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Mixed; Lagrange's and Cauchy's Remainders Form

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Abstract: ***Sometimes numerical methods are needed to solve mathematical problems, especially in applied problems. The numerical methods usually associated with errors, so the numerical solution is usually not equivalent to the exact solution, but if the error could be estimated then the exact solution could be known. The Lagrange's and Cauchy's remainders are two poplar methods to calculate the remainder and the generalization of them is known Schloemilch-Roeche's remainder. By comparing: the Lagrange's and Cauchy's remainders methods for some functions at a point x, it could be seen that the Lagrange method has more accuracy if c is in a neighborhood of x_0, while the Cauchy method gives better results if cis somewhere near the middle between $x_0$andx.***

Keywords: Taylor polynomial, Lagrange remainder, Cauchy remainder, Schloemilch-Roeche's remainder

1. Introduction

To convert a function $f$into a sample form as a polynomial form $P_N(x)$, there are some methods could be used for this purpose, one of them is known as a Taylor polynomial, its form is

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

where x_0 is a center of the polynomial [10].

The Taylor polynomial of the function f is extremely useful in all sorts of science applications, and at the same time, it is fundamental in pure mathematics, specifically in complex function theory. Recall that, if $f(x)$is infinitely differentiable at $x \in (a, b)$, the Taylor series of $f(x)$ at $x \in (a, b)$ is, by definition, $P_N(x)$ [9]. But the polynomial $P_N(x)$ which is produced, actually, does not equivalent to the original function f, so there is a difference between f and $P_N(x)$, which is known the remainder $R_N(x)$ where

$$R_N(x) = f(x) - P_N(x).$$

The most popular forms of the remainder in Taylor's formula are the classical well-known the Lagrange's and Cauchy's forms of the remainder. Lagrange's and Cauchy's forms are special cases of the Schloemilch-Roeche's remainder, [5, 6].

Every version of Taylor's Theorem says that the Taylor polynomial of some degree $P_N(x)$ about a central point x_0 can be used to approximate a given function on some (more than likely tiny) neighborhood. Thus, the most important statement, in every version of Taylor's theorem, is how to express the remainder? Taylor himself didn't, actually, incorporate an error term. It was not, until Lagrange and then Cauchy came about that Taylor's theorem was made rigorous [10]. Thus, Roche's version above can naively be appreciated as a theorem that interpolates the first rigorous expressions of the remainder [6].

Now, if the Taylor polynomial, as a method or means, is used to generate an approximation, it must able to control the accuracy of the approximation. Historically, these ideas were born early in the development of calculus born the development of modern mathematical rigor [10]. The mathematicians of the time felt that the Taylor polynomial would yield something approximately equal to the function in question.

Unfortunately, they were incorrect; since this is not always the case. The Lagrange Remainder theorem does give one the desired control. The remainder term expresses the error by breaking the series at Nth term. Thus, it is important when broken series are used for approximation; the resulting error must be evaluated.

The investigation in this paper will concern on the remainder $R_N(x)$, and comparing the Lagrange's and Cauchy's remainders of some functions in some region.

Note two things about$R_N(x)$. First, it depends on$N$; as N increases, it is expected the remainder decreases in size. Second, $R_N(x)$ depends on x, as x moves away from the center of the polynomial, the size of $R_N(x)$ will usually be expected to increase. In short, for any x-value in the domain of$f$, then $f(x) = P_N(x) + R_N(x)$,this says that the actual value of f at some x-value is equal to the polynomial approximation at x plus some remainder $R_N(x)$.

The function $R_N(x)$ is almost never be known explicitly. Coming up with such a function is just too tall an order. However, that does not mean it cannot be known anything about it. One fundamental fact about $R_N(x)$ is given by the following.

Definition: (Taylor Polynomial)
Let f be a continuous function with N continuous derivatives. Then, the Taylor polynomial of f is defined as:

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n, \qquad (1)$$

and

$$f(x) = P_N(x) + R_N(x)$$

where the remainder term is

$$R_N(x) = f(x) - P_N(x). \qquad (2)$$

The Taylor series does converge to the function itself must be a non-trivial fact. Most calculus textbooks would invoke Taylor's theorem (with Lagrange remainder)**,** and would probably mention that it is a generalization of the mean value theorem. Fortunately, a very natural derivation based only on the fundamental theorem of calculus (and a little bit of multi-variable perspective) is all one would need for most functions. It seems that Lagrange was the first to study the condition to expand a function in the Taylor series [11].

Lagrange form of the remainder:

Theorem: (Taylor's Theorem with Lagrange Remainder):
Let f be n times differentiable of the interval $[x_0, x]$ and let $f^{(n+1)}$ exists in the open interval (x_0, x). Then, for some $c \in (x_0, x)$

$$f(x) = P_N(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{(n+1)}.$$

the remainder term is:

$$R_N(x) = f(x) - P_N(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{(n+1)} \quad (3)$$

where c is a point between x_0 and x.

This formula for the remainder term is called Lagrange's form of the remainder term. Note that, this expression is very similar to the terms in the Taylor series except that $f^{(n+1)}$ is evaluated at c instead of at x_0.

All it can be said about the number c is that it lies somewhere between x and x_0 [9].

One thing that is important to realize about the theorem, that it is an existence theorem. It tells us that there must be some number c with the property expressed in equation (3), but it does not tell us how to find that number. In the vast majority of cases, in fact, it will be completely unable to find the magic$c$.

The appearance of c, a point between x and x_0, and the fact that it has been plugged into a derivative suggests that there is a connection between this result and the Mean Value Theorem. In fact, for $n = 0$ the result says that there is a number c between x_0 and x such that

$$f(x) = f(x_0) + f'(c)(x - x_0), \quad (4)$$

this is the Mean Value Theorem. On one hand, this reflects the fact that Taylor's theorem is proved using a generalization of the Mean Value Theorem. On the other hand, it shows that it can be regard a Taylor expansion as an extension of the Mean Value Theorem.

Because of the difficulties in finding the number c, the best it can be hope for is an estimate on the remainder term. This may seem like settling, but it actually makes some sense. The Taylor polynomials are used to approximate the values of functions that it cannot evaluate directly. The situation is already settling for an approximation. If the exact error in the approximation could be found, then it would be able to determine the exact value of the function, simply by adding it to the approximation. If the situation, it could find the exact value of the function, then the approximation does not need at all.

The first order of business is to convert the existence-based Lagrange's and Cauchy's remainders, and the relation between them to estimate the value of c.

Theorem: Lagrange Error Bound:
Suppose that$x > x_0$, if $f^{(n+1)}(x)$ is bounded on the interval$[x_0, x]$, i.e., if there is a positive number M such that:

$$-M \le f^{(n+1)}(t) \le M,$$

for all $t \in [x_0, x]$), then

$$|R_N(x)| \le \frac{M}{(n+1)!}|x - x_0|^{(n+1)}. \quad (6)$$

If$x < x_0$, then the interval$[x_0, \mathrm{x}]$, in question, is simply to be replaced with $[x, x_0]$. No other change is required [10].

Since it is known that $f^{(n+1)}(c)$ is no more than M in absolute value, it could replace with M, sacrificing equality for inequality.

The Lagrange error bound frees us from the need to find c, but it replaces c with M. But many times, M is no easier to find than c. The difficulty in using the Lagrange error bound is to find a reasonable upper bound—a cap—on the values of $f^{(n+1)}(t)$ on the interval in question.

Cauchy's Remainder:
Often when using Lagrange's Remainder, we'll have a bound on $f^{(n)}$, and rely of the $n!$ beating the $(x - a)^n$ as $n \to \infty$. But if $f^{(n)}$begins to provide us with an $n!$-shapedterm on top, as with the binomial expansion, we may need a better expression of $(x - a)^n$. So, in the Cauchy method, the term $(x - a)^n$ in Lagrange method was replaced by the term $(x - x_0)(x - c)^n$, as well as the value$(n + 1)!$ being replaced by the value $n!$. Cauchy's remainder after n terms of the Taylor series for a function $f(x)$ expanded about a point x_0 is given by:

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(c)}{n!}(x - x_0)(x - c)^{(n)}, (7)$$

where c is a point between x_0 and$x$[7].

Note that the Cauchy remainder R_n is also sometimes taken to refer to the remainder when terms up to the $(n - 1)$ power are taken in the Taylor series [16].

Schloemilch-Roeche's Remainder:
The Schloemilch-Roeche's remainder formula, after N terms of the series, gives

$$R_n(x) = \frac{f^{(n+1)}(c)}{n!p}(x - c)^{n+1-p}(x - x_0)^p, \quad (8)$$

For $c \in (x_0, x)$ and any $p > 0$. The choices $p = n + 1$and $p = 1$give the Lagrange and Cauchy remainders, respectively [5]. Many other researchers have developed different forms of Taylor's remainder in order to improve the bounds of error. In the literature (seefor example [1, 5, 12] many forms of the remainder in Taylor's formula are given).

New form of the remainder:

By comparing the Lagrange formula (3) and the Cauchy formula (7), the factor $\frac{f^{(n+1)}(c)}{n!}$ appears in both formulas, but the difference is the following:
From (3)

$$R_{NL}(x) = \frac{f^{(n+1)}(c)}{(n+1)n!}(x - x_0)^{(n+1)},$$

Lagrange's remainder

and from (7) $R_{NC}(x) = \frac{f^{(n+1)}(c)}{n!}(x-x_0)(x-c)^{(n)}$,

Cauchy's remainder.

Suppose that they are equal,

$$R_{NL}(x) \cong R_{NC}(x),$$

then

$$\frac{f^{(n+1)}(c)}{(n+1)n!}(x-x_0)(x-x_0)^{(n)} \cong \frac{f^{(n+1)}(c)}{n!}(x-x_0)(x-c)^{(n)},$$

and

$$\frac{(x-x_0)^{(n)}}{(n+1)} \cong (x-c)^{(n)},$$

$$x-x_0 \cong \xi(x-c),$$

where $\xi = \sqrt[n]{n+1}$, so

$$c \cong \frac{(\xi-1)x+x_0}{\xi} \qquad (9)$$

Other forms of the remainder in Taylor formula:

Theorem: If $g \in C^{n+1}(a,b)$, g^{n+2} exists on (a,b), $g^{n+1}(t) \neq 0$, and $g^{n+2}(t) \neq 0$ for all $t \in (a,b)$, then there is a number $\xi \in (a,b)$ such that

$$R_n(g,a;b) = \frac{g^{n+1}(a)(b-a)^{n+1}}{(n+1)!}\cdot\frac{(n+2)g^{n+1}(\xi)+(\xi-a)g^{n+2}(\xi)}{(n+2)g^{n+1}(\xi)+(\xi-b)g^{n+2}(\xi)}. \quad (10)$$

In many cases, this formula of the remainder gives essentially better bounds of error than all other known forms of the remainder [8].

The generalization of Taylor form

Theorem If f^n, g^n are continuous on $[a,b]$, and f^{n+1}, g^{n+1} exist on (a,b), and if $g^{(n+1)}(t) \neq 0$ for any $t \in (a,b)$, then there is a number $\xi \in (a,b)$ such that

$$\frac{R_n(f;a,b)}{R_n(g;a,b)} = \frac{f^{(n+1)}(\xi)}{g^{(n+1)}(\xi)}. \qquad (10)$$

Moreover, the remainder could be calculated by using an integral formula. In this version, the error term involves an integral, so it is assumed that $f^{(n+1)}$ is continuous, whereas previously, it is only assumed this derivative exists. However, this integral version of Taylor's theorem does not involve the essentially unknown constant c. This is vital in some applications [8].

Theorem: If $f^{(n+1)}$ is continuous on an open interval I that contains x_0 and x is in I then

$$R_n(x) = \frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}dt, \qquad (11)$$

to give an error estimate for approximating a function by the first few terms of the Taylor series, Taylor's theorem (with Lagrange remainder) provides the crucial ingredient to prove that the full Taylor series converges exactly to the function, it is supposed to be represented. [3, 11].

In the following examples a comparison between Lagrange's and Cauchy's remainders and the new form.

Example 1: consider the function:

$f(x) = e^x$, $x_0 = 0.5$, $x = 0.7$, in the interval [0,1].

The Taylor series of order 3 of the function f is

$$P_3(x) = 1.64872 + 1.64872(x-0.5) + 1.64872\frac{(x-0.5)^2}{2!} + 1.64872\frac{(x-0.5)^3}{3!}.$$

For $x = 0.7$,

$f(0.7) = 2.013752707$,

$P_3(0.7) = 2.013636693$,

then the remainder is $E(0.7) = |f(0.7) - P_3(0.7)| = 0.00011601367$.

The following tables (I) and (II) illustrate the remainders of the Lagrange and the Cauchy methods respectively

Table (I): c values and the corresponding remainder in the Lagrange formula for the function $f(x) = e^x$.

c	$R_3(0.7)$
0.52	0.000112135373
0.53	0.0001132627202
0.54	0.0001144010495
0.5405	0.0001144582443
0.55	**0.00011550259**
0.56	0.0001167115584
0.57	0.000117884529
0.6	0.0001214746474

Table (II): c values and the corresponding remainder in the Cauchy formula for the function $f(x) = e^x$.

c	$R_3(0.7)$
0.51	0.001332593769
0.54	0.00082002248
0.55	0.000682468
0.56	0.000560448623
0.57	0.0004532363164
0.6	0.0002125805
061	0.00015652869
0.62	0.000111039968
0.65	0.00002793497
0.69	0.00000023260014

Example 2: Consider the function:

$$f(x) = \cos x, \quad x \in \left[0,\frac{\pi}{2}\right], \quad x_0 = \frac{\pi}{5}, \quad x = \frac{\pi}{3}.$$

$$P_3(x) = 0.809016991 - 0.587785252\left(x-\frac{\pi}{5}\right) - 0.404508497\left(x-\frac{\pi}{5}\right)^2 + 0.097964208\left(x-\frac{\pi}{5}\right)^3$$

$$P_3\left(\frac{\pi}{3}\right) = 0.499031181,$$

$$f\left(\frac{\pi}{3}\right) = 0.5,$$

$$E_3\left(\frac{\pi}{3}\right) = \left|f\left(\frac{\pi}{3}\right) - P_3\left(\frac{\pi}{3}\right)\right| = 0.000968819,$$

Table (III): c values and the corresponding remainders in the Lagrange's and Cauchy's formulas for the function $f(x) = \cos x$

c	Lagrange $R_3\left(\frac{\pi}{3}\right)$	Cauchy $R_3\left(\frac{\pi}{3}\right)$
$0.63 \cong \frac{\pi}{5}$	0.00103650083	0.004096271712
0.7	**0.0009811039633**	0.00234802526
0.8	0.000893702943	0.0007347165858
0.9	0.0007973723385	**0.0001384066904**
$1.0 \cong \frac{\pi}{3}$	0.0006930746534	0.000003965816

2. Discussion

Table (I), for the Lagrange formula, shows that the perfect value of c is $c \cong 0.55$ which gives a suited remainder value $R_3(0.7) = 0.00011550259$, it has seven decimal places true, this value of c is closer to $x_0 = 0.5$ than $x = 0.7$.

For the Cauchy formula, table (II), the value of c which gives a good estimate of the remainder is $0.60 < c < 0.61$ which gives$0.000156528 \leq R_3(0.7) \leq 0.0002125805$, the value of c is seemed to be somewhere near the middle of $x_0 = 0.5$, and $x = 0.7$, after that the remainder decreases as c moves to $x = 0.7$. So,good results of the remainder by using Lagrange formula could be got if c is close tox_0, and by Cauchy formula, if c nearsthe middle between x_0 and x.

By checking the value of c using the equation (9),

$$c \cong \frac{(\xi - 1)x + x_0}{\xi},$$

where $x_0 = 0.5, x = 0.7, n = 3, \xi = 1.5874$,
it gives:

$$c \cong \frac{(1.5874 - 1)(0.7) + 0.5}{1.5874},$$

$$c \cong 0.574,$$

and the remainder is

$R_{3L}(0.7) = 0.000118356$ and $R_{3C}(0.7) = 0.000118379$,

which gives at least five valid digits.

Table (IV): The values of c and the remainders for the Lagrange and the Cauchy methods for the example (1)

Method	c value	The remainder	c value	The remainder
Lagrange	0.55	0.00011550259	0.574	0.000118356
Cauchy	0.61	0.00015652869		0.000118379

A table (III), it seems that $R_3(\frac{\pi}{3})$ has more accuracy when $c \approx 0.7$ is close to $x_0 = \frac{\pi}{5}$, for the Lagrangemethod, while for Cauchy method, $R_3(\frac{\pi}{3})$, has more accuracy when $c \approx 0.9$ is close to $x = \frac{\pi}{3}$. Meanwhile, the value of $c$could be estimated by the following

$$P_3\left(\frac{\pi}{5}\right) = \frac{\left(\frac{\pi}{3} - \frac{\pi}{5}\right)^4}{4!} f^{(4)}(c),$$

$$0.000968819 = 0.001282753462 f^{(4)}(c),$$
$$0.000968819 = 0.001282753462 \cos(c),$$
$$c = 0.714737734.$$

By using the equation (9)

$$c \cong \frac{[(\xi - 1)\frac{\pi}{3} + \frac{\pi}{5}]}{\xi}, \quad \text{where } \xi = \sqrt[3]{4} = 1.5874,$$
$$= 0.78333,$$

This gives a remainder?

$$E_{3L}\left(\frac{\pi}{3}\right) \cong 0.000908917 \text{ and } E_{3C}\left(\frac{\pi}{3}\right) \cong 0.000908817$$

The following tables show the values of c and the estimation of the error according to the Lagrange remainder (a)and the Cauchy remainder (b), they show that thesuitable c values which give the nearest error of polynomial P_n for the example 1, by studying higher-order of the polynomial for example 1. Now it could say that by using equation 9, it may be able toguess the value of c, than introduce a good estimation of the polynomial. Table V, shows the values of c according to higher-order for polynomial. The values of c accompanying the resulting polynomial degree for Lagrange (a) and Cauchy (b), which give close values of both cases,

Table (V- a): The value of c and Lagrange's remainder for $P_n, (n = 1, 2, .., 5)$

n	c	R_n	E_n	$\lvert E_n - R_n \rvert$
1	0.6	0.036442376	0.035287182	0.001155194
2	0.58	0.002381384	0.002312757	0.00006863
3	0.574	0.000118356	0.000114462	0.000003894
4	0.566	0.000004696	0.000004547	0.000000149
5	0.56	1.5×10^{-7}	1.5×10^{-7}	0.0

Table (V - b): The value of c and Cauchy's remainder for $P_n, (n = 1, 2, .., 5)$

n	c	R_n	E_n	$\lvert E_n - R_n \rvert$
1	0.6	0.036442376	0.035287182	0.001155194
2	0.58	0.002571895	0.002312757	0.000259138
3	0.574	0.000118379	0.000114462	0.000003917
4	0.566	0.0000044732	0.000004547	0.000000073
5	0.56	1.5×10^{-7}	1.5×10^{-7}	0.0

3. Conclusion

Interpolation, in general, is a recurring useful idea of mathematics. In this way, Roche's Theorem can be viewed as generalizing of Lagrange's and Cauchy's remainders versions in the same way that Young's Inequality generalizes [1], the same way that Holder's Inequality generalizes [5], Cauchy's inequality [8]. There are some methods could be used for this purpose, the most famous of them are the Lagrange's and Cauchy's remainders. By studying and comparing these two methods, some of them could be used; it depends on the value ofc; if c is in the neighborhood of x_0 then Lagrange method will be more suitable, and if c is near themiddle of x and x_0 however, then Cauchy method should be the choice.

The value of c can be estimated by the relation between Lagrange's and Cauchy's remainders to get a new form to define a good value of c which is used to estimate the required remainder. Now, by finding the value of c, it could help to estimate the remainder and, then produce a suitable polynomial.

Moreover, there are plenty of methods could be used to calculate the remainder; some of them use an integral form.

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