Significance of Modules with Ascending Chain Condition Which Corresponds of Definite Sorts of Annihilator

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Abstract: ***A left or right module M over a ring with identity e such that multiplication by e is the transformation $m \rightarrow em$ (respectively $m \rightarrow me$ for right modules), $m \in M$ is the identity automorphism of the group M over a commutative ring R with unit represents "$n-acc$ of $d-colons$" where unitary module's sub module P and the ascending chain R whose elements that hold distinct sequence $(a_n)_n$ such that $P: a_1 \subseteq a_1 a_2 \subseteq$........ stabilizes . Specially , in this paper we have tried to focus the acc on $d-colons$ and establish that $d-colons$ represents the acc on $n-generated$ submodules for every n, simplifying "$Commutative$ $Rings$ with ACC on $n-$Generated Ideals by W. Heinzer and David Lantz" a " generalized Nakayma's Lemma" for these modules. Moreover it has showed that every $R-\text{mod}ule$ R^I for a Noetherian ring R involves $n-acc$ and a sufficient condition for the acc on ideals to the polynomial ring. The goal of this paper is to locate on modules with acc which represents of certain sorts of $annihilators$ and $n-acc$ on $d-colons$.***

Keywords: n-acc, annihilator, d-colons

1. Introduction

Some writers [e.g. (3, 4, 5, 8, 10)] have highlighted unitary modules over commutative rings with unitary which represents the ascending chain condition on generated submodules denoted by N (or $n-acc$) for a positive integer n. In [8], Chin-Pi Lu considered modules satisfying acc on a certain type of $colons$. Moreover, he showed the provisions of modules to be $n-acc$ on $d-colons$" and acc on $d-annihilators$.

In this paper it has also been showed that a vector space V over a field F WHICH SATISFIES $n-acc$ $d-colons$ that is Noetherian when S denoted a multiplicatively closed (m.c) subset of K over a vector space V satisfying $n-acc$ on $d-colons$".

Besides these , few aspects are emphasized on $R-\text{mod}ule$ satisfying acc on $d-annihilators$. On the other hand , few perceptions have been floated that is every module M over R satisfies $n-acc$ on $d-colons$ on $d-annihilators$ ($S^{-1}R$) and $S^{-1}R$ satisfies acc on $d-annihilators$ and $\dim(S^{-1}R)=0$ and $S^{-1}R$ is a perfect ring when every flat $R-\text{mod}ule$ is projective. In addition, every module over a perfect ring and in particular in each quasilocal ring with nilpotent maximal ideal and satisfy acc on annihilator. Then, to extend this study we have found the relation of submodule N and Jacobson radical of R with acc on $d-colons$ with a module M . We have denoted the symbol by the notation $M \subseteq N$.

2. Definition

acc on $d-colons$ is being represented by modules. In this portion, it is mainly focused the type of modules defined by S.Visweswaran in [10] and it will be used the given below glossary:

$Annihilator$: A common instance is when N is the ring of integers Z and M is a ring or a field $(R,+,\times)$: Then the annihilator of D is defined as $A_{nn}(D)=\{n \in Z : \forall d \in D : n.d = 0R\}$ or when $D=R : A_{nn}(R)=\{n \in Z : \forall r \in R : n.r = 0R\}$

Consider R be a ring and a R-module which is denoted by M . If $(a_n)_n$ is a sequence of elements of a ring R, the ascending chain $A_{nn}(a_1) \subseteq A_{nn}(a_1 a_2) \subseteq$...... of submodules of M stabilizes , we say that M satisfies acc on $d-annihilators$ (ascending chain condition on annihilators of descending chains of principal ideals). It should be said that module M satisfies acc on $d-colons$, if for every submodule N of M the module M/N satisfies acc on $d-annihilators$ [this represents the provision (C) in [10]].

3. Some Provisions of Modules to be $n-acc$ on $d-colons$

Consider a module M which lies on a ring R. If module M represents $n-acc$ on $d-colons$, then it should be said that M satisfies $n-acc$ on $d-colons$ and then M corresponds acc [8,Theorem-1]. In note 2.1 (i) , we focus a specimen of a module M over a factor module denoted by M/K for every non-zero submodule K of M such that modules satisfies acc.

Note 2.1 (i):
acc is satisfied by a module denoted by M over a ring R iff so a factor module M/K for every non zero submodule K of M [8, Proposition 6].

(ii) Let P be a number which has only two factors 1 and itself and let a field denoted by F which is expressed as quotient field by the set of integer and prime number's with exponent as integer . Consider an indeterminate X over a field F. For each $1 \le n$, let $X^{\frac{1}{p^n}}$ denotes the p^n-th root of X is an algebraic closure $F(X)$.

Let $R = \bigcup_{n=1}^{\infty} F[X^{\frac{1}{p^n}}]$. Consider $S = R/\{0\}$. Then R represents $n-acc$ $d-colons$ but R does not correspond acc [3, Example 2.13].

Lemma 2.2:
Consider a vector space V over a field F .Then V represents $n-acc$ $d-colons$.

Proof:
Consider a subspace denoted by W over a vector space V and let a constant β which also belongs to F [3,Lemma 2]. It should be noted that $(W:V\beta)=V$ if β means trivial and equal to subspace where assume that $\beta \neq 0$. Let $\{\beta_n\}$ be any sequence of elements of F .When $\beta_F = 0$ for some $f \in N$ then for all $n \ge f$, $(W:V\beta_1.......\beta_n)=(W:V\beta_1.....\beta_k)=V$. If $\beta_i \neq 0$ for all $i \in N$ then $(W:V\beta_1.......\beta_i)=(W:V\beta_1.....\beta_j)=W$ for all $i.j \in N$. This proves that vectors space V represents $n-acc$ $d-colons$.

Proposition 2.3:
Vector space V is not Noetherian when S, a multiplicatively closed subset of K over a vector space V corresponds $n-acc$ $d-colons$.

Proof:
From Lemma 2.2, vectors space V represents $n-acc$ $d-colons$ [3,page 148]and so, V satisfies $n-acc$ $d-colons$ for any multiplicatively closed (m.c) subset S of K. Let S be any m.c subset of field K. Note that $S \subseteq K/\{0\} = U(K)$. Hence, for any subspace W of V and for any $s \in S$, $sW = W$. Since we are assuming that $\dim_K V$ is infinite, there exists a strictly ascending sequence of subspaces $W_1 \subset W_2 \subset W_3 \subset$.......... of V. It is clear that there exist no $k \in N$ and $s \in S$ such that $sW_n \subseteq W_k$ for all $n \ge k$. Therefore , V is not S-Noetherian for any m.c subset S of K.

4. Some aspects on R-module satisfying acc on $d-annihilators$

Consider a multiplicatively closed subset S, P is minimal prime , X is an R -module represents acc on $d-annihilators$ and X_S belongs in non zero submodule of R. We are focusing few observations in the below:

a) acc on $d-annihilators$ is the part of individual submodules of X.
b) acc on $d-annihilators$ is in the factor module $X/A_{nn}(A)$ where for every A is a proper subset of a ring R.
c) Each weakly associated prime of M is an associated prime of X.
d) P i.e. minimal prime is an associated prime of X when $X_P \neq \{0\}$ where P is a minimal prime of R.
e) If $S \subseteq R$ is multiplicatively closed , there is $s \in S$ such that $\ker(X \to X_S) = A_{nn}(S)$ and X_S also corresponds acc on $d-annihilators$.

Definition

Perfect Ring:
A ring R is called a perfect ring if every flat R-module is projective. Perfect rings can be elaborated in sundry ways. Such as, a ring R is perfect if and only if R satisfies Descending Chain Condition on principal ideals([12], p.466, Theorem 3.2) if and only if R-module M satisfies Ascending Chain Condition on submodules generated by n elements for each $n \ge 1$ ([13], p.269,Proposition1.2). Therefore , every quasi-local ring with nilpotent maximal ideal is perfect.

Theorem 3.1:
Consider a m.c. subset S of a ring R. Suppose that for any module M over a ring R and let $N \in M$, N is any submodule , $\exists$ $s \in S$ (s depends on N) such that $Sat_S(N) = (N :_M S)$. Then the following observations are true ([3],Theorem 2.8)

1) Each module M over R satisfies $n-acc$ on $d-colons$.

2) Each module M over R i.e. $S^{-1}R$ corresponds acc on $d-annihilators$.
3) $S^{-1}R$ represents $a\ c$ on $d-annihilators$ and $\dim(S^{-1}R)=0$.
4) $S^{-1}R$ is a perfect ring.

Proof:

For (1) implies (2)

Let a module M over $S^{-1}R$. It is noticeable that M can be regarded as a module over a ring R and hence by (1), M satisfies $n\ acc$ $d-annihilator$ regarded as a module over R. We know [3, Lemma 2.4] that the $S^{-1}R$-module $S^{-1}W=W$ satisfies acc $d-annihilators$. This proves that any $S^{-1}R$ module satisfies acc $d-annihilators$.
(2) implies (3)

As any $S^{-1}R$ -module satisfies acc on $d-annihilators$, it pursues that $S^{-1}R$ satisfies acc on $d-annihilators$ and any $S^{-1}R$ module satisfies acc $d-colon$. Therefore we get (2) implies (1) of [9, proposition 2.4] that $\dim(S^{-1}R)=0$. (3) implies (4) it pursues from (2) implies (3) of [10, Proposition 1.1] that $S^{-1}R$ is a perfect ring. (4) implies (1) it pursues from (3) implies (1) of [10, proposition 1.1] that M over $S^{-1}R$ satisfies $n-acc$ on $d-annihilator$ .Consider a module M over R. Now the $S^{-1}R$ -module $S^{-1}M$ satisfies $n-acc$ on $d-annihilator$. We are presuming that given any submodule N of M, there exists $s\in S$ (s depends on N) such that $Sat_s(N)=(N:_M^S)$. Therefore, we obtain from (2) implies (1) of [3, Theorem 2.7] that M satisfies $n-acc$ on $d-annihilator$. This proves that any module over R satisfies $n-acc$ on $annihilator$.

Lemma 3.2:
acc on $annihilator$ is being satisfied by every module over a perfect ring as well as in particular in each quasi local ring with nilpotent maximal ideal.

Proposition 3.3:
Consider an irreducible submodule N. Then an irreducible N of module M over a ring R which satisfies acc on $annihilator$ is initial.

Proof:
Let $pe\in N$, where $p\in R$ and $e\in M-N$. Then $N \underset{\neq}{\subset} N:p$ so that $N \underset{\neq}{\subset} N:p^k$ for every $k\geq 1$. We have [8, Proposition 1]

"Let N be a submodule of an R module M and $p\in R$. Then the following statements are equivalent
(1) The ascending chain of submodules $\{N:p^k\}_{k\in Z^+}$ terminates.
(2) There exists an $n\in Z^+$ such that $N:p^n=N:p^{n+1}$.
(3) There exists an $n\in Z^+$ such that $N=(N:p^h)\bigcap(N+p^hM)$ for every $h\geq n$."

We get $N=(N:p^n)\bigcap(N+p^hM)$ for a sufficiently large integer h. Since N is irreducible , $N=N+p^hM$ where $N\supseteq p^hM$. Hence N is initial or primary.
The above one contrast with note 2.1(i).

Theorem 3.4:
acc on $d-colons$ included a module M satisfies [Nakayama's Lemma] that is the zero submodule is the only submodule N such that $J(R)N=N$ (where $J(R)$ denotes the Jacobson radical of R)[4,page:474]."

Proof:
Consider an R-module M with acc on $d-colons$ and N a submodule of M such that $J(R)N=N$ [4, page: 474]. It is enough to prove that $N=\{0\}$, and for this M be an R -module such that every N has a initial decomposition . Then the common of the submodules initial to maximal ideals is 0. It tends to prove that submodule is bounded in any primary submodule which radical is a maximal ideal. Consider that there is a maximal ideal P of R and a P -primary submodule N' of M such that $N \underset{\neq}{\subset} N'$.

Then it is not tough to check that $N':N$ is a P-primary ideal in R . Since $PN=N$, we get $(N':N):P=N':N$ and there is [11, p.298] $a_1\in P$ such that $(N':N):a_1=N':a_1N$ is a P-primary ideal that properly bounds $N':N$. Again we have $(N':a_1N):P=N':a_1N$.

This yields an $acc(N':N):a_1\subset(N':N):a_1a_2\subset..........$ in R. But since M/N' has acc on $d-annihilators$ implies that $(R/N':N)$ has acc on $d-annihilators$, which contradicts.

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