Topological Modules

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Abstract: ***In this article we discuss the Topological Modules. If Topological Modules is Hausdorff then its structure is usual one meaning by this that there exists an isomorphism of M onto R. If M is a module then M is algebraically isomorphic to R. Consider modules over the ring R of real or complex numbers which is given the usual Euclidean Topology defined by means of the vector spaces.***

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1. Introduction

A modules M over$\mathbb{R}$ is called a topological module if M is provided with a topology τ which is compatible with the modules structure of M, i.e. τ makes the modules operations both continuous.

More precisely, the condition in the definition of topological modules requires that:

$$\begin{array}{lll} M \times M & \to & M \\ (a, b) & \to & a + b \quad \text{vector addition} \\ \mathbb{R} \times M & \to & M \\ (\mu, a) & \to & \mu a \quad \text{scalar multiplication} \end{array}$$

are both continuous when we endow M with the topology τ, $\mathbb{R}$ with the Euclidean topology, M × M and $\mathbb{R}$ × M with the correspondent product topologies.

2. Preliminaries

2.1 Definition

A nonempty set G together with a binary operation ∗:G×G→G is called a *Group* if the following conditions are satisfied

1) ∗ is associative
   i.e) a*(b*c)=(a*b) *c for all a, b, c ∈ G
2) There exists an element e ∈G such that
   a*e = e*a =a for all a ∈ G
   e is called the identity element of G.
3) For any element a in G there exists an element a' ∈ G such that
   $a*a' = a'*a = e$
   a' is called the inverse of a.

2.2. Definition

A nonempty set R together with two binary operation denoted by "+" and "." are called addition and multiplication which satisfy the following axioms is called a *Ring.*

1) (R, +) is an abelian group.
2) "." is an associative binary operation on R.
3) a.(b+c) = a.b + a.c & (a+b).c = a.c + b.c
   for all a, b, c∈R.

2.3 Definition

Let F be a field and V be a nonempty set. Then V is called a vector space over F if the following axioms hold:

- Abelian group V of "vectors".
- Field F of "scalars".
- f.v is a "scaledvector"
- Distributive properties:
  1. f. (v_1+v_2) = f.v_1+f.v_2
  2. $(f_1 + f_2)$. v = f_1.v+f_2 .v
- Associative property:
  $(f_1.f_2)$. v = f_1.(f_2.v)
- 1.v = v

2.4 Definition

Let R be a ring. A left R-module or a left module over R is a set M together with,

- Abelian group M of "elements".
- Ring R of "scalars".
- r.m is a "scaledelements".
- Distributive properties:
  1. r. $(m_1 + m_2)$ = r.$m_1 + r.m_2$
  2. $(r_1 + r_2)$.m = $r_1.m + r_2.m$
- Associative property:
  $(r_1.r_2)$.m = r_1.(r_2.m)
- 1.m = m

2.5 Definition

A *topological vector space* X is a vector space over a topological field K that is endowed with a topology such that vector addition X×X→X and scalar multiplication K×X→X are continuous functions.

2.6 Remark

If (x,τ) is a topological vector space then it is clear that $\sum_{k=1}^{N} \lambda_k^{(n)} x_k^{(n)} \to \sum_{k=1}^{N} \lambda_k x_k$ as n → ∞ with respect to τ if for each k = 1,…..,N as n → ∞ we have that $\lambda_k^{(n)} \to \lambda_k$ with respect to the Euclidean topology on $\mathbb{R}$ and $x_k^{(n)} \to x_k$ with respect to τ.

2.7 Example

a) Every vector space X over $\mathbb{K}$ endowed with the trivial topology is a topological vector space.
b) Every normed vector space endowed with the topology given by the metric induced by the norm is a topological vector space.

c) There are also examples of spaces whose topologycannot be induced by a norm or a metric but that are topological vector space, e.g. the space of infinitely differentiable functions, the space of test functions and the spaces of distributions (we will see later in details their topology).

3. Main Result

3.1 Definition

Two topological modules M and N over$\mathbb{R}$are (topologically) isomorphic if there exists a modulesisomorphism M → N which is at the same time a homeomorphism (i.e. bijective, linear, continuous and inverse continuous).

3.2 Definition

Let M and N be two topological moduli on $\mathbb{R}$.

- A topological homomorphism from M to N is a linear mapping which is also continuous and open.
- A topological monomorphism from M to N is an injective topological homomorphism.
- A topological isomorphism from M to N is a bijective topological homomorphism.
- A topological automorphism of M is a topological isomorphism from m into itself.

3.3. Proposition

Every modules M over$\mathbb{R}$ endowed with the discrete topology is not a topological module. Unless M = {0}.

Proof:
Assume by a contradiction that it is a topological moduleand take 0≠ m ∈ M. The sequence $\alpha_n = \frac{1}{n}$ in $\mathbb{R}$ converges to 0 in the Euclidean topology.

Therefore, since the scalar multiplication is continuous, $\alpha_n x \to 0$, i.e. for any neighbourhood U of 0 in M there exists m ∈ $\mathbb{N}$ such that $\alpha_n x$ ∈U for all n ≥ m.

In particular, we can take U = {0} since it is itself open in the discrete topology. Hence, $\alpha_m x = 0$, which implies that m = 0 and so a contradiction.

3.4. Proposition

Given a topological modules M we have that:

1) For any $m_0 \in$ M, the mapping m → m +m_0 is a homeomorphism of M onto itself.
2) For any 0 ≠ μ ∈ $\mathbb{R}$, the mapping m → μm is a topological automorphism of M.

Proof:
Both mappings are continuous by the very definition of topological modules. Moreover, they are bijections by the module's axioms and their inverses m → m -m_0 and m →$\frac{1}{\mu}$m is also continuous. Note that the second map is also linear so it is a topological automorphism.

3.5 Example

If a metric d on a modules M is translation invariant, i.e. d (a+c, b+c) = d (a, b) for all a, b∈ M, then the topology induced by the metric is translation invariant and the addition is always continuous. However, the multiplication by scalars does not need to be necessarily continuous (take d to be the discrete metric, then the topology generated by the metric is the discrete topology which is not compatible with the scalar multiplication see proposition 3.3).

3.5. Theorem

A filter $\mathcal{F}$ of a modules M over $\mathbb{R}$ is the filter of neighbourhoods of the origin with respect to some topology compatible with the module structure of M if and only if

1) The origin belongs to every set P ∈ $\mathcal{F}$
2) ∀ P ∈ $\mathcal{F}$, ∃ Q ∈ $\mathcal{F}$ s.t. Q + Q ⊂ P
3) ∀P ∈ $\mathcal{F}$, ∀ μ ∈ $\mathbb{R}$ with μ ≠ 0 we have μ P ∈ $\mathcal{F}$
4) ∀ P ∈ $\mathcal{F}$, P is absorbing.
5) ∀ P ∈ $\mathcal{F}$, ∃ Q ∈ $\mathcal{F}$ s.t. Q ⊂ P is balanced.

Proof

Necessity part

Suppose that M is a topological module then we aim to show that the filter of neighbourhoods of the origin $\mathcal{F}$ satisfies the properties 1, 2, 3,4,5. Let P∈ $\mathcal{F}$.

1) Obvious, since every set P∈ $\mathcal{F}$ is a neighbourhood of the origin o.
2) Since by the definition of topological modules the addition (a, b) → a+b is a continuous mapping, the preimage of P under this map must be a neighbourhood of (o, o) ∈ M×M. Therefore, it must contain a rectangular neighbourhood G×G' where G, G' ∈ $\mathcal{F}$. Taking Q = G∩ G' we get the conclusion, i.e. Q + Q ⊂ P.
3) By proposition 3.4, fixed an arbitrary 0 ≠ μ ∈ $\mathbb{R}$, the map a → μ^{-1} a of M into itself is continuous. Therefore, the pre image of any neighbourhood P of the origin must be also such a neighbourhood. This preimage is clearly μP, hence μP ∈ $\mathcal{F}$.
4) Suppose by contradiction that P is not absorbing. Then there exists y ∈ M such that ∀ n∈ $\mathbb{N}$ we have that $\frac{1}{n}y \notin$ P. This contradicts the convergence of $\frac{1}{n}y$ → 0 as n → ∞ (because P ∈ $\mathcal{F}$ must contain infinitely many terms of the sequence $(\frac{1}{n}y)_{n\in\mathbb{N}}$).
5) Since by the definition of topological modules the scalar multiplication $\mathbb{R}$ ×M→M, (μ, a) → μa is continuous, the preimage of P under this map must be a neighbourhood H×G where H is a neighbourhood of (0, o) ∈ $\mathbb{R}$ ×M. Therefore, it contains a rectangular neighbuorhood H×G where H is a neighbourhood of 0 in the Euclidean topology on $\mathbb{R}$ and G∈ $\mathcal{F}$. on the other hand, there exists ρ>0 such that $B_\rho(0)$ = {μ∈ $\mathbb{R}$: |μ| ≤ ρ} ⊆ H. Thus $B_\rho(0)$×G is contained in the preimage of P under the scalar multiplication, i.e. μG ⊂ P for all μ∈ $\mathbb{R}$ with |μ| ≤ ρ. Hence, the set Q = $\cup_{|\mu| \leq \rho}$ μG ⊂ P. Now Q ∈ $\mathcal{F}$ since each μG ∈ $\mathcal{F}$ by 3 and v is clearly balanced (since for any a ∈ Q there exists μ∈ $\mathbb{R}$ with |μ| ≤ ρ s.t. a ∈ μG and therefore for any |α| ≤ 1 we get αa ∈ αμG ⊂ Q because |αμ| ≤ ρ).

Sufficiency part.

Suppose that the conditions 1,2,3,4,5 hold for a filter $\mathcal{F}$ of the modules M. We want to show that there exists a topology τ on M such that $\mathcal{F}$ is the filter of neighbourhoods of the origin with respect to τ and (M, τ) is a topological module.

Let us define for any a ∈ M the filter $\mathcal{F}$(a) = {P+a: P ∈ $\mathcal{F}$ }.

In fact, we have:

- By 1 we have that ∀ P ∈ $\mathcal{F}$, o ∈ P, then ∀ P ∈ $\mathcal{F}$, a = o+a ∈ P+a, i.e. ∀ A ∈ $\mathcal{F}$(a), a ∈ A.
- Let A ∈ $\mathcal{F}$(a) then A = P + a for some P∈ $\mathcal{F}$. By 2, we have that there exists Q ∈ $\mathcal{F}$ s.t. Q + Q ⊂ P. Define B = Q + a ∈ $\mathcal{F}$(a) and take any y ∈ B then we have Q + y ⊂ Q + B ⊂ Q + Q + a ⊂ P + a = A. But Q + y belongs to the filter $\mathcal{F}$(y) and therefore so does A.

  There exists a unique topology τ on M such that $\mathcal{F}$(a) is the filter of neighbourhoods of each point a ∈ M and so for which in particular $\mathcal{F}$ is the filter of neighbourhoods of the origin.

  It remains to prove that the vector addition and the scalar multiplication in M are continuous with respect to τ.

- The continuity of the addition easily follows from the property 2. Indeed, let (x_0, y_0) ∈ M × M and take a neighbourhood G of its image $x_0 + y_0$.

  Then G = P + $x_0 + y_0$ for some P ∈ $\mathcal{F}$. By 2, there exists Q ∈ $\mathcal{F}$ s.t. Q + Q ⊂ P and so (Q + x_0) + (Q+y_0) ⊂ G. this implies that the preimage of G under the addition contains (Q + x_0) × (Q+y_0) which is a neighbourhood of (x_0, y_0).

- To prove the continuity of the scalar multiplication, let (μ_0, x_0) ∈ ℝ×M and take a neighbourhood P' of $\mu_0 x_0$. Then P' = P + $\mu_0 x_0$ for some P ∈ $\mathcal{F}$. By 2 and 5, there exists G ∈ $\mathcal{F}$ s.t. G + G + G⊂ Pand G is balanced. By 4, G is also absorbing so there exists ρ> 0 such that ∀ μ ∈ ℝ with |μ| ≤ ρ we have μx_0 ∈G.
  Suppose μ_0 = 0 then $\mu_0 x_0$ = 0 and P' = P. Now
  Im ($B_\rho(0)$ × (G +x_0)) ={μy + μx_0 : μ ∈ $B_\rho(0)$, y ∈ G}.
  As μ ∈ $B_\rho(0)$ and G is absorbing, μ ∈ $B_\rho(0)$. Also, since |μ| ≤ ρ ≤ 1 for all μ ∈ $B_\rho(0)$ and since G is balanced, we have μG ⊂ G. Thus Im ($B_\rho(0)$ × (G +x_0)) ⊂ G + G ⊂ G + G +G ⊂ P and so the preimage of P under the scalar multiplication contains ($B_\rho(0)$ × (G +x_0) which is a neighbourhood of (0,x_0).

  Suppose $\mu_0 \neq$ 0 and take σ = min {ρ, $|\mu_0|$}. Then Im (($B_\sigma(0)$ + μ_0) × ($|\mu_0|^{-1}$G + x_0)) = {μ$|\mu_0|^{-1} y + \mu x_0$ + μ$|\mu_0|^{-1} y + \mu x_0$: μ ∈ $B_\sigma(0)$, y∈G}. As μ ∈ $B_\sigma(0)$, $\sigma \leq \rho$ and G is absorbing, μx_0 ∈ G. Also since ∀ μ ∈ $B_\sigma(0)$ the vector space of μ$|\mu_0|^{-1}$ and $\mu_0 |\mu_0|^{-1}$ are both ≤ 1 and since G is balanced, we have μ$|\mu_0|^{-1}$G, $\mu_0 |\mu_0|^{-1}$G ⊂ G. Thus Im ($B_\sigma(0)$ + μ_0 × ($|\mu_0|^{-1}$G + x_0)) ⊂ G + G + G + $\mu_0 x_0 \subset P + \mu_0 x_0$ and so the preimage of $P + \mu_0 x_0$ under the scalar multiplication contains $B_\sigma(0)$ + μ_0 × ($|\mu_0|^{-1}$G + x_0) which is a neighbourhood of ($\mu_0 x_0$).

3.6 Proposition

1) Every linear subspace of a topological modules endowed with correspondent subspace topology is itself a topological module.
2) The closure $\overline{H}$ of a linear subspace H of a topological modules. M is again a linear subspace of M.
3) Let M, N be a two topological modules and f: M → N a linear map. f is continuous if and only if f is continuous at the origin 0.

Proof

1) This clearly follows by the fact that the addition and the multiplication restricted to the subspace are just a composition of continuous maps.
2) Let $x_0, y_0 \in \overline{H}$ and let us take any P ∈ $\mathcal{F}(o)$. There exist Q ∈ $\mathcal{F}(o)$such that Q + Q ⊂ P. Then by definition of closure points, there exist x,y ∈ H (since H is a linear subspace) and x+y ∈ (Q+x_0) + (Q+y_0) ⊂ P + $x_0 + y_0$. Hence, $x_0 + y_0 \in \overline{H}$. Similarly, one can prove that if $x_0 \in \overline{H}$, $\mu x_0 \in \overline{H}$ for any μ∈ ℝ.
3) Assume that f is continuous at o∈M and fix any x ≠ 0 in M. Let P be an arbitrary neighbourhood of f(x) ∈ M. we know that P = f(x) + Q where Q is a neighbourhood of o ∈ N. since f is linear, we have that:
   f^{-1}(P) = f^{-1}(f(x) + Q) ⊃ x + f^{-1}(P).
   By the continuity at the origin of M, we know that $f^{-1}(Q)$ is a neighbourhood of o ∈M and so x + $f^{-1}(Q)$ is a neighbourhood of x ∈ M.

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