International Journal of Science and Research (IJSR)
ISSN: 2319-7064
ResearchGate Impact Factor (2018): 0.28 | SJIF (2018): 7.426

Computational Algorithms for the Trefftz Method

Mashukov V.I.

Professor of "General Informatics" Department of The Siberian Transport University, Novosibirsk 630132, Novosibirsk, Zheleznodorozhnaya st., 200, 20

Keywords: Constraints problems, fictitious canonical domains, boundary elements, fictitious canonical domains, boundary integral equations, linear combinations algorithm, boundary condition problems.

1. Introduction

The article suggests methodological approaches to the formation of constraints in mathematical physics problems and algorithms for the numerical solution of these problems. The ideas underlying the methodological approaches and computational algorithms overlap with the ideas of Trefftz method, boundary element methods and methods of boundary integral equations. This is natural, because I began to solve applied problems using these approaches. I knew well the strengths and weaknesses of these methods. Therefore, I conducted research in order to increase the efficiency and accuracy of these methods.

The representation of constraints in the form of a matrix equation with a rectangular one, the transformation of this equation into an equation with a square matrix are considered. An iterative algorithm for solving the matrix equation and an algorithm for suppressing inaccuracies associated with the high-frequency part of the spectrum of the matrix of the equation are proposed

The results of these studies can be applied not only to solve problems of mathematical physics, but also many other linear problems with constraints.

Approaches to solving boundary value problems that are close to those described are considered in [1, 2, 3, 8].

1) Solutions of a constraint problem and equations

A solution to a constraint problem is hereinafter calling a set of functions (algorithms) that allows one to calculate the values of all functions that characterize the solution at any point in the solution definition domain. Further, linear problems are considered. I.e. a linear combination of solutions is also a solution to a constraint problem in which the constraints are a linear combination of constraints with the same coefficients. Each solution of the constraint problem from the set $\{v_i(\boldsymbol{z})\}$ satisfies the system of equations:

$$\boldsymbol{G}\boldsymbol{v}_i(\boldsymbol{z}) = 0, i = 1, \dots, N \quad (1)$$

In the equation:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{f}, \quad (2)$$

the columns of matrix A represent the effect of the solution on the constraints of the problem. $\boldsymbol{x}$ is the vector of coefficients with which the solutions enter into a linear combination. $\boldsymbol{f}$ is the vector of constraints.

$$\boldsymbol{f} = (f_1, \dots, f_M), \quad (3)$$

f_j – the value of the constraint number$j$.

$$\boldsymbol{A} = \begin{pmatrix} v_{11}(\boldsymbol{t}_1) & \cdots & v_{1N}(\boldsymbol{t}_N) \\ \vdots & \ddots & \vdots \\ v_{M1}(\boldsymbol{t}_1) & \cdots & v_{MN}(\boldsymbol{t}_N) \end{pmatrix}, \quad (4)$$

$\boldsymbol{t}_j$ is the point in the solution definition domain, $v_{ij}(\boldsymbol{t}_j)$ is the value of the function that characterizes the solutions at $\boldsymbol{t}_j$. The row of matrix $\boldsymbol{A}$ contains the values of the same function characterizing the solution of the problem at different points in the domain of definition of the solution. In the column of matrix $\boldsymbol{A}$ are the values of different functions characterizing the solution of the problem, at the same point in the domain of definition of the solution.

As can be seen from (4), the matrix $\boldsymbol{A}$ is rectangular and, in the case $N = M$, is usually not symmetric.

Replacing the variable in equation (2) leads to equation (5) with a symmetric (self-adjoint) matrix $\boldsymbol{B}$:

$$\boldsymbol{x} = \boldsymbol{A}^*\boldsymbol{y}, \boldsymbol{B} = \boldsymbol{A}\boldsymbol{A}^*, \quad \boldsymbol{B}\boldsymbol{y} = \boldsymbol{f}, \quad (5)$$

In equation (2), the vector $\boldsymbol{x}$ is the vector of coefficients. In equation (5), the coefficients of the vector $\boldsymbol{y}$ have the same physical meaning as the corresponding coefficients of the vector $\boldsymbol{f}$.

Constraints (coefficients of the vector $\boldsymbol{f}$) can have different meanings. For example, in problems of the theory of elasticity it can be:

- projections of the displacement vector on various vectors;
- force vector coefficients at a point in the solution domain and at the boundary;
- a linear combination of the displacement vector coefficients and the force vector at a point in the solution domain and at the boundary; etc.

2) The algorithm for solving the constraint problem

The boundary problem is solving by minimizing the functional:

$$(\boldsymbol{f} - \boldsymbol{f}_1, \boldsymbol{f} - \boldsymbol{f}_1) \to min, \quad (6)$$

both for equation (2) and equation (5). Where $\boldsymbol{f}_1$ are the boundary values of the approximate solution. In the case of equation (5), it should be borne in mind that the matrix $\boldsymbol{B}$ is self-adjoint.

If we denote the residual $\boldsymbol{f} - \boldsymbol{f}_1$ by $\boldsymbol{f}_{11}$ and add the correction $\boldsymbol{q}$ with coefficient α to the vector $\boldsymbol{y}$, then to determine the coefficient α in the case of equation (5), we obtain the quadratic equation:

$$\boldsymbol{q} = \boldsymbol{B}^*\boldsymbol{f}_{11}, \quad (7)$$

$$\alpha^2(\boldsymbol{B}\boldsymbol{B}^*\boldsymbol{f}_{11}, \boldsymbol{B}\boldsymbol{B}^*\boldsymbol{f}_{11}) + 2 \cdot \alpha \cdot (\boldsymbol{B}\boldsymbol{B}^*\boldsymbol{f}_{11}, \boldsymbol{f}_{11}) + (\boldsymbol{f}_{11}, \boldsymbol{f}_{11}) \to \boldsymbol{min},$$

$$\alpha^2(\boldsymbol{BB^*f_{11}}, \boldsymbol{BB^*f_{11}}) + 2\cdot\alpha\cdot(\boldsymbol{B^*f_{11}}, \boldsymbol{B^*f_{11}}) + (\boldsymbol{f_{11}}, \boldsymbol{f_{11}}) \rightarrow \boldsymbol{min},$$
$$\alpha^2(\boldsymbol{B^2f_{11}}, \boldsymbol{B^2f_{11}}) + 2\cdot\alpha\cdot(\boldsymbol{Bf_{11}}, \boldsymbol{Bf_{11}}) + (\boldsymbol{f_{11}}, \boldsymbol{f_{11}}) \rightarrow \boldsymbol{min}, \quad (8)$$

Substitution (7) makes the quadratic form positive definite, and its value reaches a minimum when:

$$\alpha = -\frac{(\boldsymbol{B^2F_{11}}, \boldsymbol{F_{11}})}{(\boldsymbol{B^2F_{11}}, \boldsymbol{B^2F_{11}(g)})}. \quad (9)$$

The solution is sought by the method of successive approximations. *l* - iteration number:

$$\boldsymbol{p^0 = 0}, \boldsymbol{p^{l+1} = p^l + \alpha^l\cdot Bf_{11}},$$
$$\boldsymbol{f_{11} = Bp^{l+1} - f},$$
$$\alpha^l = -\frac{(\boldsymbol{B^2f_{11}}, \boldsymbol{f_{11}})}{(\boldsymbol{B^2f_{11}}, \boldsymbol{B^2f_{11}})}. \quad (10)$$

Matrix $\boldsymbol{B}$ is square, self-adjoint, but without physical meaning. However, the spectrum of matrix $\boldsymbol{B}$ can be analyzed to determine the solvability of equation (5). In addition, the matrix $\boldsymbol{B}$ is smaller than the matrix $\boldsymbol{A}$ in case when the number of columns in the matrix $\boldsymbol{A}$ is greater than the number of rows. After solving equation (5), the vector of coefficients is calculated using the first of equations (5).

3) Acceleration of convergence of iterations

The iterative process presented in (9), (10) slowly converges if, for example, in the boundary heat conduction problem at the boundary of the deformation domain, mixed boundary conditions are specified (the temperature on the part of the boundary and the temperature gradient on the other part of the boundary), as well as in some cases when only the temperature is set at the boundary.

It is possible to accelerate the convergence of iterations using the following modification of the iterative process [14, 15]:

$$\lambda^0 = 0, \ \boldsymbol{p^0} = 0, \ \boldsymbol{r^0 = BF},$$
$$\boldsymbol{F_{11} = Bp^l - F},$$
$$\lambda^l = \frac{(\boldsymbol{B^2F_{11}}, \boldsymbol{Br^{l-1}})}{(\boldsymbol{Br^{l-1}}, \boldsymbol{Br^{l-1}})},$$
$$\boldsymbol{r^l = BF_{11} - \lambda^l\cdot r^{l-1}},$$
$$\alpha^l = \frac{(\boldsymbol{Br^l}, \boldsymbol{F_{11}})}{(\boldsymbol{Br^l}, \boldsymbol{Br^l})},$$
$$\boldsymbol{p^{l+1} = p^l - \alpha^l\cdot r^l}. \quad (11)$$

4) Filtering high-frequency inaccuracies

As an example of applying the filtering algorithm, the Neumann problem for the system of Lame differential equations is considering below:

$$\mu\cdot\Delta u_i(\boldsymbol{x}) + (\lambda+\mu)\cdot\sum_{j=1}^{2}\frac{\partial^2}{\partial x_i\partial x_j}u_j(\boldsymbol{x}) = 0, \ \ i = 1,2, \quad (12)$$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

The solution to system (12) is representing as the potential of a simple layer []:

$$\boldsymbol{u(x)} = \int_S \boldsymbol{\Gamma(x-y)\cdot\varphi(y)}ds. \quad (13)$$

where $\boldsymbol{\Gamma(x-y)}$is the Kelvin matrix of the fundamental solutions of system (12) [1]:

$$\boldsymbol{\Gamma(x)}_{ij} = c\cdot\delta_{ij}\cdot\ln|\boldsymbol{x}| - d\frac{x_i\cdot x_j}{|\boldsymbol{x}|^2}, \quad (14)$$
$$c = \frac{\lambda+3\cdot\mu}{2\cdot\pi\cdot(\lambda+2\cdot\mu)}, d = \frac{\lambda+\mu}{2\cdot\pi\cdot(\lambda+2\cdot\mu)}$$

Matrix$\boldsymbol{T(n(x)}$withcoefficients:

$$T_{ij}(\boldsymbol{n(x)}) = \lambda\cdot n_i\frac{\partial}{\partial x_j} + \mu\cdot n_j\frac{\partial}{\partial x_i} + \mu\cdot\delta_{ij}\cdot\frac{\partial}{\partial n(x)} \quad (15)$$

Is the stress operator [1, 2]. The vector $\boldsymbol{f(x)} = \boldsymbol{T(n(x)u(x)}$is equal to the force vector on the plane square element with the normal $\boldsymbol{n(x)}$passing through the point$\boldsymbol{x}$. Further,$\boldsymbol{n(x)}$- the normal vector to the contour of the deformation domain, external to it.

After applying operator (15) to potential (13) and calculating the limiting value on the boundary surface *S*, we obtain the integral equation [1]:

$$-\boldsymbol{\varphi(x)} + \boldsymbol{A\varphi(x)} = \boldsymbol{f(x)}, \ \ \boldsymbol{A\varphi(x)} = \int_S \boldsymbol{T(n(x)\Gamma(x-y)\cdot\varphi(y)}ds. \quad (16)$$

where $\boldsymbol{f(x)}$ is the force vector acting on the boundary of the deformation domain.

The spectrum of the integral operator $\boldsymbol{A}$ in (16) has the following characteristic features [1, 2]: eigenvalues are real;

- The spectrum lies in a circle of unit radius;
- The spectrum has two condensation points $\pm(1-2\nu)/2(1-\nu)$, where ν is the Poisson's ratio;
- Three eigenvalues are equal to 1 if the deformation domain is finite, and equal to -1 if the domain is infinite.

In the examples below, the case is considered when unit eigenvalues modulo positive.

When solving equation (16) [3, 4] numerically, the boundary of the deformation region is defined by line segments. The integral operator (6) is replacing by the sum of the values of the integrand at the midpoints of the segment multiplied by the length of this segment. As a result of such discretization, the functions at the boundary of the deformation domain are approximated by finite-dimensional vectors, and the integral operator $\boldsymbol{A}$ by the matrix $\boldsymbol{A_p}$:

$$-\boldsymbol{\varphi_p} + \boldsymbol{A_p\varphi_p} = \boldsymbol{f_p}. \quad (17)$$

Fig. 1 shows the boundary contours:

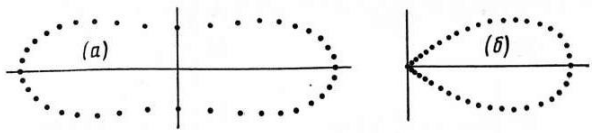

Figure 1: Boundary contours: deformed ellipse and lemniscate.

Fig. 1 shows points approximating a deformed ellipse (a) and a lemniscate (b). Contour points are obtained using conformal mappings of the appearance of a unit circle. To obtain a deformed ellipse, the Zhukovskii function is applied twice: *1.5z + 0.5 / z; 0.9z + 0.4 / z*. The lemniscate is obtained using the map $c\ (z+1)^{1/2}$.

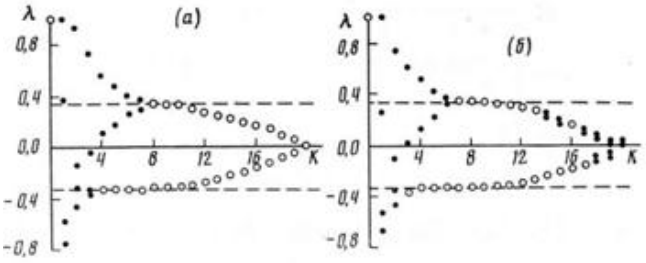

Figure 2: Spectrums of matrices

Fig. 2 shows spectrums of matrices approximating singular integral operators from the boundary equations of the corresponding boundary value problems of the plane theory of elasticity.

For these cases, the eigenvalues and eigenvectors of the matrices A_p were calculated. The eigenvalues of the matrix in the case of an ellipse are shown in Fig. 2a, in the case of lemniscates, in Fig. 2b.

The frequency number of the eigenvector is marked on the horizontal axis, and the corresponding eigenvalue is marked on the vertical axis. By the frequency of the eigenvector as a function of the length of the curve here is meant its similarity to the function sin or cos in the number of maxima or in the number of zero values. Light circles mark two eigenvalues that differ from each other by less than 0.01.

In the case under consideration, the eigenvectors are two-dimensional vector functions; therefore, four eigenvectors and, correspondingly, four eigenvalues correspond to each frequency. Zero frequency corresponds to two unit eigenvalues (hard shift), another unit eigenvalue corresponds to a unit frequency (hard turn). The dashed lines in Fig. 2 shows the expected points of the spectrum condensation.

The results of calculations of the spectrum of the matrices A_p showed the following:

- The eigenvalues of the matrices are valid;
- The spectrum lies in a circle of unit radius;
- Eigenvalues are condensed to points up to the ninth frequency of eigenvectors; three eigenvalues differ from the unit by less than 0.0002 (ellipse) and 0.013 (lemniscate).

Starting from the tenth frequency, the eigenvalues tend to zero, which is a significant difference between the spectrum of the matrix A_p and the spectrum of the integral operator A.

The concentration of the eigenvalues of the high-frequency eigenvectors to zero is explaining by the insufficient number of points for approximating such functions. For functions with such a frequency, it makes no sense to replace the integral with a finite sum without increasing the number of points. For high-frequency eigenvectors, the source of errors in the numerical solution of equation (16) is the spectrum distortion and the lack of points for approximating the high-frequency vectors of the matrix A_p.

To straighten the spectrum of the matrix A_p and at the same time reduce the computational cost of solving equation (16) allows the following technique. Point functions of length K are representing as a linear combination of a system L of orthogonal basic functions, with $L \leq N/2$. For vector functions, projections on the axes Ox_1 and Ox_2 are representing as a linear combination separately.

The high-frequency component of the grid functions is a source of inaccuracies in the numerical solution of boundary-value problems. For example, if a segment of a curve is divided into K parts (elements), then filter frequencies should be higher than $\frac{K}{4}$. This means that if a harmonic function (sine or cosine) specified on the curve segment, then at least four elements must fit into the period of such a function [5].

The filtering algorithm along the length of the curve for the function $\varphi(s)$ defined on the curve segment is as follows:

- $L = \left[\frac{K}{4}\right]$, the brackets indicate the integer part of the number;
- the projections p_{00}, p_{cj}, p_{sj} of the function $\varphi(s)$ onto the functionsфункцииa, $b \cdot cos\left(\frac{2 \cdot \pi \cdot s \cdot j}{S}\right)$и$b \cdot sin\left(\frac{2 \cdot \pi \cdot s \cdot j}{S}\right)$; $a = \frac{1}{\sqrt{K}}$, $b = \sqrt{\frac{2}{K}}$ are calculated:

$$p_{00} = a \cdot \sum_{i=0}^{K} \varphi(s_i),$$

$$p_{cj} = b \cdot \sum_{i=0}^{K} \varphi(s_i) \cdot cos\left(\frac{2 \cdot \pi \cdot (i-0.5) \cdot j}{K}\right), j = 1, \dots, L,$$

$$p_{sj} = b \cdot \sum_{i=0}^{K} \varphi(s_i) \cdot sin\left(\frac{2 \cdot \pi \cdot (i-0.5) \cdot j}{K}\right), j = 1, \dots, L,$$

- The adjusted value φ (s) of the function φ (s) is calculated:

$$\varphi_p(s_i) = a \cdot p_{00} + b \cdot \sum_{j=1}^{L} \left(p_{cj} \cdot cos\left(\frac{2 \cdot \pi \cdot (i-0.5) \cdot j}{K}\right) + p_{si} \cdot sin 2 \cdot \pi \cdot (i-0.5) \cdot jK,\right.$$

$$i = 0, \dots, K.$$

In the operator form, the above filtering algorithm looks as follows:

$$-P\boldsymbol{\varphi} + PA_p\boldsymbol{\varphi} = P\boldsymbol{f}, \boldsymbol{\varphi} = P^*\boldsymbol{\psi},$$

$$-PP^*\boldsymbol{\psi} + PA_pP^*\boldsymbol{\psi} = P\boldsymbol{f},$$

$$-\boldsymbol{\psi} + B\boldsymbol{\psi} = \boldsymbol{f}_p. \quad (18)$$

PP* = ***E*** is a consequence of the orthogonality of the considering functions.

The spectrum of matrix ***B*** from (18) is shown in Fig. 3.

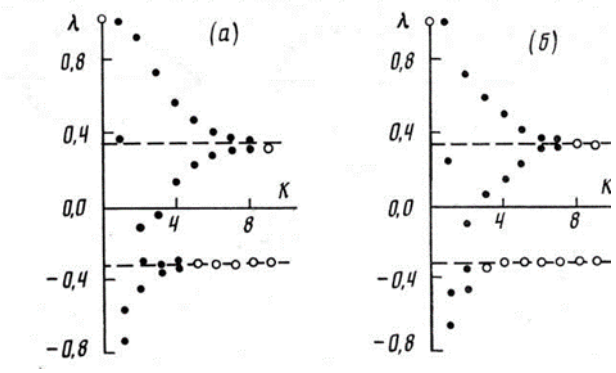

Figure 3: The spectrum of the adjusted matrix

In fig. 3 there is no condensation of eigenvalues to zero.

Таблица 1. Спектр матрицы Виз (18). Граница лемниската

	C	S	C	S
0	1.0032			0.9997
1	-0.6706	0.9789	0.2454	-0.4927
2	-0.1084	0.6994	-0.4821	-0.3684
3	0.0628	0.5762	-0.3561	-0.3467
4	0.1391	0.4886	-0.3350	-0.3347
5	0.2156	0.4097	-0.3332	-0.3331
6	0.2955	0.3624	-0.3323	-0.3319
7	0.3255	0.3400	-0.3303	-0.3294
8	0.3220	0.3299	-0.3266	-0.3252
9	0.3153	0.3217	-0.3202	-0.3184

The above procedure for the correction of the approximate operator influences the low-frequency part of the spectrum in the case of lemniscata, changing the eigenvalues by 0.01-0.02. This is due to the presence of a corner point, since in the case of an ellipse this effect is less than 0.001.

Halving the length of the row of the matrix approximating the integral operator reduces the computational cost when solving equation (17) by the iterative method by 4 times. The question of to what extent it is possible to reduce the length of the row of the matrix or to increase determines by the shape of the boundary contour and the right-hand side in equation (16). Further, in the example of calculating the spectrum of the matrix, we consider a linear combination of solutions corresponding to the application of concentrated forces at points outside the deformation domain and spaced half the length from the middle of the boundary element in the direction of the normal. The spectrum of a completely continuous operator is condensing to zero. Correspondingly, the spectrum of the matrix $\boldsymbol{B}$ from equation (5), which approximates a completely continuous operator, is condensing to zero. But, as shown above, the thickening of the spectrum of the matrix is enhanced by the inaccuracy introduced by replacing the integral by the sum of discrete terms.

Application of the filtering algorithm to the matrix $\boldsymbol{B}$ from equation (5) leads to equation (18):

$$PBy = Pf, y = P^*\varphi,$$
$$PBP^*\varphi = Pf,$$
$$B_P\varphi = f_P. \qquad (18)$$

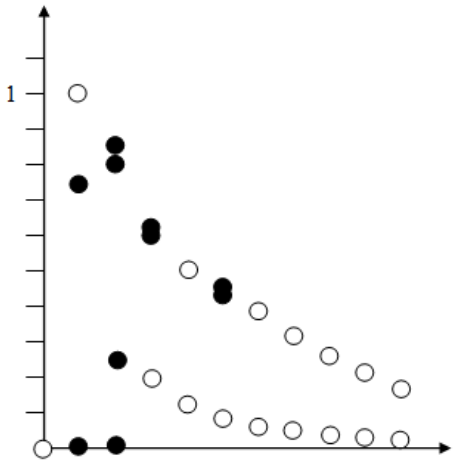

Figure 3: The spectrum of the adjusted matrix $\boldsymbol{B}_P$ for lemniscata

The light circle in the Fig. 3 indicates two close eigenvalues.

Table 2: The spectrum of the matrix $\boldsymbol{B}_P$ from (18). The border of the lemniscate

	C	S	C	S
0	0.		0.	
1	1.1138	1.1015	0.7415	0.
2	0.8657	0.7930	0.2453	0.0087
3	0.6003	0.6234	0.1890	0.1976
4	0.5137	0.5109	0.1232	0.1269
5	0.4585	0.4266	0.0939	0.0892
6	0.3838	0.3662	0.0743	0.0658
7	0.3198	0.3072	0.0581	0.0497
8	0.2735	0.2613	0.0477	0.0372
9	0.2231	0.2157	0.0320	0.0273
10	0.1675	0.1661	0.0245	0.0197

Eigenvalues in tab. 2 ordered by the frequency and likeness of the projection of the eigen function onto the *Ox* axis to the cosine (C, in the middle of the graph extreme) or sine (S, in the middle of the graph zero).

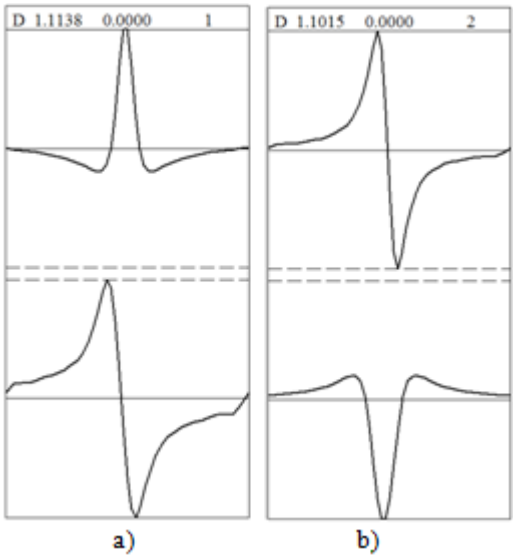

Figure 4: Examples of eigen functions. Frequency 1: a) is like a cosine, b) is like a sine

Three eigenvalues in tab. 2 are zero. They correspond to a rigid shift and rotation. In order for equation (18) to have a solution, the main vector and the main moment of the forces applied at the boundary must be zero. Of the remaining eigenvalues, the minimum is 0.0087. I.e. the spectrum of the matrix separates from zero. If filtering not carried out, then the next forty eigenvalues will thicken around zero, and the matrix will become weakly defined (ill matrix).

2. Acknowledgements

The author is very grateful to Academician A.N. Konovalov for attention to this work and the useful discussion of the results.

References

[1] Yasnitsky L.N. The method of fictitious canonical domains in continuum mechanics. // M.: Science, FIZMATLIT, 1992.128 p.
[2] Gladky S.L. Yasnitsky L.N. Solving linear thermoelasticity problems by the method of fictitious canonical regions // Dynamics and Strength of Machines. Bulletin of the Perm State Technical University. Number 4. Perm: Publishing house of the Perm State Technical University, 2003. S.79-90.
[3] Kostyuchenko C.B. Non-orthogonal series method for imperfectly conjugate elliptic problems with discontinuous coefficients. // Computational technologies. ICT SB RAS, Novosibirsk, Volume 5, N1, 2000, p. 52-64.
[4] Kupradze V.D. Potential methods in elasticity theory. // M.: Fiz.-mat GIZ, 1963, –472 p. With ill.
[5] Mashukov V.I. The outer layer method for solving boundary value problems of the theory of elasticity // Sib. journal calculation mathematics / RAS. Sib. Separation. - Novosibirsk, 2017.- T. 20. No. 3. - S. 289-296.
[6] Mashukov V.I., Osinov V.A. Spectral estimation of the accuracy of counting in problems of the theory of elasticity. MTT. - M .: Science, 1990. No. 4. p. 35-40.
[7] L. Hageman, D Young. Applied iterative methods. // M.: "World", 1986. - 446 p., With ill.
[8] Trefftz E. Mechanic der elastischen Körper. Handbuh der Physic, VI, Springer - Verlag, Berlin, 1928.

Author Profile

Mashukov Vladimir Ivanovich, Professor of "General Informatics" department of The Siberian Transport University, Novosibirsk 630132, Novosibirsk, Zheleznodorozhnaya st., 200, 20