International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2019): 7.583

Properties and Application of Beta Function

Raj Shekhar Prasad[1], Kumar Mukesh[2]

[1]Department of Mathematics, Government Engineering College, Buxar, India

[2]Department of Mathematics, R. P. Sharma Institute of Technology, Patna, India

Abstract: ***The Beta function was first studied by Euler and Legendre and was given its name by Jacques Bi-net just as the gamma function for integers describes Factorials, the gamma function can define a binomial coefficient after adjusting indices. The beta function was the first known scattering amplitude in string theory. First conjectured by Gabriele Veneziano. It also occurs in the theory of the preferred attachment process, a type of stochastic urn process. The incomplete beta function is a generalization of the beta function that replaces the definite integral of the beta function with an indefinite integral. The situation is analogous to the incomplete gamma function being a generalization of the gamma function.***

Keywords: Beta function, String theory, Preferential Attachment Process, Stochastic urn Process

1. Introduction

The Beta function was first studied by Euler and Legendre and was given its name by Jacques Binet. Just as Gamma function for integers describes factorials, the beta function can define a binomial coefficient after adjusting indices. The beta function was the first known for scattering amplitude in string theory, first conjected by Gabriele Veneziano. It also occurs in the theory of the preferential attachment process, a type of stochastic urn process. The incomplete beta function is generalization of the beta function that replaces the definite integral of the beta function with an indefinite integral. The situation is analogous to the incomplete gamma function being a generalization of gamma function.

Euler is considered to be the pre-eminent mathematician of the 18th century and one of the greatest mathematicians to have ever lived. He is also one of the most prolific mathematicians; his collected works fill 60–80 quarto volumes. Euler was born on 15 April 1707, in Basel to Paul Euler, a pastor of the Reformed Church, and Marguerite Brucker, a pastor's daughter. He spent most of his adult life in St. Petersburg, Russia, and in Berlin, Prussia.

Euler worked in almost all areas of mathematics, such as geometry, infinitesimal calculus, trigonometry, algebra, and number theory, as well as continuum physics, lunar theory and other areas of physics. He is a seminal figure in the history of mathematics; if printed, his works, many of which are of fundamental interest, would occupy between 60 and 80 quarto volumes. Euler's name is associated with a large number of topics. Euler is the only mathematician to have two numbers named after him: the important Euler's Number in calculus, e, approximately equal to 2.71828, and the Euler-Mascheroni Constant γ (gamma) sometimes referred to as just "Euler's constant", approximately equal to 0.57721. It is not known whether γ is rational or irrational. De Moivre's formula is a direct consequence of Euler's formula. Euler's contribution is endless in field of mathematics and physics and beta function is of his contribution.

In mathematics, the beta function is defined for $\{\ m,n\in C, \mathrm{Re}(m)>0,\ \ \mathrm{Re}(n)>0\ \}$ to be

$$B(m,n)=\int_0^1 t^{m-1}(1-t)^{n-1}\,dt \tag{1}$$

By putting $t=\sin^2\theta$ in equation (1), we get

$$B(m,n)=2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta \tag{2}$$

The relation between beta and gamma function is

$$B(m,n)=\frac{\Gamma m\Gamma n}{\Gamma(m+n)} \tag{3}$$

The proof of (3) is given in Higher engineering mathematics by B.S Garewal.

$$\Gamma m=\int_0^\infty e^{-t}t^{m-1}dt \tag{4}$$

To summarise the argument we put $t=y^2$ in (4)

Therefore, $dt=2ydy$

Therefore,

$$\Gamma m=\int_0^\infty e^{-y^2}\left(y^2\right)^{m-1}2ydy$$

$$\Gamma m=2\int_0^\infty e^{-y^2}y^{2m-1}dy \tag{5}$$

Similarly,

$$\Gamma n=2\int_0^\infty e^{-x^2}x^{2n-1}dx \tag{6}$$

Multiplying equation (5) and (6) we get,

$$\Gamma m\Gamma n=4\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}x^{2n-1}y^{2m-1}dxdy \tag{7}$$

put $\begin{matrix} x=r\cos\theta \\ y=r\sin\theta \end{matrix}$

therefore, $dxdy=rdrd\theta$

limit of $r\ are\ r=0\ to\ r=\infty$

International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2019): 7.583

limit of $\theta \, are \, \theta = 0 \, to \, \theta = \frac{\pi}{2}$

$$\Gamma m \Gamma n = 4\int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2} (r\cos\theta)^{2n-1} (r\sin\theta)^{2m-1} r dr d\theta$$

$$\Gamma m \Gamma n = 4\int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2} r^{2n-1} r^{2m-1} \cos^{2n-1}\theta \sin^{2m-1}\theta \, dr d\theta$$

$$\Gamma m \Gamma n = 4\int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2} r^{2(m+n)-2} \cos^{2n-1}\theta \sin^{2m-1}\theta \, dr d\theta$$

$$\Gamma m \Gamma n = 2\int_0^\infty (e^{-r^2} r^{2(m+n)-2} dr) \;\; 2\int_0^{\frac{\pi}{2}} (\cos^{2n-1}\theta \sin^{2m-1}\theta \, d\theta)$$

1.1 Application of Beta Function

1.1.1 Beta function and string theory

$$\Gamma m \Gamma n = \Gamma(m+n) B(m,n)$$

$$B(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$$

2. Result and Discussion

- Properties of Beta Function
- Application of Beta Function

2.1 Properties of Beta Function

The beta function is symmetric i.e.;

$$B(m,n) = B(m,n)$$

When m and n are positive integers, it follows from the definition of gamma function that:

$$B(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

It has many other forms, including:

$$1.\ B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$2.\ B(m,n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} dt, \qquad \mathrm{Re}(m) > 0, \mathrm{Re}(n) > ($$

$$3.\ B(m,n) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} d\theta, \mathrm{Re}(m) > 0, \mathrm{Re}(n)$$

$$4.\ B(m,n) = \frac{m+n}{mn} \prod_{p=1}^{\infty} \left(1 + \frac{mn}{p(m+n+p)}\right)^{-1},$$

The beta integrals has several more properties including

$$6.\ B(m,n) = B(m,n+1) + B(m+1,n)$$

$$7.\ B(m+1,n) = B(m,n).\frac{m}{m+n}$$

$$8.\ B(m,n+1) = B(m,n).\frac{n}{m+n}$$

$$9.\ B(m,n).\ B(m+n,1-n) = \frac{\pi}{m \sin(\pi n)}$$

An Italian theoretical physicist and the founder of string theory Gabriele Veneziano first conjectured the beta function as the Scattering amplitude
in string theory.

In 1968 a research fellow Gabriele Veneziano at CERN, observed a strange coincidence – many properties of the strong nuclear force are perfectly described by the Euler beta-function, an obscure formula devised for purely mathematical reasons two hundred earlier by Leonhard Euler. In the spurt of research that followed, Yoichoiro Nambu of the University of Chicago, Holger Nielsen of the Niels Bohr Institute, Leonhard Susskind of Stanford University revealed that the nuclear interactions of elementary particles modeled as one-dimensional strings instead of zero-dimensional particles were described exactly by the Euler beta-function. This was, in effect, the birth of string theory.

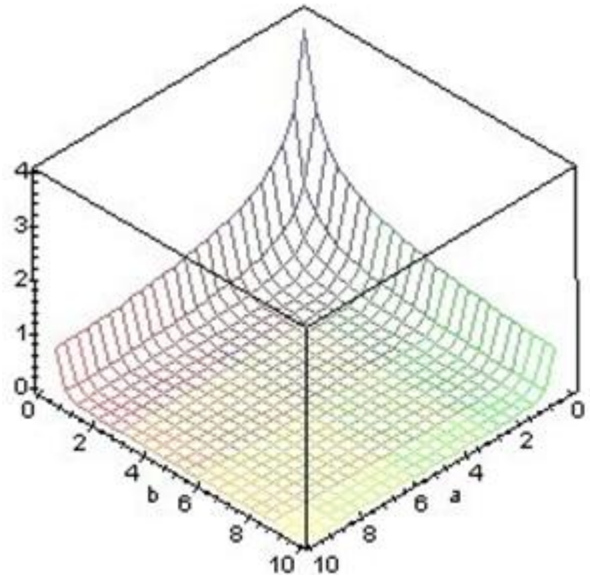
In the "dual resonance model" introduced by Veneziano in 1970th in order to fit experimental data, the Euler Beta function appeared in elementary particle physics as a model for scattering amplitude. It soon turned out that the basic physics behind this model is the string (instead of zero-dimensional mass point).

2.2.2 Preferential Attachment process

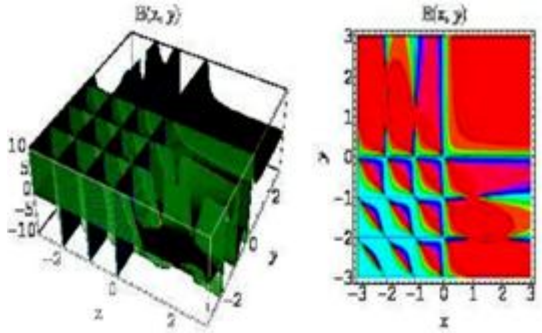
Preferential Attachment to a class of processes in which quantity, typically of wealth or credit, is distributed among a no of individual or objects according to how much they already wealthy receive more than those who are not. The principal reason for scientific interest in preferential attachment is that it can, under suitable circumstances, generate power law distributed of wealth.

2.2.3 Stochastic Urn Process and Beta Function:-

A preferential attachment process is a Increases continuously, although this is not a necessary condition for preferential attachment and examples have been studied with constant or even decreasing numbers of urns.



Graph of Beta Function



3D image of Beta Function

stochastic urn process, meaning a process in which discrete units of wealth, usually called 'balls', added in random or partly random fashion to a set of objects or containers, usually called 'urns'. A preferential attachment process is an urn process in which additional balls are added continuously to the system and are distributed among the urns as an increasing function of the number of balls the urns already have. In the most commonly studied examples, the number of urn

$$P(k) = \frac{B(k+a,\gamma)}{B(k_0+a,\gamma-1)}$$

Linear preferential attachment processes in which the number of urns increases are known to produce a distributive of balls over the urns following the so-called 'Yule distributive'. In most general form of the process, balls are added to the system at an overall rate of *m* new species for each new urn. Each newly created urn starts out with k_0 balls and further balls are added to urns at a rate proportional to the number *k* that they already have plus a constant $a > -k_0$ .With these definitions, the fraction P(*k*) of urns having k balls in the limit of long time given by

$$P(k) = \frac{B(k+a,\gamma)}{B(k_0+a,\gamma-1)}$$

for $k \geq 0$ (and zero otherwise), where $B(m,n)$ is the Euler beta function:

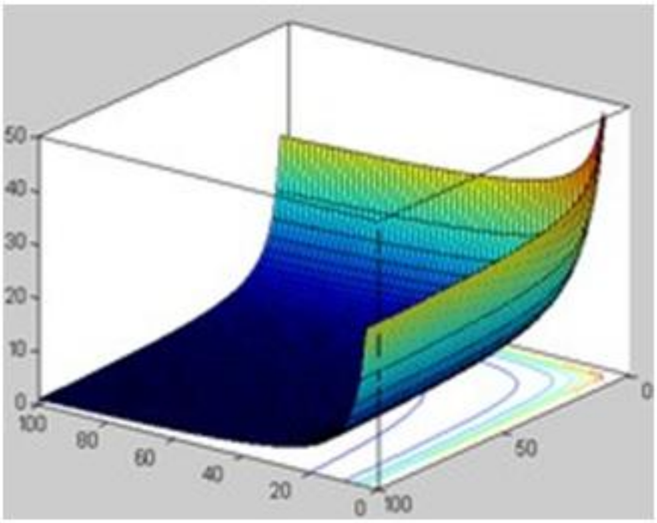
$$B(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}$$

with $\Gamma(m)$ being the standard gamma function, and

$$\gamma = 2 + \frac{k_0 + a}{q}$$

In other words the preferential attachment process generates a "long-tailed" distribution following a 'Pareto distribution'or 'power law' in its tail. This is the primary reason for the historical interest in preferential attachment. The species distribution and many other phenomena are observed empirically to follows power laws and preferential attachment process is a leading candidate mechanism to explain this behavior. Preferential attachment is considered a possible candidate for, among other things, the distribution of sizes of cities, the wealth of extremely wealthy individuals, the number of citations received by learned publication and the number of links to pages on the world wide web.

also



The plot of the beta function for positive x any values

References

[1] P. J. Davis, "Leonhard Euler's integral: a historical profile of the gamma function," *The American Mathematical Monthly, vol. 66, pp. 849–869, 1959.*

[2] W. Gautschi, "Leonhard Euler: his life, the man, and his works," *SIAM Review*, vol. 50, no. 1, pp. 3–33, 2008.

[3] M. Zelen and N.C.Severo in Milton Abramowitz and Irene A. Stegun, eds "Handbook of Mathematical Tables.

[4] Stochastics Processes by Jyotiprasad Medhi.

[5] Supersymmetry and String Theory by Michael Dine.

[6] The Beta Function from the Wolfram Functions Site. [url:- http://mathworld.wolfram.com/BetaFunction.html].