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Zeta Function

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Abstract: *Zeta function can be expressed by following equation,* $\zeta(s)=\frac{1}{2}\left\{1+\frac{1}{(s-1)}\sqrt{(|s|-1)^2+4}\right\}$

Explanation

By the definition of Zeta Function,

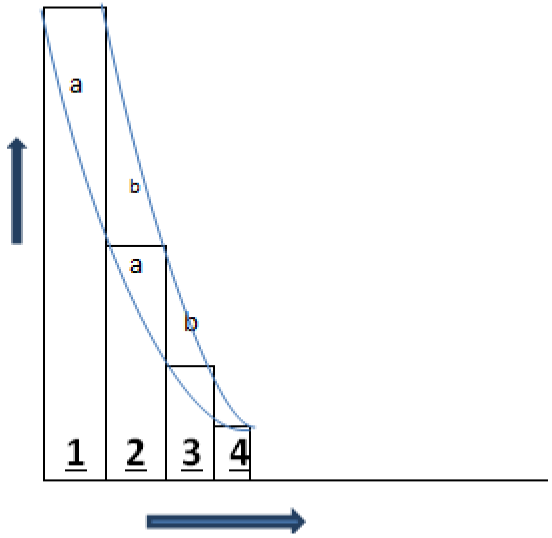
$\zeta(s)=\sum_{n=1}^{\alpha}\frac{1}{(n^s)}=\frac{1}{(1^s)}+\frac{1}{(2^s)}+\frac{1}{(3^s)}+\frac{1}{(4^s)}+\ \ldots\ldots$

$\frac{1}{(s-1)}<\zeta(s)<\frac{s}{(s-1)}\frac{1}{(n^s)}$

So, $\zeta(s)$ - Area (a) $=\frac{1}{(s-1)}\ldots.$ (1)

$\zeta(s)$ + Area (b) $=\frac{s}{(s-1)}\ldots$ (2)

Marked Area = a
Marked Area = b



For a particular "n "value

$$\frac{\text{Area (a)}}{\text{Area (b)}}=\frac{\frac{-1}{(s-1)}\left[\frac{1}{(n+1)^{(s-1)}}-\frac{1}{(n)^{(s-1)}}\right]-\frac{1}{(n^s)}}{\frac{1}{(s-1)}\left[\frac{1}{(n+1)^{(s-1)}}-\frac{1}{(n)^{(s-1)}}\right]+\frac{1}{(n+1)^s}}$$

$$=\frac{\left[-\frac{1}{(n+1)^{(s-1)}}+\frac{1}{(n)^{(s-1)}}\right]-\frac{(s-1)}{(n^s)}}{\left[\frac{1}{(n+1)^{(s-1)}}-\frac{1}{(n)^{(s-1)}}\right]+\frac{(s-1)}{(n+1)^s}}$$

$$=\frac{\left[-\frac{(n+1)}{(n+1)^s}+\frac{(n+1)}{(n)^s}\right]-\frac{s}{(n^s)}}{\left[\frac{n}{(n+1)^s}-\frac{n}{(n)^s}\right]+\frac{s}{(n+1)^s}}=\frac{\frac{1}{(n^s)}\{(n+1)\left[1-\frac{(n)^s}{(n+1)^s}\right]-s\}}{\frac{1}{(n+1)^s}\{n\left[1-\frac{(n+1)^s}{(n)^s}\right]+s\}}$$

$$=\frac{(n+1)^s\{(n+1)\left[1-\frac{(n)^s}{(n+1)^s}\right]-s\}}{(n^s)\{n\left[1-\frac{(n+1)^s}{(n)^s}\right]+s\}}$$

So, total area fraction will be,

$$\frac{\sum_{n=1}^{\alpha}Area\ (a)}{\sum_{n=1}^{\alpha}Area\ (b)}=\frac{\sum_{n=1}^{\alpha}(n+1)^s\{(n+1)\left[1-\frac{(n)^s}{(n+1)^s}\right]-s\}}{\sum_{n=1}^{\alpha}(n^s)\{n\left[1-\frac{(n+1)^s}{(n)^s}\right]+s\}}$$

$$\frac{\text{Area (a)}}{\text{Area (b)}}=\frac{\sum_{n=1}^{\alpha}\{(n+1)^{(s+1)}-n^{(s+1)}-n^s-s(n+1)^s\}}{\sum_{n=1}^{\alpha}\{n^{(s+1)}-(n+1)^{(s+1)}+(n+1)^s+sn^s\}}$$

$$=\frac{-1-\zeta(-s)-s[\zeta(-s)-1]}{1+[\zeta(-s)-1]+s\zeta(-s)}=\frac{(s-1)-(s+1)\zeta(-s)}{(s+1)\zeta(-s)}=\frac{1-\frac{(s+1)}{(s-1)}\zeta(-s)}{\frac{(s+1)}{(s-1)}\zeta(-s)}$$

So,by the afore said equation (1)&(2) we get as follows,

$\zeta(s)-\left\{1-\frac{(s+1)}{(s-1)}\zeta(-s)\right\}=\frac{1}{(s-1)}$

Or, $\zeta(s)+\frac{(s+1)}{(s-1)}\zeta(-s)=\frac{s}{(s-1)}$

So, multiplying both side by (s-1) we get,

Or, $(s-1)\,\zeta(s)+(s+1)\,\zeta(-s)=s$ ----- Analytic Continuity.

From ANALYTIC CONTINUITY Equation we get, $\frac{(s+1)}{(s-1)}=\frac{2\zeta(s)-1}{1-2\zeta(-s)}$

Let us assume k as a proportionality constant, and then we can write as follows

$2\zeta(s)-1=\frac{k}{(s-1)}$ from where one can get the following expressions

$\zeta(s)=\frac{1}{2}\left\{1+\frac{k}{(s-1)}\right\}$

…. ….(k) *is the " proportionality constant"*

And similarly,

$\zeta(-s)=\frac{1}{2}\{1-\frac{k}{(s+1)}\}$

….(k) is the " proportionality constant"

Now, as $\zeta(s)=\frac{1}{(\pi(1-p^{-s}))}=\frac{\pi p^s}{(\pi(p^s-1))}=\pi p^s\zeta(-s)$ as

$\zeta(-s)=\frac{1}{(\pi(p^s-1))}$

Thus, $\frac{\zeta(s)}{\zeta(-s)}=\pi p^s$, or, $\pi p^s=\frac{\frac{1}{2}\{1+\frac{k}{(s-1)}\}}{\frac{1}{2}\{1-\frac{k}{(s+1)}\}}$ or, $1+\frac{k}{(s-1)}=$

$\pi p^s-\frac{k\pi p^s}{(s+1)}$

So, $k=\frac{-1+\pi p^s}{\frac{1}{(s-1)}+\frac{\pi p^s}{(s+1)}}$

Now, $\zeta(s)=\frac{1}{(1^s)}+\frac{1}{(2^s)}+\frac{1}{(3^s)}+\frac{1}{(4^s)}+\ \ldots\ldots$

$=\lim_{n\to\infty}\left(\frac{1}{n}\right)^s\left[\frac{1}{\left(\frac{1}{n}\right)^s}+\frac{1}{\left(\frac{2}{n}\right)^s}+\frac{1}{\left(\frac{3}{n}\right)^s}\ \ldots\ldots\right.$

$=\lim_{n\to\infty}\left(\frac{1}{n}\right)^s\int_0^1\frac{1}{(x^s)}\,dx=\frac{1}{(s-1)}\lim_{n\to\infty}\{\frac{1}{n}-\left(\frac{1}{n}\right)^s\}\ \ldots..(3)$

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And, $\zeta(-s) = \lim_{n\to\infty}(n)^s \left[\frac{1}{(n)^s} + \left(\frac{2}{n}\right)^s + \left(\frac{3}{n}\right)^s \ldots\ldots\right.$

or, $\zeta(-s) = lim_{n\to\infty}(n)^s \int_0^1 (x)^s\ dx\ =$

$\frac{1}{(s+1)}\lim_{n\to\infty} n^s$.......(4)

SO, $\lim_{n\to\infty} n^s = (s+1)\,\zeta(-s) = S - (s-1)\,\zeta(s)$[from continuity]

Here for s=1, $\lim_{n\to\infty} n = 1$

SO, $\lim_{n\to\infty}\{\frac{1}{n} - \left(\frac{1}{n}\right)^s\} + \lim_{n\to\infty} n^s$ = S [from continuity equation]

Or, $1 - \left(\frac{1}{n}\right)^s + n^s = s$ or, $n^{2s} - (s-1)n^s - 1 = 0$

$$n^s = \frac{(s-1) \pm \sqrt{(s-1)^2 + 4}}{2}$$

Again, $\frac{\zeta(s)}{\zeta(-s)} = \pi p^s = \frac{(s+1)}{(s-1)}\lim_{n\to\infty}\left(\frac{1}{n}\right)^s \{\frac{1}{n} - \left(\frac{1}{n}\right)^s\}$[From the value of $\zeta(s)$&$\zeta(-s)$ as stated above in equation no. (3) & (4)

=>$\frac{(s+1)}{(s-1)}\lim_{n\to\infty}\left(\frac{1}{n}\right)^s \{s - n^s\}$[from continuity equation]

Explaination, $\{\frac{1}{n} - \left(\frac{1}{n}\right)^s\}$= (s-1) $\zeta(s)$= s - $(s+1)\,\zeta(-s)$ = s - n^s ... ... (5)

So, $\zeta(-s) = \frac{n^s}{(s+1)} = \frac{(s-1)\pm\sqrt{(s-1)^2+4}}{2(s+1)} = \frac{2s-(s+1)\pm\sqrt{(s-1)^2+4}}{2(s+1)}$

$\zeta(-s) = \frac{s}{(s+1)} - \frac{1}{2}\{1 \mp \frac{\sqrt{(s-1)^2+4}}{(s+1)}\}$ [by putting the value of *n^s as determined from the quadratic equation above in equation no.* (4)

And by the equation no.(4) & given relationship in equation (5),

$\zeta(s) = \frac{s}{(s-1)} - \frac{n^s}{(s-1)} = \frac{s}{(s-1)} - \frac{1}{2}\{1 \pm \frac{\sqrt{(s-1)^2+4}}{(s-1)}\}$

It's also noticed that continuity equation also holds similar for,

$(s-1)\{\frac{s}{(s-1)} - \zeta(s)\} + (s+1)\{\frac{s}{(s+1)} - \zeta(-s)\} = s$

SO, $\zeta(s) = \frac{1}{2}\left\{1 + \frac{\sqrt{(s-1)^2+4}}{(s-1)}\right\}$ AND $\zeta(-s) = \frac{1}{2}\{1 - \frac{\sqrt{(s-1)^2+4}}{(s+1)}\}$

So, in general $\zeta(s) = \frac{1}{2}\left\{1 + \frac{\sqrt{(|s|-1)^2+4}}{(s-1)}\right\}$ and, k = $\sqrt{(|s|-1)^2+4}$

So, $\frac{\zeta(s)}{\zeta(-s)} = \pi p^s = \frac{\left\{1+\frac{\sqrt{(|s|-1)^2+4}}{(s-1)}\right\}}{\{1-\frac{\sqrt{(|s|-1)^2+4}}{(s+1)}\}}$

And, $\eta(s) = (1- 2^{(1-s)})\zeta(s) = (\frac{1}{2} - \frac{1}{2^s})\left\{1 + \frac{\sqrt{(|s|-1)^2+4}}{(s-1)}\right\}$

The graph of $\eta(s)$ according to this equation shows a limiting value of

$\eta(1) = 0.693147 = \ln(2) \ldots\ldots. Hence\ Proved$

As $\zeta(-1) = \frac{1}{(\pi(p-1))}$and $(\pi(p-1))$ is a very large value thus $\zeta(-1)$ =0

As shown in the graph below.

And as shown in πp^s graph $\pi p^0 = \pi(1) = 1$, $\pi p^{-1} = \pi\left(\frac{1}{p}\right) = 0, \pi p^1$ →infinite

....
Hence also proved

Conclusion: THE CONCLUTION OF THIS TOPIC IS

$\zeta(s) = \frac{1}{2}\left\{1 + \frac{\sqrt{(s-1)^2+4}}{(s-1)}\right\}\ AND\ \zeta(-s) = \frac{1}{2}\{1 - \frac{\sqrt{(s-1)^2+4}}{(s+1)}\}$

$\eta(s)$=$(\frac{1}{2} - \frac{1}{2^s})\left\{1 + \frac{\sqrt{(|s|-1)^2+4}}{(s-1)}\right\}$ Which satisfy the value of $\eta(1)$

$$\pi p^s = \frac{\left\{1 + \frac{\sqrt{(|s|-1)^2+4}}{(s-1)}\right\}}{\{1 - \frac{\sqrt{(|s|-1)^2+4}}{(s+1)}\}}$$

Now one can calculate the largest prime number from the critical value of $\pi p^s\ WHICH\ IS$ (3.757,7.871) as shown in graph below.

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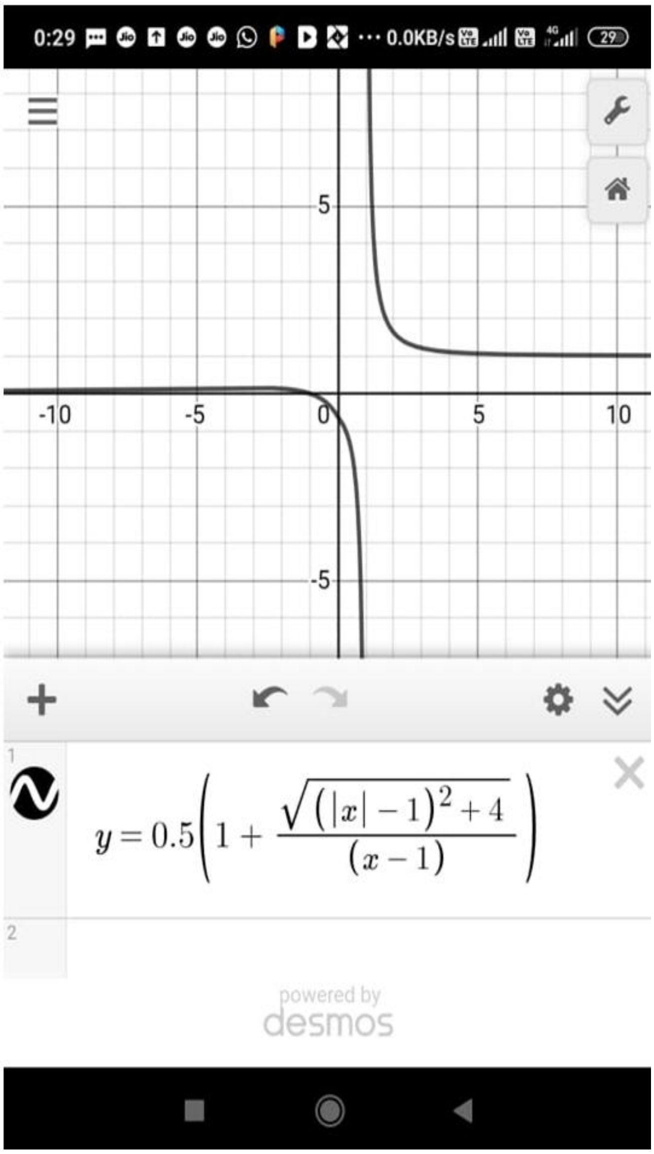

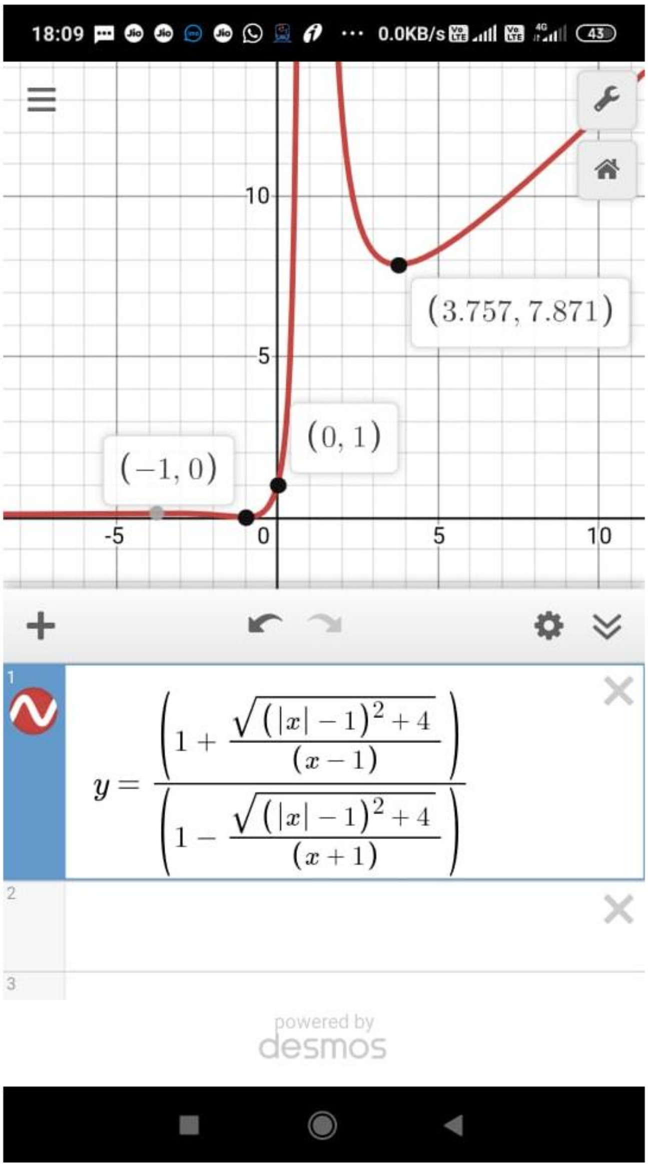

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