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Coincidence Point Theorems in D-Metric Spaces

Manoj Kumar Tiwari

Department of Mathematics, Pt. Sundarlal Sharma (Open) University, Chhattisgarh, Bilaspur, Chhattisgarh, India

Abstract: ***In this paper we used the concept of compatible mappings of type (P) in D-metric space. Our result generalize the result of Parsai V. and Singh B., Fisher and Pathak.***

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1. Introduction

In 1992, a new structure of a generalized metric space was introduced by Dhage on the line of ordinary metric space defined as under:

Let R denoted the real line and X denoted a nonempty set. Let D : XxXxX → R be a function satisfying properties:

(D_1) D(x, y ,z) ≥ 0 for all x, y, z ∈ X, equality holds if and only if x = y = z.

(D_2) D(x, y ,z) = D(x, z, y) = …………. ∀ x, y, z ∈ X,

(D_3) D(x, y, z) ≤ D(x, y, u) + D(x, u ,z) + D(u, y ,z) ∀ x, y, z, u ∈ X ,

The function D is called a D-metric for the space X and (X, D) denotes a D-metric space. Generally the usual ordinary metric is called the distance function. D-metric is called diameter function of the points of X (Daghe)

In the last three decades, a number of authors have studied the aspects of fixed point theory in the setting of D-metric spaces. They have been motivated by various concepts already known for metric space and have thus introduced analogous of various concepts in the framework of the D-metric spaces. Khan, Murthy-Chang-Cho-Sharma and Naidu-Prasad introduced the concepts of weakly commuting pairs of self mappings, compatible pairs of self mapping of type (A) in a D-metric space and notion of weak continuity of a D-metric, respectively, and they have proved several common fixed point theorems by using the weakly commuting pairs of self-mappings, compatible pairs of self-mappings of type (A) in a D-metric space and the weak continuity of a D-metric.

In this paper, we use the concept of compatible mappings of type (P) and compare these mappings with compatible mappings and compatible mappings of type (A) in D-metric spaces. In the sequel, we drive some relations between these mappings. Also, we prove a coincidence point a common fixed point theorem for compatible mappings of type (P) in D-metric spaces.

Definitions [1]: A sequence $\{x_n\}$ in a D-metric space (X, D) is said to be convergent to a point x ∈ X, denoted by $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} D(x_n,x,z) = 0$ for all z ∈ X. The point x is said to be limit of sequence $\{x_n\}$ in X.

Definition [2]: A sequence $\{x_n\}$ in a D-metric space (X,D) is called a Cauchy sequence if $D(x_m,x_n,z) \to 0$ as n, m → ∞ for all z ∈ X.

Definition [3]: A D-metric space in which every Cauchy sequence is convergent is called complete.

Remark [1]: In a D-metric space (X, D) a convergent sequence need not be a Cauchy sequence, but every convergent sequence is a Cauchy sequence when the D-metric D is continuous on X.

Definition [4]: Let S and T be mappings from a D-metric space (X,D) into itself. The mappings S and T are said to be compatible if $\lim_{n\to\infty} D(STx_n, TSx_n,z) = 0$ for all z ∈ X, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t ∈ X.

Definition [5]: Let S and T be mappings from a D-metric space (X,D) into itself. The mappings S and T are said to be compatible of type (A) if $\lim_{n\to\infty} D(STx_n, TTx_n,z) = 0$ and $\lim_{n\to\infty} D(STx_n, SSx_n,z) = 0$ for all z ∈ X, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t ∈ X.

Definition [6]: Let S and T be mappings from a D-metric space (X,D) into itself. The mappings S and T are said to be compatible of type (P) if $\lim_{n\to\infty} D(SSx_n, TTx_n,z) = 0$ for all z ∈ X, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t ∈ X.
The following propositions show that Definition [3.5] & [3.6] are equivalent under some conditions:

Proposition [1]: Let S and T be compatible mappings of type(P) from a D-metric space (X, D) into itself. If St = Tt for some t in X, Then STt = SSt = TTt = TSt.

Proof: Suppose that $\{x_n\}$ is a sequence in X defined by x_n = t , n = 1,2,3,... and St = Tt . Then we have $\text{lin}_{n\to\infty}$ Sxn = $\lim_{n\to\infty}$ Txn = St. Since S and T are compatible mappings of type (P), we have
$D(SSt, TTt, z) = \lim_{n\to\infty} D(SSx_n, TTx_n, z) = 0.$

Hence we have SSt = TTt. Therefore, STt = SSt = TTt = TSt.

Let R^+ denote the set of all non-negative real numbers and F be the family of mappings $\phi : (R^+)^5 \to R^+$ such that each ϕ is upper-semi-continuous, non-decreasing in each coordinate variable, and for any $t > 0$, $\gamma(t) = \phi(t,t,a_1t,a_2t,t) < t$, where $\gamma : R^+ \to R^+$ is a mapping with $\gamma(0) = 0$ and $a_1+ a_2 = 3$.

We have prove the following theorems:

Theorem [1.1]: Let A, B, S and T be mappings from a complete D-metric space (X, D) into itself , satisfying the following conditions:

[1.1] $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

[1.2] $S(X) \cap T(X)$ is a complete subspace of X.

[1.3] $[1+p\{D(Ax,Sx,z) + D(By,Ty,z)\}]\, D(Ax,By,z) \le p[D^2(Ax,Sx,z) +D^2(By,Ty,z)] + \phi(D(Sx,Ty,z), D(Ax,Sx,z), D(By,Ty,z), (Ax,Ty,z), D(By,Sx,z))$

for all $x,y,z \in X$, where $\phi \in F$. Then the pairs A, S and B, T have a coincidence point in X.

For our theorems, we need the following LEMMAS:

Lemma [1]: For every $t > 0$, $\gamma(t) < t$ if and only if $\lim_{n\to\infty} \gamma^n(t) = 0$, where γ^n denotes the n-times composition of γ.

Lemma [2] : Let A, B, S and T be mappings from a complete D-metric space (X, D) into itself , satisfying the conditions [3.1.1], [3.4.3]. Then we have the following :

(a) For every $n \in N0$, $D(y_n,y_{n+1},y_{n+2}) = 0$,

(b) For every $i, j, k \in N0$, $D(y_i, y_j, y_k) = 0$, where $\{y_n\}$ is the sequence in X defined by [1.4].

Proof of the Lemma: (a) By(3.1.1) since $A(X) \subset T(X)$, for any arbitrary point $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for any arbitrary point $x_1 \in X$, there exists a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

[1.4] $y_{2n} = Tx_{2n+1} = Ax_{2n}$ and $y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$ for $n = 0,1,2, \dots$

In [1.3], taking $x = x_{2n+2}$, $y = x_{2n+1}$, $z = x_{2n}$ we have,

$[1+p\{D(Ax_{2n+2},Sx_{2n+2},y_{2n}) + D(Bx_{2n+1},Tx_{2n+1},y_{2n})\}]\, D(Ax_{2n+2},Bx_{2n+1},y_{2n})$
$\le p[D^2(Ax_{2n+2},Sx_{2n+2},y_{2n}) +D^2(Bx_{2n+1},Tx_{2n+1},y_{2n})]$
$+\phi(D(Sx_{2n+2},Tx_{2n+1},y_{2n}), D(Ax_{2n+2},Sx_{2n+2},y_{2n}), D(Bx_{2n+1},Tx_{2n+1},y_{2n}), D(Ax_{2n+2},Tx_{2n+1},y_{2n}), D(Bx_{2n+1},Sx_{2n+2},y_{2n}))$

$[1+p\{D(y_{2n+2},y_{2n+1},y_{2n}) + D(y_{2n+1},y_{2n},y_{2n})\}]\, D(y_{2n+2},y_{2n+1},y_{2n})$
$\le p[D^2(y_{2n+2},y_{2n+1},y_{2n}) +D^2(y_{2n+1},y_{2n},y_{2n})]$
$+ \phi(D(y_{2n+1},y_{2n},y_{2n}), D(y_{2n+2},y_{2n+1},y_{2n}), D(y_{2n+1},y_{2n},y_{2n}), D(y_{2n+2},y_{2n},y_{2n}), D(y_{2n+1},y_{2n+1},y_{2n}))$

$[1+p\{D(y_{2n+2},y_{2n+1},y_{2n}) + 0\}]\, D(y_{2n+2},y_{2n+1},y_{2n})$
$\le p[D^2(y_{2n+2},y_{2n+1},y_{2n}) + 0] + \phi(0, D(y_{2n+2},y_{2n+1},y_{2n}), 0, 0, 0)$

$D(y_{2n+2},y_{2n+1},y_{2n}) \le \phi(0, D(y_{2n+2},y_{2n+1},y_{2n}), 0, 0, 0) < D(y_{2n+2},y_{2n+1},y_{2n})$.

which is a contradiction. Thus we have $D(y_{2n+2},y_{2n+1},y_{2n}) = 0$, similarly , we have $D(y_{2n+1},y_{2n},y_{2n-1}) = 0$.

Hence , for n = 0,1,2 …., we have [1.4] $D(y_{n+2},y_{n+1},y_n) = 0$.

(b) For all $z \in X$, let $d_n(z) = D(y_n, y_{n+1},z)$ for n = 0,1,2,….. .

By (a), we have

$D(y_n, y_{n+2},z) \le D(y_n, y_{n+2}, y_{n+1}) + D(y_n, y_{n+1},z) + D(y_{n+1}, y_{n+2},z)$

$D(y_n, y_{n+2},z) \le D(y_n, y_{n+1},z) + D(y_{n+1}, y_{n+2},z)$

$D(y_n, y_{n+2},z) \le d_n(z) + d_{n+1}(z)$

Taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in [3.1.3], we have

$[1+p\{D(Ax_{2n+2},Sx_{2n+2},z) + D(Bx_{2n+1},Tx_{2n+1},z)\}]\, D(Ax_{2n+2},Bx_{2n+1},z) \le p[D^2(Ax_{2n+2},Sx_{2n+2},z) +D^2(Bx_{2n+1},Tx_{2n+1},z)] +\phi(D(Sx_{2n+2},Tx_{2n+1},z), D(Ax_{2n+2},Sx_{2n+2},z), D(Bx_{2n+1},Tx_{2n+1},z),D(Ax_{2n+2},Tx_{2n+1},z), D(Bx_{2n+1},Sx_{2n+2},z))$

$[1+p\{D(y_{2n+2},y_{2n+1},z) + D(y_{2n+1},y_{2n},z)\}]\, D(y_{2n+2},y_{2n+1},z) \le p[D^2(y_{2n+2},y_{2n+1},z) +D^2(y_{2n+1},y_{2n},z)] + \phi(D(y_{2n+1},y_{2n},z), D(y_{2n+2},y_{2n+1},z), D(y_{2n+1},y_{2n},z), D(y_{2n+2},y_{2n},z), D(y_{2n+1},y_{2n+1},z))$

[1.5] $[1+p\{d_{2n+1}(z) + d_{2n}(z)\}]\, d_{2n+1}(z) \le p[D^2_{2n+1}(z) +D^2_{2n}(z)] + \phi(d_{2n}(z),d_{2n+1}(z), d_{2n}(z), \{d_{2n}(z)+d_{2n+1}(z)\}, 0)$

Now, we shall show that $\{ d_n(z)\}$ is a non increasing sequence in R^+ . In fact, let $d_{n+1}(z) > d_n(z)$ for some n.

By [1.5] we have, $d_{2n+1}(z) < d_{2n+1}(z)$, which is a contradiction in R^+.

Now, we claim that $d_n(y_m) = 0$ for all non negative integers m, n.

Case 1. $n \ge m$. Then we have $0 = d_m(y_m) \ge d_n(y_m)$.

Case 2. $n < m$. By (M_4), we have

$d_n(y_m) \le d_n(y_{m-1}) +d_{m-1}(y_n) \le d_n(y_{m-1}) + d_n(y_n) = d_n(y_{m-1})$

By using the above inequality repeatedly, we have

$d_n(y_m) \le d_n(y_{m-1}) \le d_n(y_{m-2}) \le \dots\dots \le d_n(y_n) = 0$,

which completes the proof of our claim.

Finally, let i, j, and k be arbitrary non-negative integers. We may assume that $i < j$. By (M_4), we have

$D(y_i,y_j,y_k) \le d_i(y_j) + d_i(y_k) + D(y_{i+1}, y_j, y_k) = D(y_{i+1},y_j, y_k)$.

Therefore, by repeatition of the above inequality, we have

$D(y_i,y_j,y_k) \le D(y_{i+1},y_j, y_k) \le \dots\dots \le D(y_i,y_j, y_k) = 0$.

This completes the proof.

Lemma [3]: Let A, B, S and T be mappings from a D-metric space (X, D) into itself satisfying the following conditions [1.1] and [1.3]. Then the sequence $\{y_n\}$ defined by [1.4] is a Cauchy sequence in X.

Proof of the Lemma: In the proof of LEMMA [2], since $d_n(z)$ is a non increasing sequence in R^+ , by [1.3], we have ,

$[1+p\{D(Ax_2,Sx_2,z) + D(Bx_1,Tx_1,z)\}]\, D(Ax_2,Bx_1,z) \le p[d^2(Ax_2,Sx_2,z) +d^2(Bx_1,Tx_1,z)] + \phi(D(Sx_2,Tx_1,z), D(Ax_2,Sx_2,z),D(Bx_1,Tx_1,z), D(Ax_2,Tx_1,z), D(Bx_1,Sx_2,z))$

$[1+p\{D(y_2,y_1,z) + D(y_1,y_0,z)\}]\, D(y_2,y_1,z) \le p[d^2(y_2,y_1,z) +d^2(y_1,y_0,z)]+ \phi(D(y_1,y_0,z), D(y_2,y_1,z), D(y_1,y_0,z), D(y_2,y_0,z),D(y_1,y_1,z))$

$[1+p\{d_1(z) + d_0(z)\}]\, d_1(z) \le p[d^2_1(z) +d^2_0(z)] + \phi(d_0(z), d_1(z), d_0(z), \{d_0(z)+d_1(z)\}, 0)$

$d_1(z) \le \phi(d_0(z), d_0(z), d_0(z), \{d_0(z)+d_0(z)\}, 0)$

$d_1(z) \le \gamma(d_0(z))$

and $d_2(z) \le \gamma(d_1(z)) \le \gamma(\gamma(d_0(z)) = \gamma^2(d_0(z))$.

In general, we have $d_n(z) \le \gamma^n(d_0(z))$.

Thus, if $d_0(z) > 0$, by LEMMA [3.1] $\lim_{n\to\infty} d_n(z) = 0$. If $d_0(z) = 0$, we have clearly $\lim_{n\to\infty} d_n(z) = 0$ since $d_n(z) = 0$ for n = 1,2, …

Now, we shall prove that $\{ y_n\}$ is a Cauchy sequence in X. Since $\lim_{n\to\infty} d_n(z) = 0$, it is sufficient to show that a subsequence $\{ y_{2n}\}$ of $\{ y_n \}$ is a Cauchy sequence in X. Suppose that the sequence $\{ y_{2n}\}$ is not a Cauchy sequence in X. Then there exist a point $z \in X$, an $\varepsilon > 0$ and strictly increasing sequences $\{m(k)\}$, $\{n(k)\}$ of positive integers such that $k \le n(k) < m(k)$,

[1.6] $(y_{2n(k)},y_{2m(k)},z) \ge \varepsilon$ and $D(y_{2n(k)},y_{(2m-2)(k)}, z) < \varepsilon$

for all k = 1,2,….. By LEMMA[3.2] anD(M_4), we have

$D(y_{2n(k)}, y_{2m(k)}, z) - D(y_{2n(k)}, y_{2m(k-2)}, z) \le D(y_{2m(k-2)},y_{2m(k)}, z) \le d_{2m(k-2)}(z) + d_{2m(k-1)}(z)$

Since $\{D(y_{2n(k)}, y_{2m(k)}, z) - \varepsilon \}$ and $\{ \varepsilon - D(y_{2n(k)}, y_{2m(k-2)}, z)\}$ are sequences in R^+ and $\lim_{n\to\infty} d_n(z) = 0$, we have

[1.7] $\lim_{k\to\infty} D(y_{2n(k)},y_{2m(k)}, z) = \varepsilon$ and $\lim_{k\to\infty} D(y_{2n(k)}, y_{2mk-2}, z) = \varepsilon$

Note that, by (M_4), we have

[1.8] $| D(x,y,a) - D(x,y,b)| \le D(a,b,x) + D(a,b,y)$

for all $x, y, a, b \in X$. Taking $x = y_{2n(k)}$, $y = a$, $a = y_{2m(k-1)}$ and $b = y_{2m(k)}$ in [1..8] and using lemma [2] and [1.7], we have

[1.9] $\lim_{k\to\infty} D(y_{2n(k)}, y_{2m(k-1)}, z) = \varepsilon$.

Once again, by using lemma [2], [1..7] and [1.8], we have

[1.10] $\lim_{k\to\infty} D(y_{2n(k)+1}, y_{2m(k)}, z) = \varepsilon$ and $\lim_{k\to\infty} D(y_{2n(k-1)}, y_{2m(k-1)}, z) = \varepsilon$.

Thus, by [1.3], we have,

[1.11]

$[1+p\{D(Ax_{2m(k)},Sx_{2m(k)},z)+D(Bx_{2n(k+1)},Tx_{2n(k+1)},z)\}]D(Ax_{2m(k)},Bx_{2n(k+1)},z)$
$\le p[d^2(Ax_{2m(k)},Sx_{2m(k)},z) +d^2(Bx_{2n(k+1)},Tx_{2n(k+1)},z)]$
$+ \phi(D(Sx_{2m(k)},Tx_{2n(k+1)},z), D(Ax_{2m(k)},Sx_{2m(k)},z),$
$D(Bx_{2n(k+1)},Tx_{2n(k+1)},z),$
$D(Ax_{2m(k)},Tx_{2n(k+1)},z), D(Bx_{2n(k+1)},Sx_{2m(k)},z))$

$[1+p\{D(y_{2m(k)},y_{2m(k-1)},z) + D(y_{2n(k+1)},y_{2n(k)},z)\}] D(y_{2m(k)}, y_{2n(k+1)},z)$
$\le p[d^2(y_{2m(k)},y_{2m(k-1)},z) +d^2(y_{2n(k+1)},y_{2n(k)},z)] + \phi(D(y_{2m(k-1)},y_{2n(k)},z),$
$D(y_{2m(k)},y_{2m(k-1)},z), D(y_{2n(k+1)},y_{2n(k)},z),D(y_{2m(k)},y_{2n(k)},z),$
$D(y_{2n(k+1)},y_{2m(k-1)},z))$

As $k \to\infty$ in [1.11] and noting that d is continuous, we have

$\varepsilon \le \phi(\varepsilon, 0, 0, \varepsilon, \varepsilon) < \gamma(\varepsilon) < \varepsilon$

which is a contradiction. Therefore, $\{ y_{2n}\}$ is a Cauchy sequence in X and so the sequence $\{ y_n\}$ is a Cauchy sequence in X. This completes the proof.

Proof of the Theorem: By lemma[3], the sequence $\{ y_n\}$ defined by [1.2] is a Cauchy sequence in $S(X) \cap T(X)$. Since $S(X) \cap T(X)$ is a complete subspace of X, $\{y_n\}$ converges to a point w in $S(X) \cap T(X)$. On the other hand, since the subsequences $\{ y_{2n}\}$ and $\{ y_{2n+1}\}$ of $\{y_n\}$ are also Cauchy sequences in $S(X) \cap T(X)$, they also converge to the same limit w. Hence there exist two points u, v in X such that Su = w and Tv = w, respectively.

By [1.3], we have

$[1+p\{D(Au,Su,z) + D(Bx_{2n+1},Tx_{2n+1},z)\}] D(Au,Bx_{2n+1},z)$
$\le p[d^2(Au,Su,z) +d^2(Bx_{2n+1},Tx_{2n+1},z)] + \phi(D(Su,Tx_{2n+1},z),$
$D(Au,Su,z),$
$D(Bx_{2n+1},Tx_{2n+1},z),D(Au,Tx_{2n+1},z),D(Bx_{2n+1},Su,z))$
$[1+p\{D(Au,Su,z) + D(y_{2n+1},y_{2n},z)\}] D(Au,y_{2n+1},z)$
$\le p[d^2(Au,Su,z) + d^2(y_{2n+1},y_{2n},z)] + \phi(D(Su,y_{2n},z), D(Au, Su,z),$
$D(y_{2n+1},y_{2n},z), D(Au,y_{2n},z),D(y_{2n+1},Su,z))$

Since $\lim_{n\to\infty} d_n(z) = 0$ in the proof of Lemma2, letting $n\to\infty$, we have

$[1+p\{D(Au, w,z) + D(w, w,z)\}] D(Au, w,z)$
$\le p[d^2(Au, w,z) +d^2(w, w, z)] + \phi(D(w, w,z), D(Au, w,z),D(w ,w,z),$
$D(Au, w,z), D(w, w,z))$
$D(Au, w,z) \le \phi(0, D(Au, w,z), 0,D(Au, w,z),0) < \gamma (D(Au, w,z)) < D(Au, w,z)$

which is contradiction . Hence Au = w = Sw, that is u is a coincidence of A and S.

Similarly, we can show that v is a coincidence point of B and T.

References

[1] Constantin, A. : Common fixed points of Weakly Commuting Mappings in D-Metric spaces, Math. Japon., 36(3) (1991), 507-514.

[2] Das B.K. and Gupta S. (1975) : An Extension of Banach Contraction Principle Through Retional Expression, Ind.

[3] J. Pure Appl. Math 6, 1455-58.

[4] Fisher , B. : Mathe Sem, Notes, Kobe Univ., 10, 17 – 26(1982).

[5] Gähler S. (1963/64) : D-Metric Raume Und Topologische Struktur, Math. Nachr. 26, 115-148.

[6] Gahler, S. : Zur geometric D-metrische Raume, Rev. Roum. De Math. Pures et Appl., 11(1968), 665-667.

[7] Imdad, M., Khan, M.S and Khan, M. D. : A common fixed point theorem in D-metric spaces, Math. Japon., 36(5)(1991), 907-914.

[8] Iseki, K. , Sharma, P. L. and Sharma, B. K. : Contraction type mappings on D-metric spaces, Math. Japon. 21(1976), 67-70.

[9] Iseki, K. : A property of orbitally continuous mappings on D- metric spaces, Math. Sem. Notes, Kobe Univ., 3(1975), 131-132.

[10] Khan, M. S. : Convergence of sequence of fixed points in D-metric spaces, Indian J. Pure Appl. Math. (10)(1979), 106D-1067.

[11] Khan, M. S. : On fixed point theorems in D-metric spaces. Pubbl. Inst. Math. (BeorgraD) (N. S.) , 41(1980), 107-112.

[12] Khan, M. S. and Swaleh : Results Concerning fixed points in D-metric spaces , Mth. Japon. , 29(1984), 519-525.

[13] Parsai V. and Singh B. (1991) : Some fixed point theorem in D- metric space, Vikarm math.soc. 33-37.

Author Profile

Manoj Kumar Tiwari is in Department of Mathematics ,Pt. Sundarlal Sharma (open) University Chhattisgarh, Bilaspur