Poisson Inverse NHE Distribution with Theory and Applications

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Abstract: ***In the presented work, a new three parameter continuous distribution with three parameter termed Poisson inverse NHE distribution using the Poisson-Generating family of distribution with the purpose of study of lifetime data. Relevant statistical as well mathematical properties pertaining to the distribution including Probability Density Function (PDF), Cumulative Distribution Function (CDF) have been described to build a clearer understanding of the proposed distribution. To estimate the model parameters of the distribution we have used well established Maximum Likelihood estimation (MLE) along with Cramer-Von-Mises estimation (CVME) and least-square estimation (LSE) methods. We have constructed the asymptotic confidence intervals and Fisher information matrix analytically to obtain the variance-covariance matrix for MLEs. These calculations are carried out in the platform of R software. The potentiality of the model that is introduced is judged by using some graphical methods and goodness-of-fit test by considering a real dataset. We have empirically proven that the proposed distribution provided a better fit and more flexible in comparison with some selected lifetime distributions.***

Keywords: Poisson-G family, Inverse NHE, Estimation, MLE

1. Introduction

Last few years, it has been observed that the many life-time models have been generated but the real data sets related to engineering, life sciences, biology, hydrology, , and risk analysis do not present a better fit in the models proposed. So, the generation of new modified distributions appears to be necessary to deal with the problems in these fields. The generalized, extended, and modified distributions are created with insertion of extra parameter or making some transformation to the baseline distribution with the purpose of achieving better fit for the data.

Kus (2007) has presented the exponential Poisson (EP) distribution with two-parameter with exponential distribution compounded to zero truncated Poisson distribution with a decreasing failure rate [7]. The CDF of PE distribution is,

$$W(t;\chi,\delta)=\frac{1}{\left(1-e^{-\delta}\right)}\left[1-e^{\left\{-\delta\left(1-e^{-\chi t}\right)\right\}}\right] \quad ;t>0,(\chi,\delta)>0 \tag{1.1}$$

While Barreto-Souza and Cribari-Neto (2009) have presented generalized EP distribution having the decreasing or increasing or upside-down bathtub shaped failure rate [3]. This is the generalization of the distribution as given by Kus (2007) adding a power parameter to this distribution [7].

Following the same trend, Cancho (2011) has developed a novel family of distribution also centered on the exponential distribution with an increasing failure rate function known as Poisson exponential (PE) distribution [5]. The CDF of PE distribution is

$$F(y;\theta,\lambda)=1-\frac{1-e^{\left\{-\theta\left(1-e^{-\lambda y}\right)\right\}}}{\left(1-e^{-\lambda}\right)} \quad ; y>0,(\lambda,\theta)>0 \tag{1.2}$$

A Poisson-exponential with two-parameter showing increasing failure rate has been defined by (Louzada-Neto et al., 2011) by using the same approach as used by (Cancho, 2011) under the Bayesian approach [5] and [10]. **Alkarni and Oraby** (2012) have given a new lifetime family of distribution with a decreasing failure rate which is obtained by compounding truncated Poisson distribution and a lifetime model [1]. The CDF of the Poisson-Generating family is given by,

$$F_P(x;\lambda,\underline{\delta})=1-\frac{1-e^{\left\{-\lambda G(y,\underline{\delta})\right\}}}{\left(1-e^{-\lambda}\right)} \quad ; \lambda>0 \tag{1.3}$$

Where $\underline{\delta}$ the parameter is space and $G(y,\underline{\delta})$ is the CDF of any distribution. Using the similar approach the Weibull power series class of distributions with Poisson has presented by (Morais & Barreto-Souza, 2011) [12]. Mahmoudi and Sepahdar (2013) have defined a new four-parameter distribution having a variety of shape of failure rate function which is named as the exponentiated Weibull–Poisson (EWP) distribution which has acquired with exponentiated Weibull (EW) compounded with Poisson distributions [14]. Similarly, Lu and Shi (2012) have created the new compounding distribution named the Weibull–Poisson distribution having the shape of decreasing, increasing, upside-down bathtub-shaped, or unimodal failure rate function [11]. Kaviyarasu and Fawaz (2017) have made an extensive study on Weibull–Poisson distribution through a reliability sampling plan [7]. Kyurkchiev et al. (2018) has used the exponentiated exponential-Poisson as the software reliability model [9].

The NHE distribution was given by (Nadarajah & Haghighi, 2011) [15]. Joshi and Kumar (2020) has introduced half-logistic NHE distribution generated from half-logistic-G family using NHE as base distribution [6]. Using this NHE distribution the inverse NHE was introduced by (Tahir et al., 2018) [21] with CDF and PDF as follows

$$G(x;\alpha,\beta)=\exp\left[1-\left(1+\frac{\alpha}{x}/\right)^{\beta}\right];x,\alpha,\beta>0$$

$$g(x;\alpha,\beta)=\alpha\beta x^{-2}(1+\alpha/x)^{\beta}\exp\left[1-(1+\alpha/x)^{\beta}\right];x,\alpha,\beta>0$$

This distribution is suitable to incorporate positive real data sets and the HRF can have decreasing and upside-down bathtub shaped for different values of shape parameter β. Hence we select this inverse NHE distribution as a baseline distribution for this study.The article shows following structure. In Section 2 we present Poisson inverse NHE and its relevant mathematical and statistical properties. To estimate the model parameters, we have comprehensively discussed some commonly used methods including Maximum likelihood estimation (MLE) also CVME and LSE in section 3 and then constructing asymptotic confidence intervals from observed information matrix for ML estimation, we present the estimated values of model parameter in section 4.Besides, we have illustrated the different test criteria to assess the goodness of fit of the proposed model. Some concluding remarks are presented in Section 5.

2. Poisson Inverse NHE distribution (PINHE)

Consider $G(y,\underline{\Theta})$ and $g(y,\underline{\Theta})$ be the baseline CDF and PDF respectively then the CDF and PDF of Poisson-G family (**Alkarni & Oraby, 2012**) [1] may be defined as,

$$F(x;\lambda,\underline{\Theta})=1-\frac{1}{\left(1-e^{-\lambda}\right)}\left[1-\exp\left\{-\lambda\left(1-G(y,\underline{\Theta})\right)\right\}\right] \quad ;x>0,\ \lambda>0 \qquad (1.4)$$

$$f(x;\lambda,\underline{\Theta})=\frac{1}{\left(1-e^{-\lambda}\right)}\lambda g(y,\underline{\Theta})\exp\left\{-\lambda\left(1-G(y,\underline{\Theta})\right)\right\} \quad ;x>0,\ \lambda>0 \qquad (1.5)$$

Where $\underline{\Theta}$ is the parameter space of base distribution. Using (1.4) and (1.5) as a baseline distribution, we can define a new distribution called Poisson inverse NHE (PINHE). The random variable $X \sim PINHE(\alpha,\beta,\lambda)$ if its CDF and PDF respectively is

$$F(x)=\frac{1-\exp[-\lambda\exp\{1-(1+\alpha/x)^{\beta}\}]}{1-\exp(-\lambda)};x>0;(\alpha,\beta,\lambda)>0$$

$$f(x)=\frac{\alpha\beta\lambda}{1-\exp(-\lambda)}x^{-2}[\{1+(\alpha/x)^{\beta-1}]\exp[1-(1+\alpha/x)^{\beta}]\exp[-\lambda\exp\{1-(1+\alpha/x)^{\beta}\};x\geq 0 \qquad (2.4)$$

The Reliability/Survival function of PINHE is

$$R(x)=1-F(x)$$

$$=\frac{\exp(-\lambda)+\exp[-\lambda\exp\{1-(1+\alpha/x)^{\beta}\}]}{1-\exp(-\lambda)};x>0;(\alpha,\beta,\lambda)>0$$

Hazard function is

$$h(x)=\frac{\alpha\beta\lambda x^{-2}[\{1+(\alpha/x)^{\beta-1}]\exp[1-(1+\alpha/x)^{\beta}]\exp[-\lambda\exp\{1-(1+\alpha/x)^{\beta}\}}{\exp(-\lambda)+\exp[-\lambda\exp\{1-(1+\alpha/x)^{\beta}\}]};x\geq 0$$

And its **quantile function** is

$$\exp[-\lambda\exp\{1\text{-}(1+\alpha/x)^{\beta}\}]\text{-p.}\exp(-\lambda)=1\text{-p}\ ;\ 0<p<1$$

Random deviation generation

$$\exp[-\lambda\exp\{1\text{-}(1+\alpha/x)^{\beta}\}]\text{-u.}\exp(-\lambda)=1\text{-u}\ ;\ 0<u<1$$

Skewness and Kurtosis of PNHE distribution

The coefficient of skewness and kurtosis are important measures of dispersion in descriptive statistics. These measures are used mostly in analysis of data for studying the shape of the distribution or data set. The Bowley's coefficient of skewness based on quartiles is,

$$S_k(Bowley)=\frac{Q(3/4)+Q(1/4)-2Q(1/2)}{Q(3/4)-Q(1/4)}$$, and

(Moors, 1988) [13] gave the Coefficient of kurtosis based on octiles which is

$$K_u(Moors)=\frac{Q(0.875)-Q(0.625)+Q(0.375)-Q(0.125)}{Q(3/4)-Q(1/4)}$$

Graphsfor probability density function and hazard rate function of PNHE(α,β,λ) with different values of parameters are presented in Figure 1.

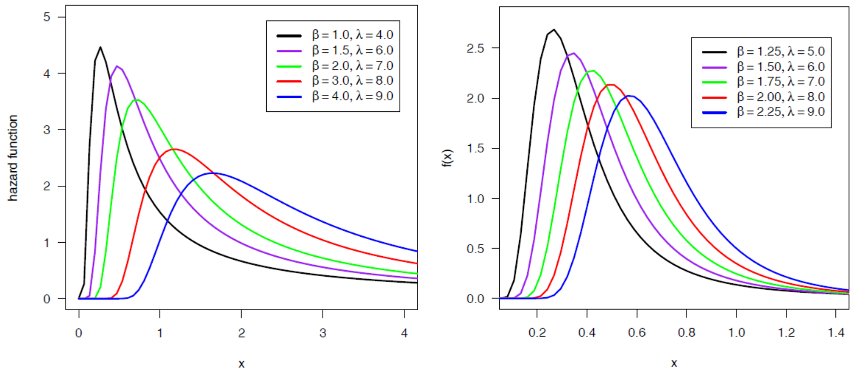


Figure 1: For fixed α and different values of β and λ, Graphs of hazard function (left side) and PDF (right side)

3. Methods of Parameter Estimation

The established estimations methods which we discuss are
a) Cramer-Von-Mises
b) Least square
c) Maximum likelihood

3.1. Maximum Likelihood Estimation (MLE) method

Consider $\underline{x} = (x_1, \ldots, x_n)$ be a random sample of size 'n' from PINHE(α, β, λ) then the log likelihood function l (α, β, λ/ $\underline{x}$) is,

$$l = n\ln\alpha + n\ln\beta + n\ln\lambda + 2\sum_{i=1}^{n}\log x_i + (\beta-1)\sum_{i=1}^{n}\log(1+\alpha/x_i)$$
$$+ n(1+\lambda) - \sum_{i=1}^{n}(1+\alpha/x_i)^{\beta} - \lambda\sum_{i=1}^{n}\exp\{1-(1+\alpha/x_i)^{\beta}\} \quad (3.1.1)$$

With the differentiation of (3.1.1) with respect to unknown parameters α, β and λ, we obtain

$$\frac{\partial l}{\partial\alpha} = \frac{n}{\alpha} + (\beta-1)\sum_{i=1}^{n}\{x_i(1+\alpha/x_i)\}^{-1} - \beta\sum_{i=1}^{n}\frac{1}{x_i}(1+\alpha/x_i)^{\beta-1}$$
$$+ \beta\lambda\sum_{i=1}^{n}\frac{1}{x_i}\exp\{1-(1+\alpha/x_i)^{\beta}\}(1+\alpha/x_i)^{\beta-1}$$

$$\frac{\partial l}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log(1+\alpha/x_i) + (\alpha+1)\sum_{i=1}^{n}(1+\alpha/x_i)^{\beta}\log(1+\alpha/x_i)$$
$$+ \lambda\sum_{i=1}^{n}\exp\{1-(1+\alpha/x_i)^{\beta}(1+\alpha/x_i)^{\beta}\log(1+\alpha/x_i)$$

$$\frac{\partial l}{\partial\lambda} = \frac{n}{\lambda} + n + \sum_{i=1}^{n}\exp\{1-(1+\alpha/x_i)^{\beta}$$

After equating these non-linear equations to zero and solving for the unknown parameters (α, β, λ) we will obtain the ML estimators of the PG distribution. Manually, it is difficult to solve hence by add of appropriate computer software one can solve these equations. Consider the parameter vector by $\underline{\Delta} = (\alpha, \beta, \lambda)$ and the corresponding MLE of $\underline{\Delta}$ as $\hat{\underline{\Delta}} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$, then the asymptotic normality results in, $\left(\hat{\underline{\Delta}} - \underline{\Delta}\right) \rightarrow N_3\left[0, \left(D\left(\underline{\Delta}\right)\right)^{-1}\right]$ here $D\left(\underline{\Delta}\right)$ is the Fisher's information matrix is

$$D\left(\underline{\Delta}\right) = -\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\beta}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\lambda}\right) \\ E\left(\frac{\partial^2 l}{\partial\beta\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\beta^2}\right) & E\left(\frac{\partial^2 l}{\partial\beta\partial\lambda}\right) \\ E\left(\frac{\partial^2 l}{\partial\alpha\partial\lambda}\right) & E\left(\frac{\partial^2 l}{\partial\beta\partial\lambda}\right) & E\left(\frac{\partial^2 l}{\partial\lambda^2}\right) \end{pmatrix}$$

In practice, we don't know $\underline{\Delta}$ hence it is useless that the MLE has an asymptotic variance $\left(D\left(\underline{\Delta}\right)\right)^{-1}$. Hence we approximate the asymptotic variance by plugging in the estimated value of the parameters. The observed fisher information matrix $O(\hat{\underline{\Delta}})$ is used as an estimate of the information matrix $D\left(\underline{\Delta}\right)$ given by

$$O(\hat{\underline{\Delta}}) = -\begin{pmatrix} \frac{\partial^2 l}{\partial\alpha^2} & \frac{\partial^2 l}{\partial\alpha\partial\beta} & \frac{\partial^2 l}{\partial\alpha\partial\lambda} \\ \frac{\partial^2 l}{\partial\beta\partial\alpha} & \frac{\partial^2 l}{\partial\beta^2} & \frac{\partial^2 l}{\partial\beta\partial\lambda} \\ \frac{\partial^2 l}{\partial\alpha\partial\lambda} & \frac{\partial^2 l}{\partial\beta\partial\lambda} & \frac{\partial^2 l}{\partial\lambda^2} \end{pmatrix}_{|(\hat{\alpha},\hat{\beta},\hat{\theta})} = -H\left(\underline{\Delta}\right)_{|(\underline{\Delta}=\hat{\underline{\Delta}})}$$

where H is the Hessian matrix.

The Newton-Raphson algorithm to maximize the likelihood produces the observed information matrix. Therefore, the variance-covariance matrix is given by,

$$\left[-H\left(\underline{\Delta}\right)_{|(\underline{\Delta}=\hat{\underline{\Delta}})}\right]^{-1} = \begin{pmatrix} \operatorname{var}(\hat{\alpha}) & \operatorname{cov}(\hat{\alpha},\hat{\beta}) & \operatorname{cov}(\hat{\alpha},\hat{\lambda}) \\ \operatorname{cov}(\hat{\alpha},\hat{\beta}) & \operatorname{var}(\hat{\beta}) & \operatorname{cov}(\hat{\theta},\hat{\lambda}) \\ \operatorname{cov}(\hat{\alpha},\hat{\lambda}) & \operatorname{cov}(\hat{\lambda},\hat{\beta}) & \operatorname{var}(\hat{\lambda}) \end{pmatrix} \quad (3.1.3)$$

Hence from the asymptotic normality of MLEs, approximate 100(1-α) % confidence intervals for α, β and θ can be constructed as,

$\hat{\alpha} \pm Z_{\alpha/2}\sqrt{\operatorname{var}(\hat{\alpha})}$, $\hat{\beta} \pm Z_{\alpha/2}\sqrt{\operatorname{var}(\hat{\beta})}$ and $\hat{\lambda} \pm Z_{\alpha/2}\sqrt{\operatorname{var}(\hat{\lambda})}$

where $Z_{\alpha/2}$ is the upper percentile of standard normal variate.

3.2. Least-Square Estimation (LSE) Method

Here weighted least square estimators and ordinary least square estimators as given by Swain et. al. (1988) for estimating Beta distribution's parameters [20]. With minimization of equation (3.2.1) with respect to unknown parameters α, β and λ, we can get least square estimators of parameters taken of the PHNE distribution which is given by,

$$B(X;\alpha,\beta,\lambda)=\sum_{i=1}^{n}\left[F(X_i)-\frac{i}{n+1}\right]^2 \qquad (3.2.1)$$

From a distribution function F(.) let $F(X_i)$ represent ordered random variables ($X_{(1)} < X_{(2)} < \ldots < X_{(n)}$)'s distribution function and $\{X_1, X_2, \ldots, X_n\}$ is a random sample (where n=size of sample). Minimization of equation (3.2.2) with respect to the unknown parameter α, β and λ, we can get respective least-square estimators $\hat{\alpha}, \hat{\beta}, and\, \hat{\lambda}$.

$$B(X;\alpha,\beta,\lambda)=\sum_{i=1}^{n}\left[1-\frac{1-\exp\left(-\lambda\exp\{1-(1+\alpha x_i)^{\beta}\}\right)}{\left(1-e^{-\lambda}\right)}-\frac{i}{n+1}\right]^2 ; x\ge 0, (\alpha,\beta,\lambda)>0. \qquad (3.2.2)$$

With respect to α, β and λ, differentiation of (3.2.2) we get,

$$\frac{\partial B}{\partial\alpha}=\frac{\beta\lambda 2}{\left(1-e^{-\lambda}\right)}\sum_{i=1}^{n}x_i\left[1-\frac{1-\exp\left(-\lambda K(x_i)\right)}{\left(1-e^{-\lambda}\right)}-\frac{i}{n+1}\right]K(x_i)\exp\left(-\lambda(x_i)K\right)(1+x_i\alpha)^{\beta-1}$$

$$\frac{\partial B}{\partial\beta}=\frac{2\lambda}{\left(1-e^{-\lambda}\right)}\sum_{i=1}^{n}\left[1-\frac{1-\exp\left(-\lambda K(x_i)\right)}{\left(1-e^{-\lambda}\right)}-\frac{i}{n+1}\right]K(x_i)\exp\left(-\lambda K(x_i)\right)(1+\alpha x_i)^{\beta}\ln(1+\alpha x_i)$$

$$\frac{\partial B}{\partial\lambda}=2\sum_{i=1}^{n}\left[1-\frac{1-\exp\left(-\lambda K(x_i)\right)}{\left(1-e^{-\lambda}\right)}-\frac{i}{n+1}\right]\left[\frac{K(x_i)\exp\left(-\lambda K(x_i)\right)}{1-e^{-\lambda}}-\frac{e^{-\lambda}\left\{1-\exp\left(-\lambda K(x_i)\right)\right\}}{\left(1-e^{-\lambda}\right)^2}\right]$$ Where

$$K(x_i)=\exp\{1-(1+\alpha x_i)^{\beta}\}$$

With minimization of following equation with respect to α, β and λ, we can get weighted LSE

$$J(X;\alpha,\beta,\lambda)=\sum_{i=1}^{n}w_i\left[F(X_{(i)})-\frac{i}{n+1}\right]^2$$

The weights w_i are $w_i=\frac{1}{Var(X_{(i)})}=\frac{(2+n)(n+1)^2}{i(1+n-i)}$

With minimization of equation (3.2.3) with respect to α, β and λ we can obtain weighted least square estimators of the following parameters

$$J(X;\alpha,\beta,\lambda)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[1-\frac{1-\exp\left(-\lambda\exp\{1-(1+\alpha x_i)^{\beta}\}\right)}{\left(1-e^{-\lambda}\right)}-\frac{i}{n+1}\right]^2$$

3.3. Method of Cramer-Von-Mises estimation (CVME)

With minimization of the following equation, the Cramer-Von-Mises estimators of α, β and λ are obtained.

$$A(X)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F(x_{i:n}\,|\,\alpha,\beta,\lambda)-\frac{2i-1}{2n}\right]^2$$

$$=\frac{1}{12n}+\sum_{i=1}^{n}\left[1-\frac{1-\exp\left(-\lambda\exp\{1-(1+\alpha x_i)^{\beta}\}\right)}{\left(1-e^{-\lambda}\right)}-\frac{2i-1}{2n}\right]^2 \qquad (3.3.1)$$

With differentiation of (3.3.1) with respect to α, β and λ we obtain

$$\frac{\partial A}{\partial\alpha}=\frac{2\beta\lambda}{\left(1-e^{-\lambda}\right)}\sum_{i=1}^{n}x_i\left[1-\frac{1-\exp\left(-\lambda K(x_i)\right)}{\left(1-e^{-\lambda}\right)}-\frac{2i-1}{2n}\right]K(x_i)\exp\left(-\lambda K(x_i)\right)(1+\alpha x_i)^{\beta-1}$$

$$\frac{\partial A}{\partial \beta}=\frac{2\lambda}{\left(1-e^{-\lambda}\right)}\sum_{i=1}^{n}\left[1-\frac{1-\exp\left(-\lambda K\left(x_i\right)\right)}{\left(1-e^{-\lambda}\right)}-\frac{2i-1}{2n}\right]K\left(x_i\right)\exp\left(-\lambda K\left(x_i\right)\right)(1+\alpha x_i)^{\beta}\ln(1+\alpha x_i)$$

$$\frac{\partial A}{\partial \lambda}=2\sum_{i=1}^{n}\left[1-\frac{1-\exp\left(-\lambda K\left(x_i\right)\right)}{\left(1-e^{-\lambda}\right)}-\frac{2i-1}{2n}\right]\left[\frac{K\left(x_i\right)\exp\left(-\lambda K\left(x_i\right)\right)}{1-e^{-\lambda}}-\frac{e^{-\lambda}\left\{1-\exp\left(-\lambda K\left(x_i\right)\right)\right\}}{\left(1-e^{-\lambda}\right)^2}\right]$$

Where $K\left(x_i\right)=\exp\{1-(1+\alpha x_i)^{\beta}\}$

By solving $\frac{\partial A}{\partial \alpha}=0, \frac{\partial A}{\partial \beta}=0 \ and \ \frac{\partial A}{\partial \lambda}=0$ simultaneously we obtain CVM estimators.

4. Application with a real dataset

For the analysis of applicability and adequacy of the PINHE distribution we are considering an actual dataset used by former researchers. The data set is originally considered by (Bader & Priest, 1982) [2] represent the strength measured in GPA for single carbon fibers of 10mm in gauge lengths with sample size 63 and they are as follows:

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020

The maximum likelihood estimates are calculated directly by using optim() function in R software (R Core Team, 2020) and (Schmuller, 2017) with maximization of likelihood function (3.1) [17]- [18]. In Table 1 we have demonstrated the MLE's and standard errors (SE) and 95% confidence interval for α, β and λ.

Table 1: MLE's and standard errors (SE) for α, β and λ

Parameter	MLE	SE
alpha	1.0174	0.9706
beta	5.1414	4.1413
lambda	23.3476	4.2268

The plots of profile log-likelihood function for the parameters α, β and λ have been displayed in Figure 2 and noticed that the ML estimates can be uniquely determined.

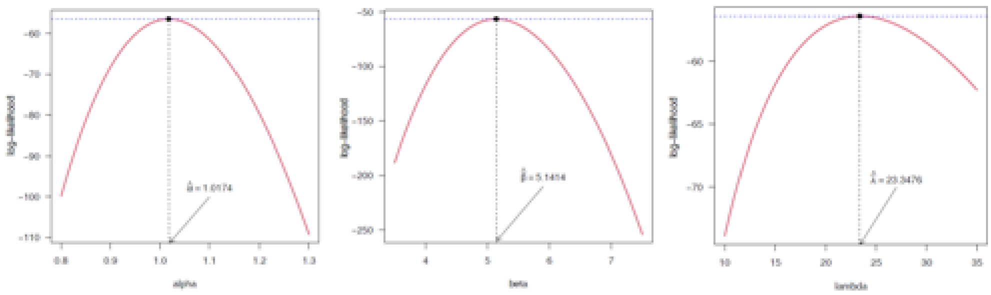


Figure 2: Graph of Profile log-likelihood function for the parameters α, β and λ.

In Figure 3 we have plotted the Q-Q plot and P-P plot and it is seen that the proposed distribution fits the data very well.

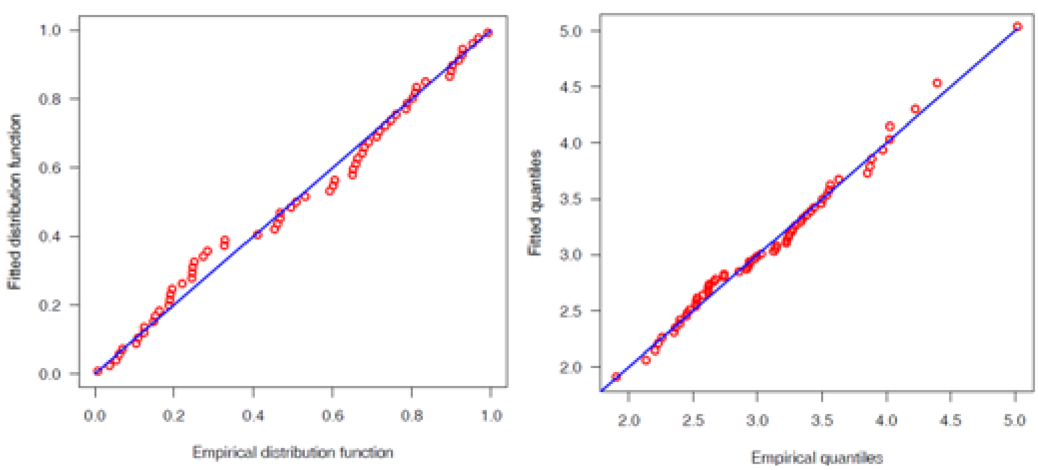


Figure 3: The P-P plot (left side) and Q-Q plot (right side) of the PINHE distribution

In Table 2 we have presented the estimated value of the parameters of PINHE distribution using MLE, LSE and CVE method and their corresponding negative log-likelihood, and AIC criterion.

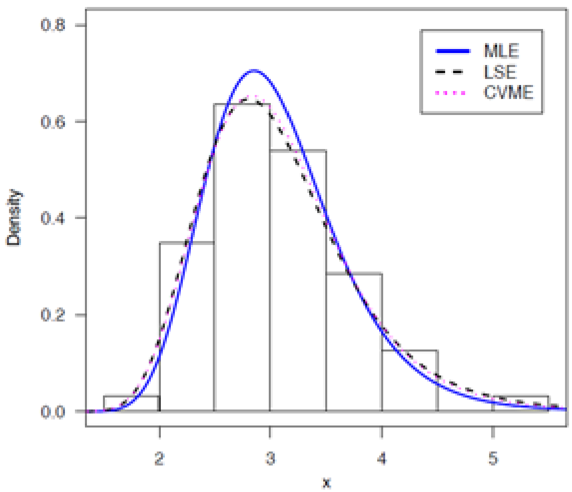
Table 2: Estimated parameters, log-likelihood, and AIC

Method of Estimation	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	-LL	AIC
MLE	1.0174	5.1414	23.3476	-56.4111	118.8222
LSE	0.9253	5.3264	17.3502	-57.1056	120.2113
CVE	1.3812	3.9023	20.9784	-56.8154	119.6307

In Table 3 we have presented The KS, W and A^2 statistics with their corresponding p-value of MLE, LSE and CVE estimates.

Table 3: The KS, W and A^2 statistics with a p-value

Method of Estimation	*KS(p-value)*	*W(p-value)*	*A^2(p-value)*
MLE	0.0821(0.7894)	0.0648(0.7859)	0.3447(0.9009)
LSE	0.0658(0.9479)	0.0506(0.8743)	0.3640(0.8831)
CVE	0.0670(0.9399)	0.0482(0.8886)	0.3214(0.9210)



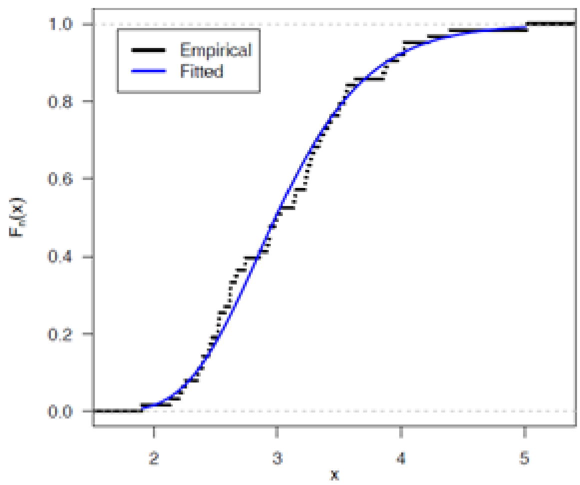


Figure 4: The Histogram and the density function of fitted distributions of estimation methods MLE, LSE and CVM (left panel) and KS plot of PNHE distribution (right panel)

In this section, we have presented the applicability of Poisson inverse NHE distribution using a real dataset used by earlier researchers. To compare the potentiality of the proposed model, we have considered the following distributions.

a) Exponentiated Exponential Poisson (EEP):
The probability density function of EEP (Ristić & Nadarajah, 2014) [16] can be expressed as

$$f(x)=\frac{\alpha\beta\lambda}{\left(1-e^{-\lambda}\right)}e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha-1}\exp\left\{-\lambda\left(1-e^{-\beta x}\right)^{\alpha}\right\} \quad ; x>0,\ \alpha>0,\lambda>0$$

b) Weibull extension (WE) distribution
The PDF of Weibull extension (WE) distribution (Tang et al., 2003) [22] with three parameters (α,β,λ) is

$$f_{WE}(x;\alpha,\beta,\lambda)=\lambda\beta\left(\frac{x}{\alpha}\right)^{\beta-1}\exp\left(\frac{x}{\alpha}\right)^{\beta}\exp\left\{-\lambda\alpha\left(\exp\left(\frac{x}{\alpha}\right)^{\beta}-1\right)\right\} \quad ; x>0$$

$\alpha>0,\beta>0$ and $\lambda>0$

c) Lindley-Exponential (LE)s distribution:
The PDF of LE (Bhati, 2015) [4]is

$$f_{LE}(x)=\lambda\left(\frac{\theta^2}{1+\theta}\right)e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\theta-1}\left\{1-\ln\left(1-e^{-\lambda x}\right)\right\};\ \lambda,\theta>0,\ x>0$$

d) Poisson–exponential distribution (PE)
The PDF of PE (Louzada-Neto et al., 2011) [10] is

$$f(x)=\frac{\beta\lambda}{\left(1-e^{-\lambda}\right)}e^{-\beta x}\exp\left(-\lambda e^{-\beta x}\right);\quad \beta>0,\lambda>0,x>0$$

e) Exponential power (EP) distribution

The PDF of EP distribution (Smith & Bain, 1975) [19] is

$$f_{EP}(x)=\alpha\lambda^{\alpha}x^{\alpha-1}e^{(\lambda x)^{\alpha}}\exp\left\{1-e^{(\lambda x)^{\alpha}}\right\} \quad ;(\alpha,\lambda)>0,\quad x\geq 0\cdot$$

where α and λ are the shape and scale parameters respectively.

For the assessment of the potentiality of the proposed model, we have calculated the Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC), and Hannan-Quinn information criterion (HQIC) which are presented in Table 4.

Table 4: Log-likelihood (LL), AIC, BIC, CAIC and HQIC

Model	-LL	AIC	BIC	CAIC	HQIC
PINHE	56.4111	118.8222	125.2516	119.2290	121.3509
EEP	57.0630	120.1261	126.5555	120.5328	122.6548
PE	57.2052	118.4105	122.6967	118.6105	120.0963
LE	57.9964	119.9929	124.2792	120.1929	121.6787
WE	61.9865	129.9731	136.4025	130.3798	132.5018
EP	69.3299	142.6598	146.9461	142.8533	144.3456

Figure 5 shows the comparision between different distributions.

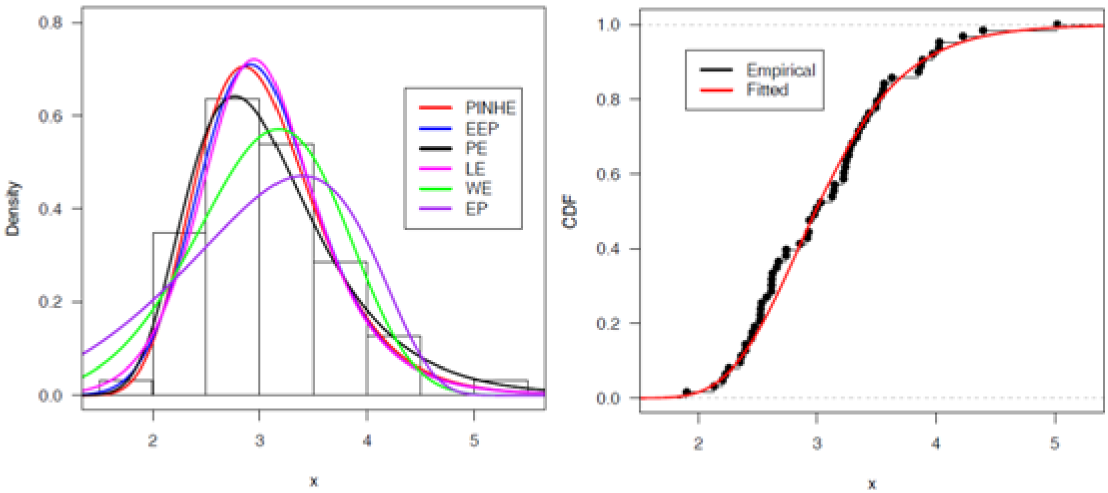


Figure 4. The Histogram and the density function of fitted distributions (left panel) and Empirical distribution function with estimated distribution function (right panel)

In Table 5, to compare the goodness-of-fit of the PNHE distribution among various distribution we have shown the value of KS, W and A^2 statistics. It is observed that the distribution of PINHE has higher *p*-value and test statistic showing minimum value thus we can derive the conclusion that the PINHE distribution shows more consistency with results with higher reliability also showing better fit for the data from others taken for comparison.

Table 5: The goodness-of-fit statistics and their corresponding p-value

Model	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
PINHE	0.0682(0.7406)	0.0740(0.7285)	0.4184(0.8300)
EEP	0.0607(0.8542)	0.0635(0.7932)	0.4212(0.8268)
PE	0.0838(0.4836)	0.1225(0.4860)	0.7042(0.5549)
LE	0.0993(0.2771)	0.1861(0.2963)	1.3081(0.2297)
WE	0.1078(0.1959)	0.2293(0.2174)	1.2250(0.2581)
EP	0.0962(0.3129)	0.2280(0.2193)	1.7537(0.1261)

5. Concluding Remarks

In this work, we put forward Poisson inverse NHE distribution having three parameter. A study of relevant statistical along with mathematical properties of the proposed distribution including the derivation of explicit expressions for its reliability function, survival function, hazard function, the quantile function which is useful for calculating partition values and skewness and kurtosis, skewness and kurtosis, and simulation of random numbers from the proposed distribution. Using a real data set we have employed established estimation methods including MLE, LSE and CVME. The graph of the PDF of the proposed distribution has shown that its shape is the skewed model and flexible for modeling real-life data. Also, the graph of the hazard function is monotonically decreasing or increasing according to the value of the model parameters. The performance of the introduced distribution has been evaluated by considering a real-life dataset and the results showed that the proposed distribution is much flexible as in contrast to some selected distributions.

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