a s^a (r^{-b} - s^{-b}) = (-1)^{a-b}.2\sqrt{2}.P_b$$

The denominator can be transformed into$(-1)^a - q_a + 1$. Using these relations the reader may verify the final formula is

$$\sum_{k=1}^{n} P_{ak-b} = \frac{(-1)^a P_{an-b} - P_{a(n+1)-b} + (-1)^{a-b} P_b + P_{a-b}}{(-1)^a - q_a + 1}$$

Thus, we found the sum of the Pell sequences starting with$a-b$indexed element continuing with$a$. With this formula particular cases can be handled.

Special case1: Let$a = 3 = b$and$n = 4$.

$$\sum_{k=1}^{4} P_{3k-3} = \frac{(-1)^3 P_9 - P_{12} + (-1)^3 P_3 + P_0}{(-1)^3 - q_3 + 1}$$
$$= \frac{-985 - 13860 - 5 - 0}{-14} = 1060$$

This result also is checked by actually summing the series:$P_0 + P_3 + P_6 + P_9 = 0 + 5 + 70 + 985 = 1060$. In a similar way, we give sums of Lucas, Pell-Lucas, Jacopsthal, generalized Pell, generalized dual Pell and generalized dual Pell quaternion sequences respectively.

2.2 Lucas(L) Summations

$$\sum_{k=1}^{n} L_{ak-b} = \sum_{k=1}^{n} r^{ak-b} + \sum_{k=1}^{n} s^{ak-b}$$
$$= \frac{(-1)^a L_{an-b} - L_{a(n+1)-b} - (-1)^{a-b} L_b + L_{a-b}}{(-1)^a - L_a + 1},$$

where$L_n = r^n + s^n$,$r = \frac{1+\sqrt{5}}{2}$, $s = \frac{1-\sqrt{5}}{2}$.Let's show this formula for$a > b$.

Special case 2: Let $a = 5$,$b = 3$and$n = 4$.

$$\sum_{k=1}^{4} L_{5k-3} = \frac{(-1)^5 L_{17} - L_{22} - (-1)^2 L_3 + L_2}{(-1)^5 - L_5 + 1}$$
$$= \frac{-3571 - 39603 - 4 + 3}{-11} = 3925$$

This result may be checked by actually summing the series:$L_2 + L_7 + L_{12} + L_{17} = 3 + 29 + 322 + 3571 = 3925$.

2.3 Pell-Lucas(q) Summations

$$\sum_{k=1}^{n} q_{ak-b} = \sum_{k=1}^{n} r^{ak-b} + \sum_{k=1}^{n} s^{ak-b}$$
$$= \frac{(-1)^a q_{an-b} - q_{a(n+1)-b} - (-1)^{a-b} q_b + q_{a-b}}{(-1)^a - q_a + 1},$$

where$q_n = r^n + s^n$,$r = 1+\sqrt{2}$,$s = 1-\sqrt{2}$. Let's show this formula for$a = b$.

Special case 3: Let$a = 4 = b$and$n = 3$,

$$\sum_{k=1}^{3} q_{4k-4} = \frac{(-1)^4 q_8 - q_{12} - (-1)^0 q_4 + q_0}{(-1)^4 - q_4 + 1}$$
$$= \frac{1154 - 39202 - 34 + 2}{-32} = 1190$$

This result also is checked by actually summing the series: $q_0 + q_4 + q_8 = 2 + 34 + 1154 = 1190$.

2.4 Jacopsthal (J) Summations;

$$\sum_{k=1}^{4} J_{5k-3} = \frac{(-2)^5 J_{17} - J_{22} + (-2)^2 J_3 + J_2}{(-2)^5 - j_5 + 1}$$
$$= \frac{-32.43691 - 1398101 + 12 + 1}{-32 - 31 + 1} = 45100$$

where $J_n = \frac{1}{3}(r^n - s^n)$, $r = 2$, $s = -1$, j_n is a Jacopsthal-Lucas sequence. Let's show this formula for $a > b$.

Special case 4: Let $a = 5$, $b = 3$ and $n = 4$.

$$\sum_{k=1}^{4} J_{5k-3} = \frac{(-2)^5 J_{17} - J_{22} + (-2)^2 J_3 + J_2}{(-2)^5 - j_5 + 1}$$
$$= \frac{-32.43691 - 1398101 + 12 + 1}{-32 - 31 + 1} = 45100$$

This result may be checked by actually summing the series: $J_2 + J_7 + J_{12} + J_{17} = 1 + 43 + 1365 + 43691 = 45100$.

2.5 Jacopsthal-Lucas (j) Summations;

$$\sum_{k=1}^{n} j_{ak-b} = \left(\sum_{k=1}^{n} r^{ak-b} - \sum_{k=1}^{n} s^{ak-b}\right)$$
$$= \frac{(-2)^a j_{an-b} - j_{a(n+1)-b} - (-2)^{a-b} j_b + j_{a-b}}{(-2)^a - j_a + 1},$$

where $j_n = r^n + s^n$, $r = 2$, $s = -1$, j_n is a Jacopsthal-Lucas sequence. Let's show this formula for $a = b$.

Special case 5: Let $a = 5 = b$ and $n = 3$,

$$\sum_{k=1}^{3} j_{5k-5} = \frac{(-2)^5 j_{10} - J_{15} - (-2)^0 J_5 + J_0}{(-2)^5 - j_5 + 1}$$
$$= \frac{-32.1025 - 32767 - 31 + 2}{-62} = 1058.$$

This result also is checked by actually summing the series: $j_0 + j_5 + j_{10} = 2 + 31 + 1025 = 1058$.

2.6 Generalized Pell($\mathbb{P}$) Summations;

The generalized Pell sequence is defined by

$$\mathbb{P}_n = 2\mathbb{P}_{n-1} + \mathbb{P}_{n-2}, \quad (n \geq 3) \tag{3}$$

with $\mathbb{P}_0 = q$, $\mathbb{P}_1 = p$, $\mathbb{P}_2 = 2p + q$, where p, q are arbitrary integers[10]. That is, the generalized Pell sequence is

$$q, p, 2p + q, 5p + 2q, 12p + 5q, 29p + 12q, \dots, (p - 2q)P_n + qP_{n+1}, \dots \tag{4}$$

Single-indexed and double-indexed summations of the Generalized Pell sequence($\mathbb{P}_n$) were studied in [10], but no generalizations were made. Let's do it now:

$$\sum_{k=1}^{n} \mathbb{P}_{ak-b} = \frac{1}{2\sqrt{2}}\left(\sum_{k=1}^{n} \bar{r}.r^{ak-b} - \sum_{k=1}^{n} \bar{s}.s^{ak-b}\right)$$

where $\mathbb{P}_n = \frac{\bar{r}r^n - \bar{s}s^n}{r-s}$, $\bar{r} = p + q(r-2) = p + \frac{q}{r}$, $\bar{s} = p + q(s-2) = p + \frac{q}{s}$, $\bar{r}.\bar{s} = e_P$. One can restate the summations on the right-hand side of the equation by using the formula for geometric progressions,

$$\bar{r}\sum_{k=1}^{n} r^{ak-b} = \bar{r}.r^{a-b}\left(1 + r^a + r^{2a} + r^{3a} + \cdots + r^{(n-1)a}\right)$$
$$= \bar{r}.r^{a-b}\left(\frac{r^{an} - 1}{r^a - 1}\right).$$

There is an entirely similar formula for the "s" summation. Substituting into the original formula and combining fractions, one obtains

$$\sum_{k=1}^{n} \mathbb{P}_{ak-b} = \frac{1}{2\sqrt{2}}\left(\bar{r}.r^{a-b}\left(\frac{r^{an} - 1}{r^a - 1}\right) - \bar{s}.s^{a-b}\left(\frac{s^{an} - 1}{s^a - 1}\right)\right)$$
$$\sum_{k=1}^{n} \mathbb{P}_{ak-b} = \frac{1}{2\sqrt{2}}\left(\bar{r}.r^{a-b}\left(\frac{r^{an} - 1}{r^a - 1}\right) - \bar{s}.s^{a-b}\left(\frac{s^{an} - 1}{s^a - 1}\right)\right)$$
$$= \frac{1}{2\sqrt{2}}\left[\frac{\bar{r}(s^a r^{an+a-b} - r^a s^{an+a-b} - r^{an+a-b} + s^{an+a-b})}{s^a r^a - s^a - r^a + 1} - \frac{\bar{s}(s^a r^{a-b} - r^a s^{a-b} - r^{a-b} + s^{a-b})}{s^a r^a - s^a - r^a + 1}\right].$$

Various simplifications result using the definitions of $\mathbb{P}_n$ and q_n in terms of $r, \bar{r}, s$, and $\bar{s}$ together with the relation $rs = -1$. For example,

$$\bar{r}s^a r^{an+a-b} - \bar{s}r^a s^{an+a-b} = r^a s^a(\bar{r}r^{an-b} - \bar{s}s^{an-b})$$
$$= (-1)^a.2\sqrt{2}.\mathbb{P}_{an-b}.$$

The denominator can be transformed into $(-1)^a - q_a + 1$. Using these relations the reader may verify the final formula is

$$\sum_{k=1}^{n} \mathbb{P}_{ak-b} = \frac{(-1)^a \mathbb{P}_{an-b} - \mathbb{P}_{a(n+1)-b} - (-1)^a \mathbb{P}_{-b} + \mathbb{P}_{a-b}}{(-1)^a - q_a + 1}$$

With this formula particular case can be handled.

Special Case 6: Let $a = 3$, $b = 2$ and $n = 2$.

$$\sum_{k=1}^{2} \mathbb{P}_{3k-2} = \frac{(-1)^3 \mathbb{P}_4 - \mathbb{P}_7 + \mathbb{P}_{-2} + \mathbb{P}_1}{(-1)^3 - q_3 + 1}$$
$$= \frac{(12p + 5q) + (169p + 70q) + (2p - 5q) - p}{14} = 13p + 5q$$

This result may be checked by actually summing the series: $\mathbb{P}_1 + \mathbb{P}_4 = p + (12p + 5q) = 13p + 5q$.

2.7 Generalized Dual Pell($\mathbb{D}_n^P$) Summations;

The n-th term of the generalized dual Pell sequence is defined by

$$\mathbb{D}_n^P = \mathbb{P}_n + \varepsilon\mathbb{P}_{n+1}.$$

Using equations (3) and (4), we get

$$\mathbb{D}_n^P = \left(p + \varepsilon(2p + q)\right)P_n + (q + \varepsilon p)P_{n-1}, \quad (n \geq 2)$$

with $\mathbb{D}_0^P = q + \varepsilon p$, $\mathbb{D}_1^P = p + \varepsilon(2p+q)$, $\mathbb{D}_2^P = (2p+q) + \varepsilon(5p+2q)$, where $\varepsilon^2 = 0$, $\varepsilon \neq 0$[10]. That is, the generalized dual Pell sequence is

$$(\mathbb{D}_n^P): q + \varepsilon p, p + \varepsilon(2p+q), (2p+q) + \varepsilon(5p+2q), (5p+2q) + \varepsilon(12p+5q), \ldots, (p + \varepsilon(2p+q))P_n + (q + \varepsilon p)P_{n-1}, \ldots .$$

No studies have been previously conducted on the sum of the generalized dual Pell sequence. Let's do it now:

$$\sum_{k=1}^{n} \mathbb{D}_{ak-b}^P = \frac{1}{2\sqrt{2}}\left(\sum_{k=1}^{n} \bar{r}.r^{ak-b} - \sum_{k=1}^{n} \bar{s}.s^{ak-b}\right),$$

where $\mathbb{D}_n^P = \frac{\bar{r}r^n - \bar{s}s^n}{r-s}$, $r = 1+\sqrt{2}$, $s = 1-\sqrt{2}$, $\bar{r} = (p - 2q + \varepsilon q) + r(q + \varepsilon p)$, $\bar{s} = (p - 2q + \varepsilon q) + s(q + \varepsilon p)$, $r.s = -1$, $\bar{r}.\bar{s} = e_P$. One can restate the summations on the right-hand side of the equation by using the formula for geometric progressions,

$$\bar{r}\sum_{k=1}^{n} r^{ak-b} = \bar{r}.r^{a-b}\left(1 + r^a + r^{2a} + r^{3a} + \cdots + r^{(n-1)a}\right) = \bar{r}.r^{a-b}\left(\frac{r^{an}-1}{r^a-1}\right).$$

There is an entirely similar formula for the "s" summation. Substituting into the original formula and combining fractions, one obtains

$$\sum_{k=1}^{n} \mathbb{D}_{ak-b}^P = \frac{1}{2\sqrt{2}}\left(\bar{r}.r^{a-b}\left(\frac{r^{an}-1}{r^a-1}\right) - \bar{s}.s^{a-b}\left(\frac{s^{an}-1}{s^a-1}\right)\right)$$
$$= \frac{1}{2\sqrt{2}}\left[\frac{\bar{r}(s^a r^{an+a-b} - s^a r^{a-b} - r^{an+a-b} + r^{a-b})}{s^a r^a - s^a - r^a + 1} - \frac{\bar{s}(r^a s^{an+a-b} - r^a s^{a-b} - s^{an+a-b} + s^{a-b})}{s^a r^a - s^a - r^a + 1}\right].$$

Various simplifications result using the definitions of $\mathbb{D}_n^P$ and q_n in terms of $r, \bar{r}, s$, and $\bar{s}$ together with the relation $rs = -1$. For example,

$$\bar{r}s^a r^{an+a-b} - \bar{s}r^a s^{an+a-b} = r^a s^a(\bar{r}r^{an-b} - \bar{s}s^{an-b}) = (-1)^a.2\sqrt{2}.\mathbb{D}_{an-b}^P.$$

The denominator can be transformed into $(-1)^a - q_a + 1$. Using these relations the reader may verify the final formula is

$$\sum_{k=1}^{n} \mathbb{D}_{ak-b}^P = \frac{(-1)^a\mathbb{D}_{an-b}^P - \mathbb{D}_{a(n+1)-b}^P - (-1)^a\mathbb{D}_{-b}^P + \mathbb{D}_{a-b}^P}{(-1)^a - q_a + 1}$$

With this formula particular cases can be handled. Now we also show that the formula works when $a = b$.

Special Case 7: Let $a = 2 = b$ and $n = 4$.

$$\sum_{k=1}^{4} \mathbb{D}_{2k-2}^P = \frac{(-1)^2\mathbb{D}_6^P - \mathbb{D}_8^P - (-1)^2\mathbb{D}_{-2}^P + \mathbb{D}_0^P}{(-1)^2 - q_2 + 1}$$
$$= \{[(169p + 70q) + \varepsilon(408p + 169q)] - [(29p + 12q) + \varepsilon(70p + 29q)] + [(5p - 12q) + \varepsilon(-2p + 5q)] - [(p - 2q) + \varepsilon q]\}/4$$
$$= (36p + 12q) + \varepsilon(84p + 36q).$$

This result also is checked by actually summing the series:

$$\mathbb{D}_0^P + \mathbb{D}_2^P + \mathbb{D}_4^P + \mathbb{D}_6^P = [(p - 2q) + \varepsilon q] + [p + \varepsilon(2p+q)] + [(5p + 2q) + \varepsilon(12p + 5q)] + [(29p + 12q) + \varepsilon(70p + 29q)] = (36p + 12q) + \varepsilon(84p + 36q).$$

2.8 Generalized Dual Pell Quaternion ($\mathbb{D}_n^P$) Summations;

The n-th term of the generalized dual Pell quaternion sequence is defined by

$$\boldsymbol{D}_n^P = \mathbb{P}_n + i\mathbb{P}_{n+1} + j\mathbb{P}_{n+2} + k\mathbb{P}_{n+3},$$

where $\mathbb{P}_n$ is the n-th Gen. Pell number, $i^2 = j^2 = k^2 = ijk = 0$ and $ij = -ji = jk = -kj = ki = -ik = 0$. The scaler and the vector part of $\mathbb{D}_n^P$ are denoted by

$$S_{D_n^P} = \mathbb{P}_n \text{ and } V_{D_n^P} = i\mathbb{P}_{n+1} + j\mathbb{P}_{n+2} + k\mathbb{P}_{n+3}$$

Thus, the generalized dual Pell quaternion $\mathbb{D}_n^P$ is given by $\mathbb{D}_n^P = S_{D_n^P} + V_{D_n^P}$. The Binet's formula for the generalized dual Pell quaternion sequence is

$$\boldsymbol{D}_n^P = \frac{\hat{r}r^n - \hat{s}s^n}{r-s},$$

where

$$\hat{r} = (p - qs) + i[p(2-s) + q] + j[p(5-2s) + q(2-s)] + k[p(12-5s) + q(5-2s)],$$
$$\hat{s} = (qr - p) + i[p(r-2) - q] + j[p(2r-5) + q(r-2)] + k[p(5r-12) + q(2r-5)],$$
$$r = 1+\sqrt{2},\ s = 1-\sqrt{2}[14].$$

Following summations of the Pell quaternion sequence(QP_n) are shown in [2];

$$\sum_{s=1}^{n} QP_s,\ \sum_{s=1}^{n} QP_{2s},\ \sum_{s=1}^{n} QP_{2s-1},$$

and following summations of the generalized dual Pell quaternion sequence ($\mathbb{D}_n^P$) are shown in [14];

$$\sum_{s=1}^{n} \boldsymbol{D}_s^P,\ \sum_{s=0}^{n} \boldsymbol{D}_{n+s}^P,\ \sum_{s=1}^{n} \boldsymbol{D}_{2s-1}^P,\ \sum_{s=1}^{n} \boldsymbol{D}_{2s}^P$$

Now, we calculate the sums the $\sum_{k=1}^{n} \boldsymbol{D}_{ak-b}^P$:

$$\sum_{k=1}^{n} \boldsymbol{D}_{ak-b}^P = \frac{1}{2\sqrt{2}}\left(\sum_{k=1}^{n} \hat{r}.r^{ak-b} - \sum_{k=1}^{n} \hat{s}.s^{ak-b}\right)$$

One can restate the summations on the right-hand side of the equation by using the formula for geometric progressions,

$$\hat{r}\sum_{k=1}^{n} r^{ak-b} = \hat{r}.r^{a-b}\left(1 + r^a + r^{2a} + r^{3a} + \cdots + r^{(n-1)a}\right) = \hat{r}.r^{a-b}\left(\frac{r^{an}-1}{r^a-1}\right).$$

There is an entirely similar formula for the "s" summation. Substituting into the original formula and combining fractions, one obtains

$$\sum_{k=1}^{n} \boldsymbol{D}_{ak-b}^P = \frac{1}{2\sqrt{2}}\left(\hat{r}.r^{a-b}\left(\frac{r^{an}-1}{r^a-1}\right) - \hat{s}.s^{a-b}\left(\frac{s^{an}-1}{s^a-1}\right)\right)$$
$$= \frac{1}{2\sqrt{2}}\left[\frac{\hat{r}(s^a r^{an+a-b} - s^a r^{a-b} - r^{an+a-b} + r^{a-b})}{s^a r^a - s^a - r^a + 1} - \frac{\hat{s}(r^a s^{an+a-b} - r^a s^{a-b} - s^{an+a-b} + s^{a-b})}{s^a r^a - s^a - r^a + 1}\right].$$

Various simplifications result using the definitions of $\mathbb{D}_n^P$ and q_n in terms of r, $\hat{r}, s,$ and $\hat{s}$ together with the relation $rs = -1$. For example,

$$\hat{r}s^a r^{an+a-b} - \hat{s}r^a s^{an+a-b} = r^a s^a(\hat{r}r^{an-b} - \hat{s}s^{an-b}) = (-1)^a . 2\sqrt{2}. \boldsymbol{D}_{an-b}^P.$$

The denominator can be transformed into $(-1)^a - q_a + 1$. Using these relations the reader may verify the final formula is

$$\sum_{k=1}^{n} \boldsymbol{D}_{ak-b}^P = \frac{(-1)^a \boldsymbol{D}_{an-b}^P - \boldsymbol{D}_{a(n+1)-b}^P - (-1)^a \boldsymbol{D}_{-b}^P + \boldsymbol{D}_{a-b}^P}{(-1)^a - q_a + 1}$$

Let's confirm the sum $\sum_{s=1}^{n} \mathbb{D}_{2s-1}^P$ in [14] with the general formula that we found. This summation corresponds to $a = 2$ and $b = 1$.

$$\sum_{k=1}^{n} \boldsymbol{D}_{2k-1}^P = \frac{(-1)^2 \boldsymbol{D}_{2n-1}^P - \boldsymbol{D}_{2(n+1)-1}^P - (-1)^2 \boldsymbol{D}_{-1}^P + \boldsymbol{D}_1^P}{(-1)^2 - q_2 + 1} = \frac{\boldsymbol{D}_{2n}^P - \boldsymbol{D}_0^P}{2},$$

where $\mathbb{D}_0^P = p\mathbf{D}_0^P + q\mathbf{D}_{-1}^P$, $\mathbf{D}_n^P$ is the n-th dual Pell quaternion. This is the same as the result in [14].

Special Case 8: Let $a = 3$, $b = 2$ and $n = 3$.

$$\sum_{k=1}^{3} \boldsymbol{D}_{3k-2}^P = \frac{(-1)^3 \boldsymbol{D}_7^P - \boldsymbol{D}_{10}^P - (-1)^3 \boldsymbol{D}_{-2}^P + \boldsymbol{D}_1^P}{(-1)^3 - q_3 + 1}$$

$$= \{(2548p + 1050q) + i(6146p + 2548q) + j(14840p + 6146q) + k(35826p + 14840q)\}/14$$
$$= (182p + 75q) + i(439p + 182q) + j(1060p + 439q) + k(2559p + 1060q).$$

This result may be checked by actually summing the series:

$$\boldsymbol{D}_1^P + \boldsymbol{D}_4^P + \boldsymbol{D}_7^P = [p + (12p + 5q) + (169p + 70q)] + i[(2p + q) + (29p + 12q) + (408p + 169q)] + j[(5p + 2q) + (70p + 29q) + (985p + 408q)] + k[(12p + 5q) + (169p + 70q) + (2378p + 985q)]$$
$$= (182p + 75q) + i(439p + 182q) + j(1060p + 439q) + k(2559p + 1060q).$$

3. Conclusion

In this paper, the finite Pell sums and infinite Pell sums have been considered. In addition, Lucas, Pell-Lucas, Jacopsthal, generalized Pell, generalized dual Pell, generalized dual Pell quaternion sums have been made. These finite sums found are almost identical, showing slight differences in markers and coefficient originating from the denominator. So, these results may lead us to the following potential question: Can a single formula be formed for the finite sums of special number sequences? One possible indication regarding the issue is that this formula works out in thecases except for the ones that make the denominator zero. It is of great importance to pay attentionalthough this is a rare case. An example of this is given as in the below:

The $a = 4,\ b = 2,\ n = 3$ values in the Jacopsthal (J) summations and the Jacopsthal-Lucas (j) summations.

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