Coincidence Point Theorems in D-Metric Spaces

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Abstract: In this paper we used the concept of compatible mappings of type (P) in D-metric space. Our result generalize the result of Parsai V. and Singh B., Fisher and Pathak.

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1. Introduction

In 1992, a new structure of a generalized metric space was introduced by Dhage on the line of ordinary metric space defined as under:

Let R denote the real line and X denoted a nonempty set. Let D : XxXxX → R be a function satisfying properties:

(D₁) D(x, y, z) ≥ 0 for all x, y, z ∈ X, equality holds if and only if x = y = z.
(D₂) D(x, y, z) = D(y, z, x) = .............. ∀ x, y, z ∈ X,
(D₃) D(x, y, z) ≤ D(x, u, z) + D(u, y, z) ∀ x, y, z, u ∈ X ,

The function D is called a D-metric for the space X and (X, D) denotes a D-metric space. Generally the usual ordinary metric is called the distance function. D-metric is called diameter function of the points of X (Dughe).

In the last three decades, a number of authors have studied the aspects of fixed point theory in the setting of D-metric spaces. They have been motivated by various concepts already known for metric space and have thus introduced analogous of various concepts in the framework of the D-metric spaces. Khan, Murthy-Chang-Cho-Sharma and Naidu-Prasad introduced the concepts of weakly commuting pairs of self mappings, compatible pairs of self mapping of type (A) in a D-metric space and notion of weak continuity of a D-metric, respectively, and they have proved several common fixed point theorems by using the weakly commuting pairs of self-mappings, compatible pairs of self-mappings of type (A) in a D-metric space and the weak continuity of a D-metric.

In this paper, we use the concept of compatible mappings of type (P) and compare these mappings with compatible mappings and compatible mappings of type (A) in D-metric spaces. In the sequel, we drive some relations between these mappings. Also, we prove a coincidence point a common fixed point theorem for compatible mappings of type (P) in D-metric spaces.

Definitions [1]: A sequence {xₙ} in a D-metric space (X, D) is said to be convergent to a point x ∈ X, denoted by limₙ→∞ xₙ = x, if limₙ→∞ D(xₙ,xₙ,z) = 0 for all z ∈ X. The point x is said to be limit of sequence {xₙ} in X.

Definition [2]: A sequence {xₙ} in a D-metric space (X,D) is called a Cauchy sequence if D(xₙ,xₙ,z) → 0 as n, m → ∞ for all z ∈ X.

Definition [3]: A D-metric space in which every Cauchy sequence is convergent is called complete.

Remark [1]: In a D-metric space (X, D) a convergent sequence need not be a Cauchy sequence, but every convergent sequence is a Cauchy sequence when the D-metric D is continuous on X.

Definition [4]: Let S and T be mappings from a D-metric space (X,D) into itself. The mappings S and T are said to be compatible if \( \lim_{n \to \infty} D(STx_n, TSx_n, z) = 0 \) for all \( z \in X \), whenever \( \{x_n\} \) is a sequence in X such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in X \).

Definition [5]: Let S and T be mappings from a D-metric space (X,D) into itself. The mappings S and T are said to be compatible of type (A) if

\[
\lim_{n \to \infty} D(STx_n, SSx_n, z) = 0
\]

for all \( z \in X \), whenever \( \{x_n\} \) is a sequence in X such that

\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t
\]

for some \( t \in X \).

Definition [6]: Let S and T be mappings from a D-metric space (X,D) into itself. The mappings S and T are said to be compatible of type (P) if

\[
\lim_{n \to \infty} D(SSx_n, TTx_n, z) = 0
\]

for all \( z \in X \), whenever \( \{x_n\} \) is a sequence in X such that

\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t
\]

for some \( t \in X \).

The following propositions show that Definition [3.5] & [3.6] are equivalent under some conditions:

Proposition [1]: Let S and T be compatible mappings of type(P) from a D-metric space (X, D) into itself. If \( S \circ T = T \circ S \) for some \( t \in X \), Then \( STt = SST = Tt = TSt \).

Proof: Suppose that \( \{x_n\} \) is a sequence in X defined by \( x_n = t \), \( n = 1, 2, 3, \ldots \) and \( S \circ T = T \circ S \). Then we have \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = Tt = S \circ T \circ t \). Since S and T are compatible mappings of type (P), we have

\[
D(SSx_n, TTx_n, z) = 0
\]

Hence we have \( SST = Tt \). Therefore, \( S \circ T = SST = Tt = TSt \).
Let $R^*$ denote the set of all non-negative real numbers and $F$ be the family of mappings $\phi: (R^*)^2 \rightarrow R^*$ such that each $\phi$ is upper-semi-continuous, non-decreasing in each coordinate variable, and for any $t > 0$, $\gamma(t) = (\phi(t,a_1,a_2,t) \leq t$, where $\gamma: R^* \rightarrow R^*$ is a mapping with $\gamma(0) = 0$ and $a_1 + a_2 = 3$.

We have proved the following theorems:

**Theorem [1.1]:** Let $A$, $B$, $S$, and $T$ be mappings from a complete D-metric space $(X, D)$ into itself, satisfying the following conditions:

[1.1] $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$.

[1.2] $S(X) \cap T(X)$ is a complete subspace of $X$.

[1.3] $[1 + p[D(Ax,Sx,z) + D(By,Ty,z)]] D(Ax,By) \leq p[D^2(Ax,Sx,z) + D^2(By,Ty,z)] + \phi(D(Sx,Ty,z), D(Ax,Sx), D(By,Ty), (Ax, Ty), d(By,Sx,z))$

for all $x, y, z \in X$, where $\phi$ is F. Then the pairs $A, S$, and $T$ have a coincidence point in $X$.

For our theorems, we need the following LEMMAS:

**Lemma [1]:** For every $t > 0, \gamma(t) < t$ if and only if lim$_{n \to \infty}$ $\gamma^n(t) = 0$, where $\gamma^n$ denotes the n-times composition of $\gamma$.

**Lemma [2]:** Let $A$, $B$, $S$, and $T$ be mappings from a complete D-metric space $(X, D)$ into itself, satisfying the conditions [1.1], [1.2], and [1.3]. Then we have the following:

(a) For every $n \in N_0, D(y_n,y_{n+1},y_{n+2}) = 0$.

(b) For every $i, j, k \in N_0, D(y_i, y_j, y_k) = 0$, where $y_i$ is the sequence in $X$ defined by [1.4].

**Proof of the Lemma:** (a) By (3.1.1) since $A(X) \subseteq T(X)$, for any arbitrary point $x_0 \in X$, there exists a point $x_1 \subset X$ such that $Ax_0 = Tx_1$. Since $B(X) \subseteq S(X)$, for any arbitrary point $x_1 \in X$, there exists a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in $X$ such that

$[1.4] y_n = Tx_{2n+1} = Ax_n$ and $y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$ for $n = 0, 1, 2, \ldots$

In [1.3], taking $x = x_{2n+2}, y = x_{2n+1}, z = x_n$, we have,

$[1 + p[D(Ax_{2n+2},Sx_{2n+2},z) + D(By_{2n+1},Ty_{2n+1},z)]] D(Ax_{2n+2},By_{2n+1}) \leq p[D^2(Ax_{2n+2},Sx_{2n+2},z) + D^2(By_{2n+1},Ty_{2n+1},z)] + \phi(D(Sx_{2n+2},z), D(Ax_{2n+2},z), D(By_{2n+1},z), D(Ax_{2n+2},z), D(By_{2n+1},z), D(By_{2n+1},Ty_{2n+1},z))$

which is a contradiction. Thus we have $D(y_{2n+2},y_{2n+1},y_{2n+2}) = 0$ and similarly, we have $D(y_{2n+1},y_{2n+1},y_{2n+2}) = 0$. Hence, for $n = 0, 1, 2, \ldots$, we have $[1.4] D(y_n,y_{n+1},y_{n+2}) = 0$.

(b) For all $z \in X$, let $d_z(z) = D(y_n, y_{n+1}, z)$ for $n = 0, 1, 2, \ldots$.

By (a), we have

$D(y_n, y_{n+1}, z) \leq D(y_n, y_{n+1}, y_{n+2}) + D(y_{n+1}, y_{n+2}, z)$

$D(y_{n+1}, y_{n+2}, z) \leq d_z(z) + d_{y_{n+2}}(y_{n+2})$.

Taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in [1.3.1], we have

$[1 + p[D(Ax_{2n+2},Sx_{2n+2},z) + D(By_{2n+1},Ty_{2n+1},z)]] D(Ax_{2n+2},By_{2n+1}) \leq p[D^2(Ax_{2n+2},Sx_{2n+2},z) + D^2(By_{2n+1},Ty_{2n+1},z)] + \phi(D(Sx_{2n+2},z), D(By_{2n+1},z), D(By_{2n+1},Ty_{2n+1},z))$

$[1 + p[D(y_{2n+2},y_{2n+2},z) + D(y_{2n+1},y_{2n+2},z)]] D(y_{2n+2},y_{2n+1},z) \leq p[D^2(y_{2n+2},y_{2n+2},z) + D^2(y_{2n+1},y_{2n+2},z)] + \phi(D(y_{2n+2},y_{2n+2},z), D(y_{2n+1},y_{2n+2},z), D(y_{2n+1},y_{2n+2},z), D(y_{2n+1},y_{2n+2},z), 0)$

Now, we shall show that $d_z(z)$ is a non-increasing sequence in $R^*$. In fact, let $d_z(z)$ be some n. By [1.5] we have, $d_z(z) < d_{z+1}(z)$, which is a contradiction in $R^*$.

Thus, we claim that $d_z(z) = 0$ for all non-negative integers $m$, $n$.

Case 1. $n \geq m$. Then we have $0 = d_z(y_m) \geq d_z(y_n)$.

Case 2. $n < m$. By (3.1), we have

$d_z(y_n) \leq d_z(y_{n+1}) + d_z(y_{n+2}) \leq d_z(y_n) + d_z(y_m) = d_z(y_m)$.

By using the above inequality repeatedly, we have

$d_z(y_n) \leq d_z(y_{n+1}) \leq d_z(y_{n+2}) \leq \ldots \leq d_z(y_m) = 0$.

This completes the proof.

**Finaly, let i, j, and k be arbitrary non-negative integers. We may assume that $i < j$. By (3.1), we have

$D(y_j, y_i, y_k) \leq d_j(y_j) + d_j(y_i) + d_j(y_k) = D(y_i, y_j, y_k)$.

Therefore, by repetition of the above inequality, we have $D(y_j, y_i, y_k) \leq \ldots = D(y_i, y_j, y_k) = 0$.

This completes the proof.

**Lemma [3]:** Let $A$, $B$, $S$, and $T$ be mappings from a D-metric space $(X, D)$ into itself satisfying the following conditions [1.1] and [1.3]. Then the sequence $\{y_n\}$ defined by [1.4] is a Cauchy sequence in $X$.

**Proof of the Lemma:** In the proof of LEMMA [2], since $d_z(z)$ is a non-increasing sequence in $R^*$, by [1.3], we have

$[1 + p[D(Ax_{2n},Sx_{2n},z) + D(By_{2n+1},Ty_{2n+1},z)]] D(Ax_{2n},By_{2n+1}) \leq p[D^2(Ax_{2n},Sx_{2n},z) + D^2(By_{2n+1},Ty_{2n+1},z)] + \phi(D(Sx_{2n},z), D(By_{2n+1},z), D(By_{2n+1},Ty_{2n+1},z))$

$[1 + p[D(y_{2n},y_{2n},z) + D(y_{2n+1},y_{2n+1},z)]] D(y_{2n},y_{2n+1},z) \leq p[D^2(y_{2n},y_{2n},z) + D^2(y_{2n+1},y_{2n+1},z)] + \phi(D(y_{2n},y_{2n},z), D(y_{2n+1},y_{2n},z), D(y_{2n+1},y_{2n},z), D(y_{2n+1},y_{2n},z), 0)$

which is a contradiction. Thus, we have $D(y_{2n},y_{2n+1},y_{2n+1}) = 0$ and similarly, we have $D(y_{2n},y_{2n+1},y_{2n+1}) = 0$. Hence, for $n = 0, 1, 2, \ldots$, we have $[1.4] D(y_n,y_{n+1},y_{n+2}) = 0$.

Therefore, by repetition of the above inequality, we have $D(y_j, y_i, y_k) \leq \ldots = D(y_i, y_j, y_k) = 0$.

This completes the proof.

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Now, we shall prove that \( \{ y_n \} \) is a Cauchy sequence in \( X \). Since \( \lim_{n \to \infty} d_n(z) = 0 \), it is sufficient to show that a subsequence \( \{ y_{n_k} \} \) of \( \{ y_n \} \) is a Cauchy sequence in \( X \). Suppose that the sequence \( \{ y_{n_k} \} \) is not a Cauchy sequence in \( X \). Then there exist a point \( z \in X \), an \( \varepsilon > 0 \) and strictly increasing sequences \( \{ m(k) \} \), \( \{ n(k) \} \) of positive integers such that \( k \leq n(k) < m(k) \),

\[
\{ y_{n(k)} \}, \{ y_{m(k)} \} \geq \varepsilon \quad \text{and} \quad D(y_{n(k)}, y_{m(2n(k) + k \cdot z)} < \varepsilon \quad \text{for all} \quad k = 1, 2, \ldots \}
\]

By Lemma [3, 2] and \( D(M_d) \), we have

\[
D(y_{n(k)}, y_{m(k)}) - D(y_{n(k)}, y_{m(2n(k) + k \cdot z)}) \leq D(y_{m(2n(k) + k \cdot z)}, y_{m(k)}) < \varepsilon
\]

Since \( D(y_{m(2n(k) + k \cdot z)}, y_{m(k)}) \) and \( \varepsilon < D(y_{n(k)}, y_{m(2n(k) + k \cdot z)}) \) are sequences in \( \mathbb{R}^2 \) and \( \lim_{n \to \infty} d_n(z) = 0 \), we have

\[
[1.7] \lim_{n \to \infty} D(y_{2n(k)+1}, y_{m(k)}) = \varepsilon \quad \text{and} \quad \lim_{n \to \infty} D(y_{2m(k)+2}, y_{2m(k)-k}) = \varepsilon.
\]

Thus, by [1.3], we have

\[
[1.11] \{ [1+pD(Ax_{2m(k)}, y_{2m(k)}) + D(Bx_{2m(k)+1}, y_{2m(k)+1})] D(Ax_{2m(k)}, y_{2m(k)}) \} \leq \{ p[d^2(Ax_{2m(k)}, y_{2m(k)})] + d^2(Bx_{2m(k)+1}, y_{2m(k)+1})] \}
\]

As \( k \to \infty \) in [1.11] and noting that \( d \) is continuous, we have

\[
\varepsilon \leq \phi(\varepsilon, 0, 0, \varepsilon) < \gamma(\varepsilon) < \varepsilon
\]

which is a contradiction. Therefore, \( \{ y_n \} \) is a Cauchy sequence in \( X \) and so the sequence \( \{ y_n \} \) is a Cauchy sequence in \( X \). This completes the proof.

**Proof of the Theorem:** By lemma [3], the sequence \( \{ y_n \} \) defined by [1.2] is a Cauchy sequence in \( S(X) \cap T(X) \). Since \( S(X) \cap T(X) \) is a complete subspace of \( X \), \( y_n \) converges to a point \( w \) in \( S(X) \cap T(X) \). On the other hand, since the subsequences \( \{ y_{n_1} \} \) and \( \{ y_{n_2} \} \) of \( \{ y_n \} \) are also Cauchy sequences in \( S(X) \cap T(X) \), they also converge to the same limit \( w \). Hence there exist two points \( u, v \) in \( X \) such that \( S = w \) and \( T = w \), respectively.

By [1.3], we have

\[
\{ [1+pD(Au, Su, z) + D(Bx_{2n_1}, T_{2n_1})] D(Au, Su, z) \} \leq \{ p[d^2(Au, Su) + d^2(Bx_{2n_1}, T_{2n_1})] + \phi(Du, T_{2n_1}) \}
\]

\[
[1+pD(Au, Su, z) + D(y_{2n_1}, y_{2n_2})] D(Au, Su, z)
\]

\[
D(y_{2n_1}, y_{2n_2}) \quad \text{(Au, Su, z), D(y_{2n_1}, y_{2n_2}), D(Au, Su, z), D(y_{2n_1}, y_{2n_2}), D(Au, Su, z))
\]

Similarly, we can show that \( v \) is a coincidence point of \( B \) and \( T \).

**References**


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