Properties and Applications of Gamma Function

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Abstract: ***Lots of data and information from various resources like books, internet and research papers, were collected to know about the properties, logarithmic derivatives, and applications of gamma function.It is used in various integration techniques, beta function, andto calculate power series.***

Keywords: Gamma Function, Factorials, Gamma Distributions, Logarithmic Derivatives

1. Introduction

It was found necessary to generalize the factorial function to allow for non-whole numbers. e.g. (2.33)! The common method for determining the value of n! , found by {1 x 2 x 3… x (n-1) x n} was found inefficient for large n. The solution was found by Leohard Euler in 1729. He expressed n! As both an infinite sum and an integral, both of which were outlined in his famous paper "*De progressionibustranscendentibusseuquarum termini generals algebraic darinequent*" ***[1]***

The main objectives of the research are –
- To be aware of history and origin of gamma function.
- To study the various expressions of gamma function.
- To study the properties of gamma function in detail.
- To have a deep knowledge about digamma functions or logarithmic derivatives of gamma function.
- To find out the applications of gamma function.
- To focus and study the main applications of gamma function.

2. Methodology

Research methodology is a way to solve research problems systematically.

Our basic steps throughout the whole research process were:
- To collect data and information from various resources like books, internet and research papers to know about the properties of gamma function.
- To find the logarithmic derivatives of gamma function.
- To know about all the applications of gamma function but to focus on any four.
- To arrange all the data to conclude about the said topic.

3. Results and Discussions

The gamma function has following definitions:

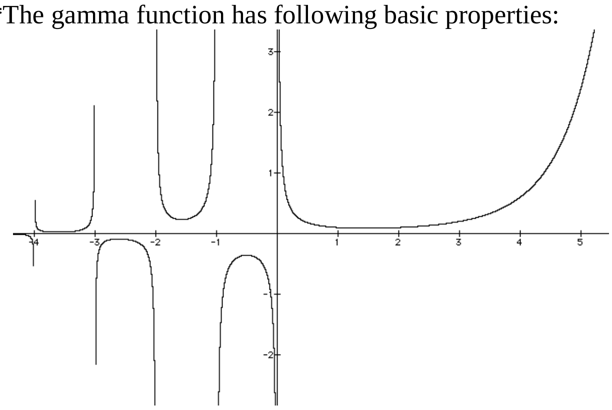
$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\; n^x}{x\,(x+1)(x+2)\cdots(x+n)}$$

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt$$

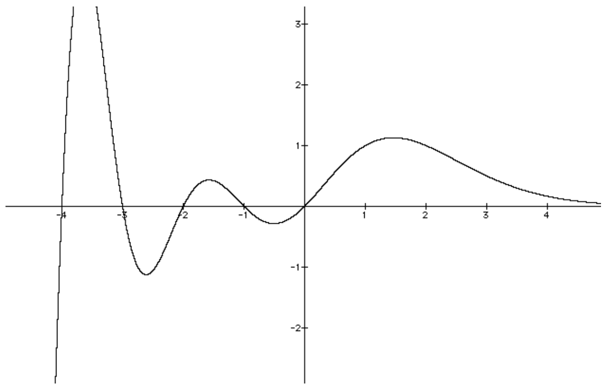
$$\frac{1}{\Gamma(x)} = x\, e^{\gamma x} \prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right) e^{\frac{-x}{n}}$$

$$\Gamma(\text{x}) = \frac{1}{x}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{x}\left(1+\frac{x}{n}\right)^{-1}$$

*The gamma function has following basic properties:



Graph 1: Graph of $\Gamma(x)$



Graph 2: Graph of $1/\Gamma(x)$

$$\Gamma(x+1) = x\Gamma(x)$$
$$\Gamma(x+1) = x!$$
$$\Gamma(x+n) = x(x+1)\cdots(x+n-1)\Gamma(x)$$
$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$
$$\lim_{n\to\infty} \Gamma(x+n)/\Gamma(x)n^x = 1$$

*Logarithmic derivatives:

$$\psi(x) \equiv \frac{d}{dx} log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

$$\lim_{n\to 1}\Gamma^{'}(x) = -\gamma$$

$$\psi(x+1) = \frac{1}{x} + \psi(x)$$

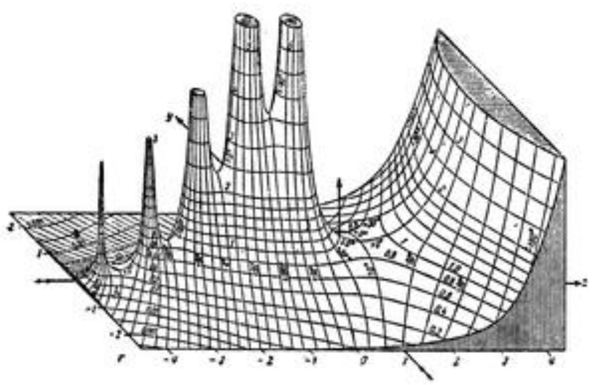
$$\psi(x+n) = \sum_{k=0}^{n-1}\frac{1}{x+k} + \psi(x)$$

$$\psi(x) - \psi(1-x) = \frac{-\pi}{tan\pi x}$$

*Applications:

There are many applications of the gamma function, including its use in various integration techniques and its importance to the beta function. It is also used frequently in combinatory to calculate power series and in analytic number theory to allow further study into the famousRiemann zeta function.***[2]***One interesting theorem regarding the gamma function is Hölder's Theorem, which states that the gamma function does not satisfy any algebraic differential equation. The gamma function does not appear to satisfy any simple differential equations, but this has currently not been proven.***[3]***

The gamma function can be defined for complex numbers as well as real numbers, although the real part of the complex number must be strictly positive***[4]***



Graph 3: Graph of $\Gamma(x)$ in complex plane

1) Volume of N-dimensional ball

Let us define the open ball of radius r of dimension n, Bn(r), to be the set of points such that, for 1≤ j≤n,

$$\sum x_j^2 < r^2$$

Its volume will be referred to as Vn(r).

By definition,***[5]***

$$V_n(r) = \iiint \dots \int_{B_n(r)} dx_1 dx_2 \dots dx_n$$

When n=1,

$$V_n = 2V_{n-1}\int_0^{\pi/2} cos^n\theta \, d\theta$$

David Singmaster uses the following formula for the volume of an n-dimensional ball:

$$V_n(r) = \frac{\pi^{\frac{n}{2}} r^n}{\Gamma\left(\frac{n}{2}+1\right)}$$

2) The Packing Problem

The following problem: "which fits better, a round peg in a square hole or a square peg in a round hole?" can easily be solved once one arrives at the following mathematical formulation of the problem. Which is larger: the ratio of the area of a circle to the area of the circumscribed square or the ratio of the area of a square to the area of the circumscribed circle ***[6].***

For the unit ball, the edge of the circumscribed cube is necessarily length 2, since it is equal in length to a diameter of the unit ball. The edge of the n-cube inscribed in the unit n-ball has length $2/\sqrt{n}$, since the diagonal of an n-cube is $\sqrt{n}$ times its edge.

So, we construct formulas for the volume of the relevant balls and cubes using and the facts which we have just stated:

$$V(n) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$$

$$V_c(n) = 2^n$$

$$V_i(n) = \frac{2^n}{n^{n/2}}$$

where V(n) represents the volume of the unit n-ball, Vc(n) the volume of the circumscribed cube, and Vi(n) the volume of the inscribed cube.

Now,

$$R_1(n) = \frac{V(n)}{V_c(n)} = \frac{\pi^{\frac{n}{2}}}{2^n\Gamma\left(\frac{n}{2}+1\right)}$$

$$R_2(n) = \frac{V_i(n)}{V(n)} = \frac{2^n\Gamma\left(\frac{n}{2}+1\right)}{\pi^{\frac{n}{2}} n^{\frac{n}{2}}}$$

He then takes $\frac{R_1(n)}{R_2(n)}$and applies Stirling's approximation for the gamma function.

The n-ball fits better in the n-cube better than the n-cube fits in the n-ball if and only if n≤ 8.

3) In Computation of infinite sums

An infinite series whose general term is a rational function in the index may always bereduced to a finite series of psi and polygamma functions ***[7].***

$$\frac{1}{A} = \int_0^\infty e^{-Ax}dx$$

Also,

$$\sum_{n=1}^{N} x^n = \frac{x(1-x^N)}{1-x}$$

For a≠b, and {a, b $\in \mathbb{C}; Re(a), Re(b) > 0\}$,

$$\sum_{n=1}^{\infty}\frac{1}{(n+a)(n+b)} = \frac{\psi(b+1)-\psi(a+1)}{b-a}$$

4) The Gamma Distributions

For positive values of the parameters α andβ, the *gamma family of probability distributions*has the density function

$$f(y) = Ky^{\alpha-1}e^{-\frac{y}{\beta}}$$

for $y \geq 0$; $f(y) = 0$ elsewhere.

The constant *K* that causes this function to

Integrate to 1 over the positive half line is

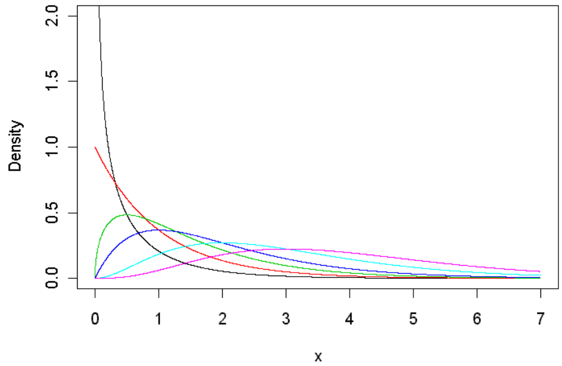
$$K = [\beta^{\alpha}\Gamma(\alpha)]^{-1}$$

For b = 1, this is obvious from looking at the definition of Γ(a), and it can easily be seen for other values of b by making the change of variable $x = y/b$.

The parameter 'a' governs the shape of the gamma density and b is a scale parameter.

The gamma family of distributions can take five fundamentally different shapes, depending on a.

I) 0 < a < 1: vertical asymptote near the origin (black)
II) a = 1 (exponential): $f(0+) = 1/b$, $f'(0+) = -b-2$ (red)
III) 0 < a < 2: $f(0+) = 0$, $f'(0+) = \infty$, mode exists, 1 inflection pt. (green)
IV) a = 2: $f(0+) = 0$, $f'(0+) = b-2$, mode exists, 1 inflection pt. (blue)
V) a> 2: $f(0+) = 0$, $f'(0+) = 0$, mode exists, 2 inflection pts. (cyan, violet),where 0+ indicates taking a limit as the argument approaches 0 through positive values.



Graph 4: Graph showing the distributions

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