A Mathematics Letter Lecture Note on Some Variety of Algebraic Γ-Structures

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Abstract: ***Algebraic Γ-structures represent a natural generalization of classical algebraic struc- tures. Results studied in semigroups are particular cases of those studied in Γ-semigroups as every semigroup is a Γ-semigroups but not vice-versa. This research paper is based on the introduction and initiation of rectangular Γ-semigroups, quasi-rectangular Γ-semigroups, total Γ-semigroups, viable Γ-semigroups and idempotent Γ-semigroups. Among lots of results, we prove that a rect- angular Γ-semigroup is the direct product of a left singular and a right singular Γ-semigroups. Moreover, this product is unique up to isomorphism.***

Keywords: Γ-semigroup, rectangular bands, rectangular Γ-semigroup, total Γ-semigroup, viable, Γ-semigroup, quasi-rectangular Γ-semigroup, singular Γ-semigroup, Γ-ideal

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1. Introduction and Fundamental Definitions

The concept of a Γ-ring was given by Nobusawa [18] as a more general form than that of a ring. Thereafter, Barnes [29] studied Γ-rings in a way that takes different approach than to that of Nobu- sawa. Motivated by those generalizations of rings, a lot of algebraists authored generalized results from rings and semigroups to Γ-rings, Γ-semigroups, and other algebraic structures as well. The detailed deliberation on Γ-semigroups was done by certain algebraists which are parallel to those results in semigroup theory, for instance, one can see [15], [16], [17], [19], [21], [30]. Recently, on the globe some new papers appeared, such as , [23], [24], [25], [26], [27]. For some most recent study of the theory, one can refer the elaborative exposition work of Basar et al, [1], [2], [3], [4], [5], [6], [7], [8], [10], [11], [12], [28].

One can see that Γ-semigroup is a generalization of semigroups. Suppose A and B are two nonempty sets. Let S be the set of all mappings from A to B and Γ be the set of all mappings from B to A. Now, the usual mapping product of two elements of S cannot be defined. However, if we consider f, g from S and α, β from Γ, then the usual mapping products $f\,\alpha\, g$ and $\alpha\, f\,\beta$ are defined. Moreover, $f.\ \alpha.\ g \in S$ and $\alpha.\ f.\ \beta \in \Gamma$ and $f.\ \alpha.\ (g.\ \beta.\ h) = f.\ (\alpha.\ g.\ \beta).h = (f.\ \alpha.\ g).\ \beta.\ h$ for all $f,\ g,\ h \in S$ and $\alpha,\ \beta \in \Gamma$. As such, the notion of Γ-semigroup was defined by Sen [13] is a generalization of a semigroup. A Γ-semigroup is ordered triplets (S, Γ,) consisting of two sets S and Γ and a ternary operation S x Γx $S \to S$ with the property that $(axb)yc = ax(byc)$ for all $a,\ b,\ c \in S$ and $x,\ y \in \Gamma$. Let A be a nonempty subset of (S, Γ,·). Then, A is called a sub-Γ-semigroup of (S, Γ,·) if $a\ \gamma\ b \in A$ for all $a,\ b \in A$ and $\gamma \in \Gamma$. Furthermore, a Γ-semigroup S is called commutative if $a.\ \gamma.\ b = b.\ \gamma.\ a$ for all $a,\ b \in S$ and $\gamma \in \Gamma$. If we consider, Γ= {1} in the definition, then one can see that every semigroup is a Γ-semigroup

Example 1.1. [22] *Let S = [0, 1] and* $\Gamma = \{\frac{1}{n}$: *n is a positive integer}. Then, S is a Γ-semigroup under the usual multiplication. Next, let K = [0, 1/2]. We have K is a nonempty subset of S and* $a \cdot \gamma \cdot b \in K$ *for all* $a,b \in K$ *and* $\gamma \in \Gamma$. *Then, K is a sub Γ-semigroup of S.*

The above example shows that every semigroup is a Γ-semigroup and not conversely, and thus, Γ-semigroup is a generalization of semigroup.

The notion of a viable semigroup was introduced by Putcha and Weissglass[14].

Definition 1.1. *A Γ-semigroup S is called viable if* $a\alpha b = b\beta a$ *whenever* $a\alpha b$ *and* $b\beta a$ *are idempotents for* $\alpha,\beta \in \Gamma$.

A group S is called Γ-group if $h\alpha S = S\beta g$ = S for all (h,g) ∈ S^2 and for α,β ∈ Γ.

The concept of idempotent semigroups was introduced by McLean [9].

Definition 1.2: *An idempotent Γ-semigroup or band is a Γ-semigroup S which satisfies* $h^2 = h$ *for all* $h \in S$.

Definition 1.3: *A Γ-semigroup satisfying* $h\alpha b\beta g = h$ ($h\alpha g = h$, $g\beta h = h$) for $\alpha, \beta \in \Gamma$ *is called rectangular (left singular Γ-semigroup, right singular Γ-semigroup) Γ-semigroup. These Γ-semigroups are all idempotent. A left (right) singular Γ-semigroup is rectangular Γ-semigroup.*

Definition 1.4: *A Γ-semigroup S is called total if every element of S can be written as the product of two elements of S, that is,* $S^2 = S$.

Definition 1.5: *Suppose a,b ∈ S. Then, a | b if there exists h, g ∈ S such that* $a\alpha h = g\beta a = b$ *for* $\alpha,\beta \in \Gamma$. Furthermore, the set-valued function R on S is defined as follows:

$$R(h) = \{e \mid e \in E,\ h \mid E\}.$$

The relation δ on H is defined as follows:

$$h\delta g \text{ if } R(h) = R(g).$$

Definition 1.6: *A Γ-semigroup S is called I -indecomposible if it has no proper semilattice de- composition.* Suppose the set-valued functions I and Ī on

Γ-semigroup S are defined as follows for α,β ∈ Γ:

$$I(h,S) = \{e \mid e \in E(S)\} = e\alpha h\beta e,$$

and

$$\bar{I}(h,S) = \{g \mid g \in S, g = g\alpha h\beta g\},$$

respectively, for $\alpha, \beta \in \Gamma$. We denote by E, $I(h)$ and $\bar{I}$ for $E(S)$, $I(h, S)$ and $I(\bar{h}, S)$, respectively, when there is no possibility of ambiguity. Let τ be a congruence on S. If S/τ is a semilattice, τ is called a semilattice congruence on S. Let ρ be the smallest congruence on S and σ denote the relation on S defined by

$$h\sigma g \Longleftrightarrow I(h) = I(g) \text{ for } \alpha \in \Gamma.$$

If $\rho = SxS$, then S is called s-indecomposible. Furthermore, for any congruence τ on a Γ- semigroup S, we denote by $\tau| E$ the restriction of τ to E and by $h\tau$ the equivalence class mod τ containing an element h.

Definition 1.7: *A Γ-semigroup S is quasi-rectangular if and only if E(S) is nonempty and* $e = e\alpha h\beta e$ for every $e \in E(S)$, $h \in S$ and α, β ∈ Γ.

This paper is based on some notions in [20], [14] and [31] in the context of a wide class of the theory of Γ-semigroups.

2. Various Classes of Γ-Semigroups

We now begin proving the main results.

Theorem 2.1: *A rectangular Γ-semigroup is the direct product of a left singular and a right singular Γ-semigroups. Also, this factorization is unique up to isomorphism.*

Proof: Suppose S is a rectangular *Γ*-semigroup. Then, for $h, g \in S$ and α, β ∈Γ, we have the following:

$h\Gamma S \supset h\Gamma(g\Gamma S) = (h\alpha g)\Gamma S \supset (h\beta g)\Gamma(h\Gamma S) = (h\Gamma g\Gamma h)\Gamma S = h\Gamma S,$

We obtain the following:

$$h\alpha g\Gamma S = h\Gamma S \quad (1)$$

and

$$S\Gamma h\alpha g = S\beta g \quad (2)$$

Also, we have the following:

$$(h\Gamma S)\Gamma(g\Gamma S) = (h\Gamma S)\Gamma(g\Gamma h\Gamma S) = (h\Gamma S\Gamma g\Gamma h)\Gamma S = h\Gamma S \quad (3)$$

Dually, we have the following:

$$(S\Gamma h)\Gamma(S\Gamma g) = S\Gamma g \quad (4)$$

Let P(Q) be the set of all subsets of S of the form hΓS(resp. SΓh). Then, P(Q) forms a left(right) singular Γ-semigroup with respect to the usual multiplication induced by that of S by (3) and (4). Suppose

$$f_1 : S \to P(\, f_2 : S \to Q)$$

are the mappings defined as follows:

$$f_1(h) = h\Gamma S(\, f_2(h) = S\Gamma h)$$

Then, by (1), (2), (3) and (4), f_1 *and* f_2 *are onto homomorphisms.*
Suppose

$$r: S \to P \times Q$$

is the mapping defined as follows:

$$r(h) = (\, f_1(h), f_2(h)).$$

Therefore, r is a homomorphism. Consider any element of P Q, i.e., (hΓS, SΓg). It follows by (1) and (2) that

$$r(h,g) = (h\Gamma g\Gamma S, S\Gamma h\Gamma g) = (h\Gamma S, S\Gamma g).$$

Therefore, r is onto.

Also, if

$$r(z) = (h\Gamma S, S\Gamma g),$$

then

$$z\Gamma S = h\Gamma S$$
and
$$S\Gamma z = S\Gamma g$$

Then, by rectangularity, we have the following:
$h\alpha g = (h\beta\ S\gamma h)\theta\ (g\gamma_1 S\gamma_2 g) = (z\alpha S\beta h)\gamma(g\gamma_1 S\gamma_2 z) = z\alpha(S\beta h\gamma g\theta\ S)\gamma_1 z = z.$
for $\alpha,\beta,\gamma,\theta,\gamma_1,\gamma_2 \in \Gamma$*. Therefore, r is an isomorphism between S and PxQ, where P(Q) is left(right) singular.*

Suppose $r^j : S \to P^j \text{ X } Q^j$ *is an isomorphism, where* P^j (Q^j) *is left(right)singular.*
Define $f_3 : S \to P^j$ *and* $f_4 : S \to Q^j$
by $r^j(h) = (f_3(h), f_4(h))$,
therefore, they are onto homomorphisms. If $f_1(h) = f_1(g)$*, that is,* $h\Gamma S = g\Gamma S$*, then*

$$f_3(h\alpha S) = f_3(h)\beta f_3(S) = f_3(h)$$

and

$$f_3(g\gamma S) = f_3(g).$$

Therefore, $f_3(h) = f_3(g)$*, that is,* $h\alpha S = g\beta\ S$*, which follows that*

$$f_3(h\gamma S) = f_3(h)\, f_3(S) = f_3(h)$$

and

$$f_3(g\Gamma S) = f_3(h)$$

Therefore, $f_3(h) = f_3(g)$*. Thus, we have an onto homomorphism:*

$$f : P \to P^j\ (\, f_5 : Q \to Q^j\,)$$

such that

$$f_3 = f\, f_1(\, f_4 = f_5\, f_2).$$

Now, we show that f (f_5*) is one-to-one. Let*

$$h\Gamma S \neq g\Gamma S,\ f(h\Gamma S) = f(g\Gamma S).$$

Then,

$$h\Gamma g\Gamma S = h\Gamma S = g\Gamma S,$$

Therefore,

$$h\gamma g \neq g.$$

But

$$\begin{aligned} f_3(h\Gamma g) &= f f_1(h\Gamma g) = f(h\Gamma g\Gamma S) \\ &= f(h\Gamma S) \\ &= f(g\Gamma S) \\ &= f f_1(g) = f_3(g), \end{aligned}$$

$$\begin{aligned} f_4(h\Gamma g) &= f_5\, f_2(g\Gamma h) \\ &= f_5(S\Gamma h\Gamma g) \\ &= g(S\Gamma g) \\ &= f_5\Gamma\, f_2(g) = f_4(g). \end{aligned}$$

Therefore, $r^j(h\gamma g) = r^j(g)$*, which contradicts the assumption that* r^j *is an isomorphism. Hence, f and* f_5 *are isomorphisms.*

N. B.: The above defined $P(Q)$ is the set of all minimal right(left) Γ-ideals of S.

Lemma 2.1: *A band is rectangular if and only if it satisfies*

$$a\alpha b\beta c = a\gamma c.$$

for $\alpha,\beta,\gamma \in \Gamma$.

Proof: *Suppose the band S satisfies the given identity, then substituting c = a, proves that S is a rectangular Γ-semigroup.*

Conversely, let S be a rectangular band, then

$$a\alpha(b\beta c)\vartheta a = a.$$

for $\alpha,\beta,\theta \in \Gamma$. Therefore, we have the following:

$$a\alpha b\beta c = a\vartheta b\lambda\,(c\gamma a\gamma_1 c) = (a\alpha b\beta c\vartheta a)\gamma c = a\alpha c$$

for $\alpha,\beta,\theta,\gamma,\gamma_1,\lambda \in \Gamma$. This completes the proof.

Lemma 2.2. *A total Γ-semigroup is rectangular if and only if it satisfies the following:*

$$a\alpha b\beta c = a\vartheta c$$

for $\alpha,\beta,\theta \in \Gamma$.

Proof: *Suppose S is total, and*

$$a\alpha b\beta c = a\gamma c.$$

Let $h \in H$, then $h = m\gamma n$ for some elements m,n and $\alpha,\beta,\gamma \in \Gamma$. Then, we have the following:

$$h^2 = (h\alpha g)^2 = (h\beta g)(h\gamma g) = h\alpha(g\beta h)\gamma g = h\gamma g = h.$$

for $\alpha,\beta,\gamma \in \Gamma$. So, S is a band. Thus, by Lemma 2.1, S is a rectangular Γ-semigroup. Since, any rectangular Γ-semigroup satisfies the given identity by Lemma 2.1, the converse part proves.

Lemma 2.3: *Suppose H is a viable Γ-semigroup. If $a\alpha b = e$ E, then $b\alpha e\beta a = e$.*

Proof:

$$(b\alpha e\beta a)2 = b\gamma e\alpha a\beta b\gamma e\theta a = b\alpha e\beta a.$$

Hence, $b\alpha e\beta a \in E$.

But, clearly

$$a\alpha b\beta e = e \in E$$

Hence, $b\alpha e\beta a = a\gamma b\theta e = e$ for $\alpha, \beta, \gamma, \theta \in \Gamma$

Lemma 2.4: Suppose H is a viable Γ-semigroup and $h \in S$ and $e \in E$. Then, h | e if and only if $e \in S\alpha h\beta$ S for $\alpha,\beta \in \Gamma$.

Proof: *If h | e, then by the definition, we have the following:*

$$e \in S\Gamma h\Gamma S$$

Conversely, let $e = s\alpha h\beta t$ with $s, t \in S$. By Lemma 2.3

$$h\alpha t\beta e\gamma s = e$$
$$t\alpha e\beta s\gamma h = e.$$

Hence,

$$h \mid e.$$

Theorem 2.2. *Suppose S is a viable Γ-semigroup. Then, we have the following:*

1) δ is a congruence relation on S containing Green's relation S.
2) S/δ is a semilattice.
3) each δ-class contains at most one idempotent and a Γ-ideal wherever it contains an idem- potent.

Proof: *(i) Obviously, we see that δ is an equivalence relation. We need to prove that δ is right compatible. Let $a\delta b$. If $a\gamma c|e \in E$, then $a\alpha c\beta x = e$ for some $x \in S$ and $\alpha,\beta \in \Gamma$. By Lemma 2.3, we have $c\alpha x\beta e\gamma a = e$. Hence, a e. Thus, b e, so $y\gamma b = e$ for some y S. Thus, $y\alpha b\beta c\gamma x\lambda e\theta a = e$, for $\alpha,\beta,\gamma,\lambda,,\theta$ Γ therefore, $b\gamma c$ e by Lemma 2.4. Hence,*

$$\mathrm{R}(a\gamma c) \in \mathrm{R}(b\beta c).$$

Similarly, $\mathrm{R}(b\gamma c) \subseteq \mathrm{R}(a\gamma c)$ and hence, $a\alpha c\delta b\beta c$,

That δ is left compatible follows analogously. Consequently, δ is a congruence relation on Γ- semigroup S. Hence, we have

(ii) *We need to prove that S/δ is a band.*
Let $a \in S$. If a2 | $e \subseteq E$, then by Lemma 2.4, we have a |e.
Hence,

$$\mathrm{R}(a^2) \subseteq \mathrm{R}(a).$$

Suppose a | e ∈ E, and $a\alpha x = y\beta a = e$, $x,y \in S$ and $\alpha,\beta \in \Gamma$. Therefore,

$$y\alpha a^2\beta x = e.$$

Again, applying Lemma 2.4, a^2 e.
Therefore, $\mathrm{R}(a^2) = \mathrm{R}(a)$ and $a\delta a^2$.
Hence, S/δ is a band.
Now, suppose $a,b \in S$. If $e \in \mathrm{R}(a\gamma b)$, then there exists $x,y \in S$ such that

$$a\alpha b\beta x = y\theta a\lambda b = e.$$

for $\alpha,\beta,\gamma,\theta,\lambda \in \Gamma$. Therefore,

$$y\alpha a\lambda\,(b\beta a)\gamma b\theta x = e,$$

and by Lemma 2.4, $e \in \mathrm{R}(b\gamma a)$. Therefore, $\mathrm{R}(a\alpha b) \in \mathrm{R}(b\beta a)$

By symmetry, we have

$$\mathrm{R}(b\alpha a) \in \mathrm{R}(a\beta b).$$

Hence, $a\alpha b\delta b\beta a$ and S/δ is a semilattice.

(iii) Let $e_1\delta e_2$ with e_1,e_2 E. Then, $e_1 \in \mathrm{R}(e_1) = \mathrm{R}(e_2)$, therefore, $e_2| e_1$. In a similar fashion, $e_1|e_2$. Hence, by Lemma 2.4, $e_1 = e_2$.

Therefore, each δ-class contains at most one idempotent. Now, let A be a δ-class containing an idempotent e. Suppose $a \in$ A. As, $e \in \mathrm{R}(e) = \mathrm{R}(a) = \mathrm{R}(a^2)$, there exists $x \in S$ such that $a^2\gamma x = e$. Now, $a\delta a^2$ a ⇒ $a\alpha x\delta a^2\gamma x$. Therefore, $a\gamma x\delta e\delta a$.

Hence, $a\gamma x \in A$ and $a\alpha(a\beta x) = e$ ⇒ e is a right zeroid of A.

In a similar fashion, e is a left zeroid and by Lemma 2.4, A has a Γ-group Γ-ideal.

Proposition 2.1. *The following assertions are equivalent:*
(i) *$I(h) \cap I(g) = I(h\gamma g)$ for some $h,g \in S$ and $\gamma \in \Gamma$,*
(ii) *$I(h) \cap I(g) = I(h\gamma g)$ for some every $h,g \in S$. In this case, we further have $\bar{I}(h) = I(h)$ for every $h \in S$.*

Proof: (i) ⇒ (ii). *It follows from $\bar{I}(h) \cap E = I(h)$ for every h*

$\in S$.

(ii)⇒ (i). We will prove that $\bar{I}$ (h) = I(h) for every h ∈ S. Let a ∈ I(h). Then, a = aαhβa. Hence,

$$a\alpha h = (a\beta h)\gamma(a\lambda h) = (a\alpha h)\beta\ (a\gamma h)\lambda\ (a\theta h).$$

Therefore, $a\underline{\alpha}h\beta a = (a\gamma h\lambda a)\theta\ (a\gamma_1 h\gamma_2 a) \Rightarrow a = a^2$ *for* $\alpha, \beta, \gamma, \lambda, \theta, \gamma_1, \gamma_2 \in \Gamma$. *So,* $a \in \bar{I}(h) \cap E = I(h)$. *Thus,* $\bar{I}(h) \subseteq I(h)$. *Obviously,* $I(h) \subseteq I(h)$. *Hence,* $I(h) = I(h)$ *for every* $h \in S$.

Proposition 2.2. *Suppose N ⊆S such that $\bar{I}$ (x) = ∅. If N is nonempty, then N is a Γ-ideal of S and idempotent free.*

Proof: *Let N be a nonempty set. It is easy to observe that N is idempotent free. Let x ∈ N and y ∈ S. If x, y ∈/ N, there exists a ∈ H such that a = aαxβγyx. Hence, yαa = (yβa)γxλ (yθa) and therefore, yγa ∈ I(x). This contradicts the fact that $\bar{I}$ = ∅. Therefore, xγy ∈ N. In a similar fashion, yγx ∈ N.*

Lemma 2.5. *Suppose N is an idempotent free Γ-ideal of Γ-semigroup S. Then, S satisfies the following:*

$$I(x,S) \cap I(y,S) = I(x\gamma y,S)$$

for every x,y ∈ S if and only if the Rees factor Γ-semigroup S/N satisfies the following:

$$I(x,S/N) \cap I(y,S/N) = I(x\gamma y,S/N)$$

for every x,y S/N and γ ∈Γ.

Proof: *Suppose $\bar{n}$ be the equivalence class N in S/N. Since, N is idempotent free, we have*

$$E(S/N) = E(S) \cup \{\bar{n}\}.$$

If a, x / N, then a ∈ I (x,S) if and only if a ∈ I (x, S|N).

Furthermore, I ($\bar{n}$, S|N) = $\bar{n}$ and I (z,S) = ∅ for z ∈ N, since N is an idempotent free Γ-ideal of S.

Hence,

$$I(x,S) \cup \{n\} = I(x,S/N)$$

for every x ∈ S, where x = x if x ∈/ N and $\bar{x}$ = $\bar{n}$ if x ∈ N. This proves the Lemma.

Combining Proposition 2.1, Proposition 2.2 and Lemma 2.5, we have the following:

Theorem 2.3. *Suppose E(S) is a nonempty set. Then, the following are equivalent:*

(i) (x,S) ∩ I(y,S) = I(xγy,S) for every x,y∈ S and γ ∈ Γ;

(ii) S is a Γ-ideal extension of an idempotent free Γ-semigroup(possibly empty) by a Γ-semigroup T such that

$$I(x,T) \cap I(y,T) = I(x\gamma y,T),$$

and

$$I(x,T)\ f = \emptyset$$

for every x,y ∈ T.

Theorem 2.4. *The following are equivalent:*

(i) I(x) ∩ I(y) = I(xγy) for every x,y ∈ S and γ Γ;

(ii) (a) σ is a semilattice congruence on S;

(b) each σ-class is either idempotent free or a quasi-rectangular Γ-semigroup;

(iii) H is a semilattice of s-indecomposable Γ-semigroups, each of which is either idempotent free or quasi-rectangular;

(iv) H is a semilattice of Γ-semigroups each of which is either idempotent free or quasi- rectangular. In this case, for a semilattice congruence τ on S induced by the decomposition in (iv), we have ρ ⊆ τ ⊆ σ and ρ | E = τ | E = σ | E. Furthermore, for every a,b ∈ E, we have

$$a\sigma b \Longleftrightarrow a = a\alpha b\beta a$$

and

$$b = b\alpha a\beta b.$$

for α,β ∈ Γ.

Proof: *(i) ⇐⇒ (ii). is straight forward.*

(i) ⇐⇒ (iii). S is a semilattice of s-indecomposable Γ-semigroups. Also, since S satisfies:

I(x) ∩ I(y) = I(xγy)

for some every x, y S, any Γ-subsemigroup of S satisfies also the same. Therefore, if we consider the congruence σ on each component of S, it follows from (ii)(b), that any component is idempotent free or quasi-rectangular. Hence, (iii) holds.

(ii)⇐⇒(iv) and (iii) ⇐⇒(iv) are straightforward.

(iv) ⇐⇒ (i). Let τ be the congruence induced by the decomposition in (iv) and suppose x, y ∈S. If a ∈ I(x) ∩ I(y), we have the following:

$$a = a\alpha x\beta a = a\gamma y\delta a.$$

Since, τ is a semilattice congruence on S, we have

$$a\tau a\alpha x\tau a\beta y.$$

Thus, aαxβy ∈ aτ. Also, a ∈ aγτ ∩ E.

Hence,

$$a = a\alpha(a\beta x\delta y)\theta a = a\alpha x\beta y\gamma a.$$

So,

$$a \in I(x\gamma y).$$

Conversely, if a ∈ I(xγy), we have the following:

$$a = a\alpha x\beta y\gamma a.$$

Hence,

$$a\tau a\alpha x\beta y$$

Thus,

$$A\alpha y\tau a\beta x\gamma y^2\tau a\alpha x\beta y.$$

Hence, aγy ∈ aτ. Since, a ∈ aτ ∩ E, aτaαxβy.

$$a = a\alpha(a\beta y)\gamma a = a\alpha y\beta a.$$

Hence, a ∈ I(y). In a similar fashion, a ∈ I(x). Hence, a ∈ I(x) ∩ I(y). Therefore, I(x) ∩ I(y) = I(xγy), i.e., (i) holds. Now, suppose x,y ∈ S such that xτy. Let a ∈ I(x). Then, a = aαxβa. Hence, aαx ∈ aβxτ ∩ E, and aαy ∈ aγxτ. Since, aγxτ is quasi-rectangular,

$$a\alpha x = (a\beta x)\gamma(a\alpha y)\beta\ (a\gamma x).$$

Hence, $a = a\alpha x\beta a = (a\gamma x)\theta\ (a\lambda y)(a\gamma_1 x)\gamma_2 a = (a\alpha x\beta a)\gamma y\lambda\ (a\theta x\gamma_1 a) = a\gamma_2 y\gamma_3 a$.

Therefore, a ∈ I(y). Thus, I(x) ⊆ I(y). By symmetry, we have I(y) ⊆ I(x). Hence, I(x) = I(y). Thus, xτy. This shows that τ ⊆ σ. On the other hand, clearly, ρ ⊆ τ. Now, let a, b ∈ E. If aσ | E(b), then a,b ∈ I(a) = I(b). Hence, a = aαbβa and b = bαaβb.

Conversely, if a = aαbβa and b = bαaβb, we have aρ | E(b) since ρ is a semilattice congruence on S. On the other hand, ρ ⊆ τ ⊆ σ. Hence, ρ | E = τ | E = σ | E.

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References

[1] Abul Basar, On some power joined Γ-semigroups, International Journal of Engineering, Science and Mathematics, 8(12)(2019), 53-61.
[2] Abul Basar and M. Y. Abbasi, on some properties of normal Γ-ideals in normal Γ-semigroups, TWMS Journal of Applied Engineering and Mathematics, 9(3)(2019), 455-460.
[3] Abul Basar and M. Y. Abbasi, On generalized bi-Γ-deals in Γ-semigroups, 23(2)(2015), 181-186.
[4] Abul Basar and M. Y. Abbasi, On covered Γ-ideals in Γ-semigroups, Global Journal of Pure and Applied Mathe- matics, 13(5) (2017),1465-1472.
[5] Abul Basar, A note on (m, n)-Γ-ideals of ordered LA-Γ-semigroups, Konuralp Journal of Mathematics, 7(1)(2019), 107-111.
[6] Abul Basar, Application of (m, n)-Γ-Hyperideals in Characterization of LA-Γ-Semihypergroups, Discussion Mathematicae General Algebra and Applications, 39(1)(2019), 135-147.
[7] Abul Basar, M. Y. Abbasi and Bhavanari Satyanarayana, On generalized Γ–hyperideals in ordered Γ- semihypergroups, Fundamental Journal of Mathematics and Applications, 2(1)(2019), 18-23.
[8] Abul Basar, Shahnawaz Ali, Mohammad Yahya Abbasi, Bhavanari Satyanarayana and Poonam Kumar Sharma, On some hyperideals in ordered semihypergroups, Journal of New Theory, 29(2019), 42-48.
[9] David McLean, Idempotent semigroups, Amer. Math. Monthly, 61(1954), 110-113.
[10] M. Y. Abbasi and Abul Basar, On Ordered Quasi-Gamma-Ideals of Regular Ordered Gamma-Semigroups, (2013) 1-7.
[11] M. Y. Abbasi and Abul Basar, Weakly Prime Ideals in Involution po-Γ-Semigroups, Kyungpook Mathematical Journal, 54(4)(2014), 629-638.
[12] M. Y. Abbasi and Abul Basar, On Generalizations of Ideals in LA-Γ-Semigroups, 39(1)(2015), 1-12.
[13] M. K. Sen. On Γ–semigroups. In Algebra and its applications (New Delhi, 1981), 301-308., Lecture Notes in Pure and Appl. Math., volume 91. Dekker, New York, 1984.
[14] M. S. Putcha and J. Weissglass, A semilattice decomposition into semigroups having at most one idempotent, Pacific J. Math., 39(1)(1971), 1-4.
[15] M. K. Sen and N. K. Saha. On Γ-semigroup, I. Bull. Calcutta Math. Soc., 78(1986)180–186.
[16] M. K. Sen and A, Seth. Radical of Γ-semigroup, Bull. Calcutta Math. Soc., 80(3) 189–196.
[17] M. K. Sen and S. Chattopadhyay, Semidirect Product of a Monoid and a Γ-semigroup, East-West J. of Math. 6, 2 (2004), 131–138.
[18] N. Nobusawa, On a generalization of the ring theory, Osaka J. Math., 1(1964) 81–89.
[19] N. K. Saha. The maximum idempotent separating congruence on an inverse Γ-semigroup. Kyungpook Math. J., 34(1)(2994), 59–66.
[20] N. Kimura, The structure of idempotent semigroups, Pacific J. Math., 8(2)(1958), 1-23.
[21] N. K. Saha, On Γ-semigroup III. Bull. Cal. Math. Soc., 80 (1988), 1–12.
[22] R. Chinram and C. Jirojkul, On bi-Γ-ideals in Γ-semigroups, Songklanakarin J. Sci. Technol., 29(1)(2007), 231- 234.
[23] R. Chinram, On Quasi-gamma-ideals in Gamma-semigroups, ScienceAsia, 32 (2006), 351-353.
[24] R. Chinram and P. Siammai, On Green's relations for Γ-semigroups and redutive Γ-semigroups, Int. J. Algebra, 2(2008)187–195.
[25] R. Chinram and K. Tinpun, Isomorphism theorems for Γ-semigroup and ordered Γ-semigroup, Thai J. Math., 7(2)(2009), 231–241.
[26] S. Chattopadhyay, Right orthodox Γ-semigroup, Southeast Asian Bull. Math., 29(2005), 23– 30.
[27] S. Chattopadhyay, Right inverse Γ-semigroup, Bull. Cal. Math. Soc 93, 6(2001), 435–442.
[28] Satyanarayana Bhavanari, Mohammad Yahya Abbasi, Abul Basar and Syam Prasad Kuncham, International Jour- nal of Pure and Applied Mathematical Sciences,7(1)(2014), 43-49.
[29] W. E. Barnes, On the Γ-rings of Nobusawa. Pacific J. Math., 18(3)(1966), 411–422.
[30] W. Jantanan and T. Changphas, On (m, n)-Regularity of Γ-Semigroups, Thai Journal of Mathematics Volume 13(1) (2015), 137–145.
[31] Z. Goseki, on semigroups in which $x = xyx = xzx$ if and only if $x = xyzx$, Pacific J. Math 60(1)(1975), 2-11.