International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2019): 7.583

On Classification of Moduli Space and its Implications

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Abstract: ***The properties of moduli space and Teichmuller space has been studied mathematically and physically to show their importance and applications in metric structure. Riemann considered the space M of all complex structures on an orientable surface modulo the action of orientation preserving diffeomorphisms and derived the dimension of this space which expressed as dimg. M=6g-6…..(I) where g ≥ 2 is the genus of the topological surface. Moduli spaces of Riemann surfaces have also been studied in algebraic geometry by F.O. Gardina. The geometric invariant theory developed by Mumfor is major achievement. Deligne and Mumford studied the projective property of the Moduli space and they showed that the moduli space is quasi-projective and can be compactified naturally by adding in the stable nodal surfaces. The classical metrics on the Teichmuller space and the moduli spaces have also been studied independently. Each metric has played mportant role in the study of the geometry and topology of the moduli and Teichmuller spaces. We consider some of them such as (India) finsler metrics: (ii) the Teichm "uller metric ll lle (iii) They are all complete metrics on the Teichmuller space and are invariant under the moduli group action.***

Keywords: Teichmuller space, Moduli space, Finsler metrics, curvature, Poincare metric, and Kahler-Einstein metric

1. Introduction

We derive interesting applications in algebraic geometry. K.Liu suitabley applied it algebraic geometry. Teichmuller considered a cover of *M* by taking the quotient of all complex structures by those orientation preserving diffeomorphims which are isotopic to the identity map. The Teichm uller space T_g is a contractible in set in C^{3g-3}. Also, it is a pseudo convex domain. Teichm uller also introduced the Teichm uller metic by first takeing the L1 norm on the contangent space of T_g and then taking the dual norm on the tangent space, which is a Finsler metric. Another finsler metrics are the Carath eodorymetic and Kobayashi metric. These Finsler metries have been powerful tools to study the hyperbolic property of the moduli and the Teichm uller spaces and the mapping class groups Royden proved that the Teichm uller metric and the Kobayashi metric are the same. He has shown the holomorphic automorphism group of the Teichm uller space is exactly the mapping class group. Well Peterson introduced the fisrt derive Hermitian metric on the Teich'muller space. Ahlfors shown that the Well- Patersson metric is K ahler and its holomorphic sectional curvature is negative. Ahlofors and Beys derived on the solutions of Beltrami equation which put a solid forundation of the theory of Fugian space. Under the moduli group action. The sxistence of the K"ahler-Einstein metric is used here to derive algebraic and geometic applications including as its properties like the curvature and the behavours near the comactification divisor. S.T. Yau developed the concept that the K" ahler-Einstein metric is equivalent to the Teichm uller metric and the Bergman metric. McMullen introduced a metric known as the McMullen metric by perturbing the Well-Petersson metic go get a complete K" ahler metric which is complete and K"ahler hyperbolic. Thus the lowest eigen value of the Laplace operator is positive and the L^2-Cohomology is trivial except for the middle dimension. The moduli space has been widely used in geometry, topologhy, algebraic geometry to number theory. Faltings' proof of the Mordell conjecture depends heavily on the moduli space which can be defined over the integer ring. Moduli space also paly significant role in many areas of theoretical physics. Such as in string theory in where many computations of path integrals are reduced to integrals of Chern classes on the moduli space. In physical theories. Physicists have made several interesting conjectures about generating series of Hodge integrals for all genera and all marked points on the moduli spaces. We discuss the basic concepts of teichmuller theory introduced by A.J. Trombe and its impact on moduli space. Let ∑ be an orientable surface with genus g ≥ 2. A complex structure on ∑ is a covering of ∑ by charts such that the transition functions are holomorphic. By the uniformzation theorem. If we put a complex structure on ∑ then it can be viewed as aquotient of the phperbolic plane H^2 by a Fuchsian group. Thus there is a unique K" ahler-Einstein metric. or the hyperbolic metric on ∑ Let Country be the set of all complex structures on ∑ Let $Diff^+$ (∑) be the group of orientation preserving diffeomorphisms and let $Diff^+_0$ (∑) be the subgroup of $Diff^+$(∑) consisting of those elements which are isotopic to dentity. The groups $Diff^+$ (∑) and $Diff^+_0$(∑) act on the space Country by pull-back. The Teichmuller space is s a quotient of the space Country

$$T_g = C/Diff^+_0(\Sigma). \qquad ...(1)$$

By an appropriate application of Bers embedding theorem, it is known that T_g can be embedded into C^{3g-3} as a pseudoconves domain and is contractible. Let us put

$$Mod_g = Diff^+ (\Sigma)/Diff^+_0(\Sigma)$$

be the group of isotopic classes of diffeomorphisms. This groups is called the (Teichm"uller) moduli group or the mapping class group. Its representations are helpful in topology and in quantum field theory. The moduli space M_g is the space of distinct complex structures on ∑ defined as follows.

$$M_g = C/Diff^+(\Sigma) = T_g/Mod_g.$$

The moduli space is a complex orbifold. For any point $\in M_g$, let $X = X_s$ be a representative of the corresponding class of

Riemann surfaces. Combining properties of the Kodair-Spencer deformation theory and the Hodge theory, we get.

$$T_xM_g \cong H^1(X,T_x) = HB\,(X) \qquad ....(3)$$

where HB(X) is the space of harmonic Beltrami differentials on X.

$$T_xM_g \cong Q(X)$$

where Q(X) is the space of holomorphic qualdratic differentials on X. Let us μ ∈ HB (X) and ∅ μ Q(X). If we fix a holomorphic local coordinate z on X. we can write μ = μ (z)$\frac{\partial}{\partial z}$ ⊗ $\overline{dz}$ and μ = (z)dx^2 Thus, the quality between T_xM_g and $T^*{}_xM_g$ is

$$[\mu : \varphi] \int_x \mu(z)\varphi(z)dz\overline{dz}$$

By suitable application the Riemann-Roch theorem, we obtain

$$\dim C\ HB(X) = Q(X) = 3g- 3$$

which implies

$$\dim_C T_g = \dim_C M_g = 3g- 3$$

2. Classification of Teichmuller Space and the Moduli Space

We describe norm structure of seven types of Kahler metrics (india) the Weil-Petersson metric ω wp which is incomplete, (ii) the Cheng-Yau's K"ahler-Einstein metric ω KE, (iii) the McMullen metric ω c, (iv) the Bergman metric ω b, (v) the asymptotic Poincar'e metric on the moduli space ω p, (vi) the Ricci metric , and (vii) the perturbed Ricci metric ω_T Six metices (viii) are complete. Ricci metric and perturbed Ricci metric have been studied by S.I. yau. The Teichmuller metric was first introduced by Teichmuller to be L^1 norm in the cotangent space. For each $\varphi = \varphi$ (z) dz_2 ∈ Q (X) ≅ $T^*{}_xM_g$, the Teichm"uller norm of φ is expressed as

$$\|\varphi\|T = \int_x|\mu(z)|dzd\bar{z}.$$

By using the duality. For each $\mu \in HB(X) \cong T_xM_g$.

$$\|\mu|\mu = \sup\{Re[\mu;\varphi] \setminus \|\varphi\|\ |r = 1\}$$

F.P.Gardner derived that Teich"uller metric has constant holomorphic sectional curvature 1.

The Kobayashi and the carath eodory metrics defined it for any complex space as follows. Let Y be a complex manifold and of dimension n. let Δ_R be the disk in Country with radius R. Let $\Delta = \Delta_1$ and let ρ be the Poincar'e metric on Δ. Let P φ Y be a point and let v φ T_p Y be a holomorphic tangent vector. Let Hol (Y, Δ_R) and Hol (Δ_R, Y) be the spaces of holomorphic maps from Y to Δ_R and from Δ_R to Y respectively. The Caratheodory norm of the vector v is defined as follows.

$$\|\vartheta\|c = \sup_{f \in Hol(Y,\Delta)} \|f^*\vartheta\|\Delta_1\rho \qquad ....(5)$$

and the Kabayashi norm of v is expressed as

$$\|\vartheta\|k = \frac{inf}{f \in Hol\,(\Delta R\ Y), f(0)=P1\ \ f1(0)=\vartheta}\ \frac{2}{R}.$$

The Bergman (pseudo) metric has been defined for any complex space Y provided the Bergman Kernel is positive. Let Ky be the canonical bundle of Y and let W be the space of L2 holomorphic sections of Ky in the sense that if $\sigma \in W$, then

$$\|\sigma\|_L^2 = (\sqrt{-1)})^{n2}\ \sigma\ \Delta\ \bar{\sigma} < \infty$$

The inner product on W is defined to be

$$(\sigma,\rho) = (\sqrt{-1})n2\sigma\ \Delta\ \bar{\rho}$$

for all σ ρ ∈ W. Let $\sigma_1\ \sigma_2 \cdots$ be an othonormal basis of W. The Bergman Kernel form is the non-negative (n, n)- form

$$\beta y = \sum_{0;1}^{\infty}(\sqrt{-1})n2\ a_j\ \Delta\ a_j$$

With a suitable choice of local coordinates $z_1 \cdots z_n$. we get $\beta Y = BE_Y(..........)\ (\sqrt{-})n2\ dz1\ \Delta ... \Delta\ d2n\ \Delta\ d21\ \Delta ... \Delta\ \overline{d2}n$ where BE_Y (z, $\bar{z}$) is called the Bergman kernel function. if the Bergman kernel B_Y is positive, Bergman metric is defined as follows.

$$B\overline{\iota j} = \frac{\partial 2 logBEY\ \ (z,\bar{z})}{\partial 2i\partial \bar{z}j}$$

The Bergman metric is well-defined and is non degenerate if the elements in W separate points and the first jet of Y. Hence, the Bergman metric is a K"ahler metric. We thus conclude that both the Teichmuller space and the moduli space are equipped with the Bergman metrics. However, the Bergman metric on the moduli space is different from the metric induced from the Bergman metric of the Teichm"uller space. The bergman metric defined on the moduli space is incomplete due to the fact that the moduli space is quasi-projective and any L2 holomorphic section the canonical bundle can be extended over. We state here the basic properties of the Kobayashi, the Carath eodory and the Bergman metrics which have been studied by S. Kobayash. We reformulate some of the theorem in its complex metric structure.

3. Theorem

Let X be a complex space. Then following relations holds

(i) ‖ . ‖c. x ≤ ‖ . ‖K,X;
(ii) Let Y be another coplex space and f : X Y be a holomorphic map. Let p ∈ X and v ∈ T_pX. Then‖f+(ϑ)‖C,Y f(p) ≤ ‖ϑ‖C,X, p and ‖f*(ϑ)‖K.Y f(p) ≤ ‖ϑ‖ K,X,P;
(iii) If X ⊂ Y is a connected open subset and z ∈ X is a point. Then with any local coordinates we have BE_Y (z) ≤ BE_X (z);
(iv) If the Bergman kernel is positive, then at each point z ∈ X a peack section σ at z exists. Such a peak section is unique up to a constant factor c with norm 1. furthermore, with any choice of local coordinates, we have BEX(z) = | $\sigma(z)|^2$;
(v) If the Bergman kernel of X is positive, the ‖ . ‖C,X ≤ 2‖ . ‖B.X;
(vi) If X is a bounded convex domain C^n, then‖ . ‖ C.X = ‖ . K,X;
(vii) Let | . | be the Euclidean norm and let B_r be the open ball with centre 0 and radius r in C^n. Then for any holomorphic tangent vector v at 0,

$$|\vartheta|C.Br0 = \|\vartheta\|K.Br.0 = \frac{2}{r}|\vartheta|$$

where |v| is the Euclidean norm of v.

Proof

The three Finsler metrics (i) - (iii) paly a significant role in understanding the hyperbolic geometry of the doduli spaces, and the mapping class group. It has been derived that the Bergman metric on the Teichmuller space is complete The Weil-Petersson metric is the first K"ahler metric defined on the Teichmuller and the moduli space. It has been defined by using the L^2 inner product on the tangent space in the following way: let $\mu, \vartheta \in T_x M_g$ Two gangent vectors and let λ be the unique K"ahler-Enistein metric on X. Then the Weil-Pertersson metric is given by

$$h(\mu.\vartheta) = \int_x \mu\bar{\upsilon}\, d\vartheta$$

where $dv = \frac{\sqrt{-1}}{2} \lambda\ dz\ \Delta$ is the volume form. The curvatures of the Weil-Pertesson metric have been investigated by Ahlors. Royden and Wolpert, its Ricci and holomophic sectional carvatuure are all negative with negative upper bound, but with bno nower bound. Its boundary behavior is incomplete metric from which we find that it is an inmomplete metric. The xistence of the K"ahler-Einstein metric was studied by Cheng-yau. Its Ricci curvature is -1. The expression for Ricci-curvature is given by

$$\partial\bar{\partial}\log\omega^{n}{}_{KE} = \omega_{KE}$$

where n = 3g – 3. it has been found that a bounded domain in Cn admits a complete K"ahler-Einstien metric if and only if it is pseudoconvex. The McMullen 1/1 metric defined has been Weil-Pertersson metric by adding a K"ahler form whose potention involves the short geodesic length functions on the Riemann surfaces. for each simple colosed curve y let, 1_y (X) be the length of the unique geodesic in the homotopy class of y with respect to the unique K:ahler-Einstein metric. Then the McMullen metric is defined as follows.

$$\omega\vartheta = \omega wp - i\delta \sum \partial\bar{\partial}\mathrm{Log}\frac{\epsilon}{\varphi}$$

where $\in$ and δ are small positive constants and Log(x) is a smooth function defined as follows.

$$\mathrm{Log}(r) = \{\begin{matrix}\log\ r & .r \geq 2 \\ 0 & r \geq 1.\end{matrix}\}$$

This metric is Kahler hyperbolic imples that it satisfies the following conditions:

(i) (Mg ω1/1) has finite volume:
(ii) The sectional Curvature of (Mg, ω 1/1) is bounded above and below;
(iii) The injectivity radius of (T_g, ω 1/1) is bounded below;
(iv) On T_g, the K"ahler form ω1/1 can be written as ω1/1 where α is a bounded 1- form.

We find that the Kahler hyperbolicity is that the L^2 -cohomology is trivial except for the middle dimension. The asymptotoic Poincar'e metric is defined as a complete K"ahler metric on a complex manifold M which is obtained by removing a divisor Y with only normal crossings from a compact K"ahler manifold ($\bar{M}$, ω). The Kahler hyperbolicity satisfies the above conditions (i) – (iii).

4. Theorem

The curvature of the Weil-Petersson metric is given by

$$RJ,KT= \int_x (J\int kt + \pi \int kj)\ dv$$

Proof

We show that the Ricci and the holomorphic sectional curvature have explicit negative upper bound. we establish the curvature formula of the Ricci metric and introduce more operators. Firstly, the Commutator of the operator V_K and (+ 1) play an important role. we view the vector field V_K as a operator acting on functions. Let us define.

$$\xi_k = [\overline{\vartheta 1}\ \xi_k] \qquad \text{….(9)}$$

By simple computation reduces to

$$\in K = A_K P.$$

Lte us define the commutator of $\overline{\vartheta 1}$ and $\in_K$.Let us put

$$Q\bar{k}j = (\overline{\vartheta j}.\in_K)$$

We obtain the relation

$$Q\bar{k}l\ (f) = \bar{P}\ (e\bar{k})P(f) - 2f\,\overline{Ki}$$
$$f + \lambda^{-1}\,\partial_z f\bar{k}i\ \partial_z f\bar{k}i\ \partial\bar{z}f$$

for any smooth function f. Let us introduce the symmertization operator of the indices. Let U be any quantity which depends on indicesi i,k,$\alpha,\bar{j},\bar{l},\bar{\beta}$. The symmetrization operator σ_1 is defined by taking summation of all orders of the triple (i, k,). Fin other words,

$$\sigma 1\ (U(i,k,\alpha,\bar{j},\bar{l},\bar{\beta})) = U(i,k,\alpha,\bar{j},\bar{l},\bar{\beta}) + U(i,k,\alpha,\bar{j},\bar{l},\bar{\beta})$$
$$+ U(i,k,\alpha,\bar{j},\bar{l},\bar{\beta}) + U(i,k,\alpha,\bar{j},\bar{l},\bar{\beta}) + U(i,k,\alpha,\bar{j},\bar{l},\bar{\beta})$$
$$+ U(i,k,\alpha,\bar{j},\bar{l},\bar{\beta}) + U(i,k,\alpha,\bar{j},\bar{l},\bar{\beta}) \qquad \text{…(10)}$$

Similarly, σ_2 is the symmetrization operator of $\bar{j}$ and $\tilde{\beta}$ and $\tilde{\sigma}$ is the symmetrization operator of $\tilde{j}\ \tilde{l}$ and $\tilde{\beta}$. K. Liu an S.T. Yau derived the curvature formulas stated here as a theorem.

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