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Method for Solving Fractional Bernoulli's Differential Equation

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Abstract: ***In this paper, we use a new multiplication of fractional functions, the chain rule for fractional derivatives, and the result of first order linear fractional differential equation, regarding the Jumarie type of modified Riemann-Liouville (R-L) fractional derivative, to obtain the general solution of fractional Bernoulli's differential equation. In addition, an example is proposed to illustrate our result.***

Keywords: New multiplication, Chain rule, Modified R-L fractional derivative, Fractional Bernoulli's differential equation

1. Introduction

Fractional differential equations (FDEs) are generalization of the classical differential equations of integer-order. Fractional derivatives are useful in describing the memory and hereditary-properties materials and processes. FDEs are widely used as models to express much important phenomena in natural science such as chemistry, biology, mathematics, communication, physics, engineering, diffusion process, and porous media. The interested reader is also referred to numerous applications of fractional calculus in different areas of science [1-4]. The study of analytical and numerical solutions of FDEs became an important issue and matter of interest for researchers in the last decades. Many definitions of fractional integration and differentiation operators have been utilized. The Riemann-Liouville (R-L) definition could be considered as a famous one [4], which has been applied successfully in various fields of science and engineering. During the 1980s fractional calculus attracted researchers and explicit applications began to appear in several fields. We mention the doctoral thesis, published as an article [5], which seems to be the first one in the subject and the classical book by Miller and Ross [6], where one can see a timeline from 1645 to 1974. After the decade of 1990, completely consolidated, there appeared some specific journals and several textbooks were published. These facts lent a great visibility to the subject and it gained prestige around the world. An interesting timeline from 1645 to 2010 is presented in references [7-9]. We recall here that an important advantage of using fractional differential equations in applications is their non-local property. The use of fractional calculus is more realistic and this is one reason why fractional calculus has become more popular. However, it led to the result that the fractional derivative of a constant function is not zero. In the last few years, Jumarie type of modified R-L fractional derivative to new formulas that are suitable for continuous and non-differentiable functions, with a zero value derivative for a constant function [10]. This modified R-L fractional definition has been used effectively in various problems. Other papers on fractional differential equations can refer to [15-18].

In this paper, we solve the fractional Bernoulli's differential equation, regarding the Jumarie type of modified R-L fractional derivatives. We define a new multiplication of fractional functions, and use the chain rule for fractional derivatives and the result of first order linear fractional differential equation [11] to obtain the general solution of fractional Bernoulli's differential equation. In fact, this result is the generalization of Bernoulli's ordinary differential equation. Moreover, an example is given to demonstrate the advantage of our result.

2. Preliminaries

Firstly, the fractional calculus used in this paper is introduced below.

Definition 2.1: If α is a real number and m is a positive integer.The modified R-L fractional derivatives of Jumarie type ([12]) is defined by

$$
{}_aD_x^{\alpha}[f(x)] = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_a^x (x-\tau)^{-\alpha-1} f(\tau)d\tau, & \text{if } \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-\tau)^{-\alpha}[f(\tau)-f(a)]\,d\tau & \text{if } 0 \le \alpha < 1 \\ \frac{d^m}{dx^m}\left({}_aD_x^{\alpha-m}\right)[f(x)], & \text{if } m \le \alpha < m+1 \end{cases} \tag{1}
$$

where $\Gamma(y) = \int_0^{\infty} t^{y-1}e^{-t}dt$ is the gamma function defined on $y > 0$. If $\left({}_aD_x^{\alpha}\right)^n[f(x)] = \left({}_aD_x^{\alpha}\right)\left({}_aD_x^{\alpha}\right)\cdots\left({}_aD_x^{\alpha}\right)[f(x)]$ exists, then $f(x)$ is called n-th order α-fractional differentiable function, and $\left({}_aD_x^{\alpha}\right)^n[f(x)]$ is the n-th order α-fractionalderivative of $f(x)$. We note that $\left({}_aD_x^{\alpha}\right)^n \neq {}_aD_x^{n\alpha}$ in general. On the other hand, we define the fractional integral of (x), ${}_aI_x^{\alpha}[f(x)] = {}_aD_x^{-\alpha}[f(x)]$, where $\alpha > 0$, and $f(x)$ is called α-integral function. We have the following property [13].

Proposition 2.2: *Suppose that* α, β, c *are real constants and* $0 < \alpha \le 1$, *then*

$$
{}_0D_x^{\alpha}\left[x^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \ if \beta \ge \alpha \tag{2}
$$

$$ {}_0D_x^\alpha[c] = 0, \tag{3}$$

and

$$\left({}_0I_x^\alpha\right)[x^\beta] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}x^{\beta+\alpha}, \quad if\beta > -1. \tag{4}$$

In the following, we define a new multiplication of fractional functions.

Definition 2.3 ([14])**:** Let λ, μ, z be complex numbers, $0 < \alpha \le 1$, j, l, k be non-negative integers, and a_k, b_k be real numbers, $p_k(z) = \frac{1}{\Gamma(k\alpha+1)}z^k$ for all k. The $\otimes$ multiplication is defined by

$$p_j(\lambda x^\alpha) \otimes p_l(\mu y^\alpha)$$

$$= \frac{1}{\Gamma(j\alpha+1)}(\lambda x^\alpha)^j \otimes \frac{1}{\Gamma(l\alpha+1)}(\mu y^\alpha)^l$$

$$= \frac{1}{\Gamma((j+l)\alpha+1)}\binom{j+l}{j}(\lambda x^\alpha)^j(\mu y^\alpha)^l, \tag{5}$$

where $\binom{j+l}{j} = \frac{(j+l)!}{j!\,l!}$.

If $f_\alpha(\lambda x^\alpha)$ and $g_\alpha(\mu y^\alpha)$ are two fractional functions,

$$f_\alpha(\lambda x^\alpha) = \sum_{k=0}^\infty a_k\, p_k(\lambda x^\alpha) = \sum_{k=0}^\infty \frac{a_k}{\Gamma(k\alpha+1)}(\lambda x^\alpha)^k, \tag{6}$$

$$g_\alpha(\mu y^\alpha) = \sum_{k=0}^\infty b_k\, p_k(\mu y^\alpha) = \sum_{k=0}^\infty \frac{b_k}{\Gamma(k\alpha+1)}(\mu y^\alpha)^k, \tag{7}$$

then we define

$$f_\alpha(\lambda x^\alpha) \otimes g_\alpha(\mu y^\alpha)$$

$$= \sum_{k=0}^\infty a_k\, p_k(\lambda x^\alpha) \otimes \sum_{k=0}^\infty b_k\, p_k(\mu y^\alpha)$$

$$= \sum_{k=0}^\infty \left(\sum_{m=0}^k a_{k-m} b_m\, p_{k-m}(\lambda x^\alpha) \otimes p_m(\mu y^\alpha)\right). \tag{8}$$

Proposition 2.4: $f_\alpha(\lambda x^\alpha) \otimes g_\alpha(\mu y^\alpha)$

$$= \sum_{k=0}^\infty \frac{1}{\Gamma(k\alpha+1)} \sum_{m=0}^k \binom{k}{m} a_{k-m} b_m\, (\lambda x^\alpha)^{k-m}(\mu y^\alpha)^m. \tag{9}$$

Definition 2.5: Let $\left(f_\alpha(\lambda x^\alpha)\right)^{\otimes n} = f_\alpha(\lambda x^\alpha) \otimes \cdots \otimes f_\alpha(\lambda x^\alpha)$ be the n times product of the fractional function $f_\alpha(\lambda x^\alpha)$. If $f_\alpha(\lambda x^\alpha) \otimes g_\alpha(\lambda x^\alpha) = 1$, then $g_\alpha(\lambda x^\alpha)$ is called the $\otimes$ reciprocal of $f_\alpha(\lambda x^\alpha)$, and is denoted by $\left(f_\alpha(\lambda x^\alpha)\right)^{\otimes -1}$.

Definition 2.6: If $f(z) = \sum_{k=0}^\infty a_k z^k$, $g_\alpha(\mu x^\alpha) = \sum_{k=0}^\infty b_k\, p_k(\mu x^\alpha)$, then

$$f_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right) = \sum_{k=0}^\infty a_k \left(g_\alpha(\mu x^\alpha)\right)^{\otimes k}. \tag{10}$$

Theorem 2.7 (chain rule for fractional derivatives) ([14])**:** *Let* $f(z) = \sum_{k=0}^\infty a_k z^k$, $g_\alpha(\mu x^\alpha) = \sum_{k=0}^\infty b_k\, p_k(\mu x^\alpha)$. *If* $f_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right) = \sum_{k=0}^\infty a_k \left(g_\alpha(\mu x^\alpha)\right)^{\otimes k}$, $f'_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right) = \sum_{k=1}^\infty a_k\, k\left(g_\alpha(\mu x^\alpha)\right)^{\otimes(k-1)}$, *then*

$$\left({}_0D_x^\alpha\right)\left[f_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right)\right]$$

$$= f'_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right) \otimes \left({}_0D_x^\alpha\right)[g_\alpha(\mu x^\alpha)]. \tag{11}$$

The following is the general solution of first order linear FDE.

Theorem 2.8 ([11])**:** *Let* $0 < \alpha \le 1$, C *be a constant and* $P(x^\alpha), Q(x^\alpha)$ *be* α*-integral functions. Then the first order linear* α*-fractional differential equation*

$$ {}_aD_x^\alpha[y(x^\alpha)] + P(x^\alpha) \otimes y(x^\alpha) = Q(x^\alpha) \tag{12}$$

has the general solution

$$y(x^\alpha) = E_\alpha\left(-{}_aI_x^\alpha[P(x^\alpha)]\right) \otimes \left({}_aI_x^\alpha\left[Q(x^\alpha) \otimes E_\alpha\left({}_aI_x^\alpha[P(x^\alpha)]\right)\right] + C\right). \tag{13}$$

3. Method and Result

Definition 3.1: Suppose that $0 < \alpha \le 1$, $r \ne 0,1$, and $P(x^\alpha), G(x^\alpha)$ are α-fractional functions. The α-fractional Bernoulli's differential equation is a first-order fractional differential equation

$$ {}_aD_x^\alpha[y(x^\alpha)] + P(x^\alpha) \otimes y(x^\alpha) = G(x^\alpha) \otimes y^{\otimes r}. \tag{14}$$

The following is the method for solving α-fractional Bernoulli's differential equation.

Theorem 3.2: *The* α *-fractional Bernoulli's differential equation Eq. (14) has the general solution*

$$y^{\otimes(1-r)}(x^\alpha) = E_\alpha\left(-{}_aI_x^\alpha[(1-r)P(x^\alpha)]\right)$$

$$\otimes \left({}_aI_x^\alpha\left[(1-r)G(x^\alpha) \otimes E_\alpha\left({}_aI_x^\alpha[(1-r)P(x^\alpha)]\right)\right] + C\right). \tag{15}$$

Proof Since

$$y^{\otimes(-r)}(x^\alpha) \otimes {}_aD_x^\alpha[y(x^\alpha)] + P(x^\alpha) \otimes y^{\otimes(1-r)}(x^\alpha) = G(x^\alpha). \tag{16}$$

Let $z(x^\alpha) = y^{\otimes(1-r)}(x^\alpha)$, by chain rule for fractional derivatives, we have

$$ {}_aD_x^\alpha[z(x^\alpha)] = (1-r)\cdot y^{\otimes(-r)}(x^\alpha) \otimes {}_aD_x^\alpha[y(x^\alpha)]. \tag{17}$$

Therefore,

$$ {}_aD_x^\alpha[z(x^\alpha)] + (1-r)\cdot P(x^\alpha) \otimes z(x^\alpha) = (1-r)\cdot G(x^\alpha). \tag{18}$$

Eq. (18) is a first order α-fractional differential equation about z. By Theorem 2.8, we get the desired result.

Q.e.d.

Example 3.3: Consider the following $1/4$ -fractional Bernoulli's differential equation

$$ {}_0D_x^{1/4}[y(x^{1/4})] - \frac{1}{2}x \otimes y(x^{1/4}) = 2x^{1/4} \otimes y^{\otimes(-1)}(x^{1/4}). \tag{19}$$

By Eq. (15), the general solution of Eq. (19) is

$$y^{\otimes 2}\left(x^{1/4}\right)$$
$$= E_{1/4}\left({}_0I_x^{1/4}[x]\right) \otimes \left({}_0I_x^{1/4}\left[4x^{1/4} \otimes E_{1/4}\left({}_0I_x^{1/4}[-x]\right)\right] + C\right)$$
$$= E_{1/4}\left(\frac{1}{\Gamma\left(\frac{9}{4}\right)}x^{5/4}\right) \otimes \left({}_0I_x^{1/4}\left[4x^{1/4} \otimes E_{1/4}\left(-\frac{1}{\Gamma\left(\frac{9}{4}\right)}x^{5/4}\right)\right] + C\right). \tag{20}$$

4. Conclusion

The general solution of fractional Bernoulli's differential equation can be obtained by using the chain rule for fractional derivatives and the result of first order linear fractional differential equation. In fact, the applications of these two methods are extensive, and can be used to easily solve many fractional differential equations; On the other hand, our result is the generalization of classical Bernoulli's differential equation. Moreover, the new multiplication we defined is a natural operation in fractional calculus. In the future, we will use the Jumarie type of modified R-L fractional derivatives and the new multiplication to extend the research topics to the problems of engineering mathematics and fractional calculus.

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