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Ciric Quasi-Contraction Fixed Point Mappings in Generalized Metric Spaces

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Abstract: ***In this article we study the Ciric quasi-contraction fixed point mappings on generalized metric spaces and then we present some examples of them.***

Keywords: fixed point, Ciric quasi-contraction generalized mappings, generalized metric spaces

1. Introduction

Fixed-point theorem or Banach presents, if the complete metric space of (X, d) and $T : X \to X$ be a mapping as it includes $k \in [0, 1)$ for a fixed and $x, y \in X$ for each:
$d(Tx, Ty) \leq kd(x, y)$.

Hence, T includes a unique fixed point of $z \in X$. Besides, for each $x_0 \in X$ the $\{T^n x_0\}$ iterated sequence is convergent to z. Many generalizations from above Banach contraction principle is introduced in recent years. The following generalization is indicated by Ciric (Ciric L. B., 1974).

1.1. Theorem

Suppose that (X, d) is a complete metric space and $T : X \to X$ is a Ciric quasi-contraction map; it means there is a $k \in [0, 1)$, so that for each $x, y \in X$
$d(Tx, Ty)$
$\leq k \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$.

Hence, T includes a unique fixed point of $z \in X$. In addition for each $x_0 \in X$, the $\{T^n x_0\}$ is convergent to z (Amini-Harandi, 2011).

Refer to Amini-Harandi (2011) for other generalization from Banach contraction principle.

2. Main Result

Assume that X is not an empty set and $D : X \times X \to [0, \infty]$ is a map. In this case, we consider the following set for each $x \in X$.
$C(D, X, x) = \{\{x_n\} \subset X : \lim_{n\to\infty} D(x_n, x) = 0\}$.

We call the $\{x_n\}$ sequence in X , D-convergent to x , if $\{x_n\} \in C (D, X, x)$.

2.1 Definition

We call D a generalized meter on X, if the following conditions apply for each $x, y \in X$:
(D_1) If $D(x, y) = 0$ hence ; $x = y$
(D_2) $D(x, y) = D(y, x)$;
(D_3) There is a real fixed $C > 0$ so that we have $x, y \in X$ for each and $\{x_n\} \in C (D, X, x)$ sequence:
$D(x, y) \leq C \limsup_{n\to\infty} D(x_n, y)$.

In this case, the (X, D) paired is a generalized metric space.

Obviously, if the $C (D, X, x)$ set be empty, the (X, D) is a generalized metric space when the (D_1) and (D_2) conditions apply on it. We can easily see that every meter is a generalized meter. For example, assume that X is arbitrary set and Y is the sort of all X to $\mathbb{R}$ functions. $D : Y \times Y \to \mathbb{R}$ function with regulation of $D(f, g) sup|f(x) - g(x)|$ is also a generalized meter on Y.

In this section we want to explain Banach contraction principle on generalized metric spaces.

2.2. Definition

Suppose that (X, D) is a generalized metric space, $f : X \to X$ is a function and $k \in (0, 1)$, we say that f is a $k-$ contraction if we have $x, y \in X$ for each as:
$D(f(x), f(y)) \leq kD(x, y)$.

2.3. Proposition

Assume that for each $f, k \in (0, 1)$, there is $k-contraction$. In this case for each $w \in X$ from f, if $D(w, w) < \infty$, hence $D(w, w) = 0$.

Proof: Suppose that $w \in X$ is a fixed point of f. So that $(w, w) < \infty$, since f is a k-contraction then we have:
$D(w, w) = D(f(w), f(w)) \leq kD(w, w)$, then it results $(w, w) = 0$.

Next, we apply for every $\in X$:

$$\delta(D, f, x) = sup\left\{D\left(f^i(x), f^j(x)\right) : i, j \in \mathbb{N}\right\}.$$

Where $f^i(x)$ the i combination denotation f mapping order is into itself. In the next theorem we describe Banach contraction principle on generalized.

2.4 Theorem

Suppose the following conditions applied (Jleli & Samet, 2015):
(1) (X, D), -D is complete;
(2) for a $k \in (0,1)$, f is a k-contraction;
(3) There is $x_0 \in X$ so that $(D, f, x_0) < \infty$.

Hence, the $\{f^n(x_0)\}$ sequence is convergent to a single fixed point of $w \in X$ from f. Also, if the $w' \in X$ be the other single fixed point of f so that $D(w, w') < \infty$, since $w = w'$.

Most of the metric spaces concept such as sequences, convergent sequence and complement of metric spaces, are definable as they are in generalized metric spaces. The main purpose of this article is to find for the Ciric quasi-contraction mappings on generalized metric spaces, and Ciric quasi-contraction generalized mappings on generalized metric spaces. Then, we will present the mentioned concept for quasi-contraction mappings on generalized metric space and next its generalization.

2.5. Definition

Assume that (X, D) is a generalized metric space and $f: X \to X$ is a self-mapping and $k \in (0,1)$. We say , is $k - quasi\ contraction$ or Ciric quasi-contraction if the following condition applies for each x and y into X:
$D(fx, fy) \leq$
$kmax\ \{D(x,y), D(x,fx), D(y,fy), D(x,fy), D(y,fx)\}$.

2.6. Proposition

Suppose that f is a k-quasi contraction for a $k \in (0,1)$ if $w \in X$ be a fixed point of f which $D(w,w)$, hence $D(w,w) = 0$.

Proof: Assume that $w \in X$ is a fixed point of f so that $D(w,w) < \infty$. Where f is k-quasi contraction, so;
$D(w,w) = D(fw,fw) \leq kD(w,w)$.
Where $k \in (0,1)$, so $D(w,w) = 0$ is obtained.

If we replace in theorem 2.4. the k-quasi contraction condition instead of f k-contraction condition, The following theorem is obtained.
(1) (X, D), is D-complement;
(2) for a $k \in (0,1)$, f is a k-quasi contraction,
(3) there is $x_0 \in X$ so that $\delta\ (D, f, x_0) < \infty$.

Since, the $\{f^n x_0\}$ sequence is convergence to a $w \in X$. If $D(x_0, fw)$ and $D(w, fw) < \infty$, hence w is a fixed point of f. In addition, if $w' \in X$ be the other fixed point f so that $D(w, w') < \infty$ and $(w', w') < \infty$, then $w = w'$.

Proof: Assume that $n \in \mathbb{N}$ where f is a k-quasi contraction mapping, for each $i, j \in \mathbb{N}$, we have:

$$\begin{aligned} &D(f^{n+i}x_0, f^{n+j}x_0) \\ &\leq k \max\{D(f^{n+i-1}x_0, f^{n+j-1}x_0), D(f^{n+i-1}x_0, f^{n+i}x_0) \\ &\quad D(f^{n+i-1}x_0, f^{n+j}x_0), D(f^{n+j-1}x_0, f^{n+j}x_0), \\ &\quad D(f^{n+j-1}x_0, f^{n+i}x_0)\}, \end{aligned}$$

Which results:

$$\delta(D, f, f^n x_0) \leq k\delta(D, f, f^{n-1}x_0) \leq k^2\delta(D, f, f^{n-2}x_0)$$

Therefore, for analysis of each $n \geq 1$, we have:

$$\delta(D, f, f^n x_0) \leq k^n \delta(D, f, x_0)$$

So for every $m, n \in \mathbb{N}$:
$D(f^n x_0, f^{n+m}x_0) \leq \delta(D, f, f^n x_0) \leq k^n \delta(D, f, x_0)$.
Whereas $\delta(D, f, x_0) < \infty$ and $k \in (0,1)$, then it obtains:

$$\lim_{n,m\to\infty} D(f^n x_0, f^{n+m}x_0) = 0.$$

Which results $\{f^n x_0\}$ is D-Cauchy sequence. Where (X, D), is D-complete there is a $w \in X$ in a way that $\{f^n x_0\}$ sequence is D-convergent to it.

Now, we assume $D(x_\circ fw) < \infty$. Since we have the following unequal for each of the $m, n \in \mathbb{N}$

$$D(f^n x_\circ, f^{n+m}x_\circ) \leq k^n \delta(D, f, x_\circ). \qquad (2\text{-}1)$$

By application of definition 2.1. (D_3) condition, a $C > 0$ fixed exists so for each of $n \in \mathbb{N}$, we have:

$$D(w, f^n x_0) \leq C \limsup_{m\to\infty} D\,(f^n x_0, f^{n+m}x_0) \leq Ck^n \delta(D, f, x_0). \qquad (2\text{-}2)$$

On the other hand, we have:
$D(fx_0, fw) \leq k \max\{D(x_0, w), D(x_0, fx_0), D(w, fw), D(fx_0, w), D(x_0, fw)\}$.
By application of (2-1) and (2-2) unequal, we obtain:
$D(fx_0, fw) \leq$
$k \max\{C\delta(D, f, x_0), \delta(D, f, x_0), D(w, fw), D(x_0, fw)\}$,
Hence,
$D(f^2 x_0, fw) \leq$
$k^2 \max\{C\delta(D, f, x_0), \delta(D, f, x_0), D(w, fw), D(x_0, fw)\}$.
Following this induction process we observe that for each $n \geq 1$, we have:
$D(f^n x_0, fw) \leq$
$k^n \max\{C\delta(D, f, x_0), \delta(D, f, x_0), D(w, fw), D(x_0, fw)\}$,
In this case, for each $n \geq 1$, we have:

$$\limsup_{n\to\infty} D\,(f^n x_0, fw) \leq kD(w, fw),$$

Whereas $D(x_0, fw) < \infty$ and $\delta(D, f, x_0) < \infty$ by (D_3) condition application, it obtains:

$$D(fw, w) \leq \limsup_{n\to\infty} D\,(f^n x_0, fw) \leq kD(w, fw),$$

Which results $D(fw, w)$, it means w is a fixed point of f. Also by application of 2.6. proposition, we have $D(w, w) = 0$. Finally, we assume w' is another fixed point of f in a way which $D(w, w') < \infty$ and $D(w', w') < \infty$, by application of 2.6. proposition we have $D(w', w')$. Whereas f is k-quasi contraction, it obtains:
$D(w, w') = D(fw, fw') \leq kD(w, w')$,
Which is resulted $w = w'$.

Now, we introduce the generalized Ciric quasi-contraction mapping on generalized metric spaces.

2.7. Definition

Suppose that (X, D) is generalized metric space and $T: X \to X$ is a self-mapping. T is called generalized Ciric quasi-contraction, if the following condition matches with each of $x, y \in X$:

$$\begin{aligned} D(Tx, Ty) \leq \alpha\big(D(x,y)\big) \max\{&D(x,y), D(x,Tx), \\ &D(y,Ty), D(x,Ty), D(y,Tx)\}. \end{aligned}$$

In which $\alpha: [0, \infty] \to [0,1)$ is a mapping.

The following simple example denotes that theorem 1.1. for Ciric generalized mappings do not apply, even if we suppose α is connected and ascendant (Sastry & Naidu, 1980).

2.8. Definition

Suppose $X = [0, \infty]$ is with ordinary meter and $T : X \to X$ is with $Tx = 2x$ criterion. $\alpha: [0, \infty] \to [0,1)$ is introduced with

$\alpha(t) = \frac{2t}{1+2t}$. It is clear that α is connected and ascendant and for each $x, y \in X$,

$$|Tx - Ty| \leq \alpha(|x - y|) \max\{|x - y|, |x, -Tx|, |y - Ty|, |x - Ty|, |y - Tx|\}.$$

But T does not have fixed point.

Now there is a common question, what condition should to be imposed to $T\ or\ \alpha$ to certify the existence of fixed point for T? for finding the answer of this question and the application of quasi-contraction mappings refer to Ciric, Hussain and Cakic (2010).

The next theorem which presented with a brief change here, shows that a generalized Ciric self-mapping on generalized metric space has a fixed point (Kiany & Amini-Harandi, 2013).

2.9. Theorem

Suppose (X, D) is a complete generalized metric space and $T : X \to X$ is a generalized Ciric quasi-contraction self-mapping in a way that for each $r \in [0, \infty)$,

$$\limsup_{t \to r} \alpha(t) < 1.$$

Suppose that there is a $x_0 \in X$, with boundary circuit, it means that $\{T^n x_0\}$ is boundary. If for each $\in X$, $D(x, Tx) < \infty$, hence T has a fixed point of $x' \in X$ and $\lim_{n\to\infty} T^n x_0 = x'$. In addition, if y' be another fixed point of T, hence $D(x', y') = 0$ or $x' = y'$.

Proof: If for a $n_0 \in \mathbb{N}$, $T^{n_0 - 1}x_0 = T^{n_0}x_0 = T(T^{n_0-1}x_0)$, hence for each $\geq n_0$, $T^n x_0 = T^{n_0-1}x_0$. therefore $(T^{n_0-1}x_0)$ is a fixed point of T and $\{T^n x_0\}$ sequence is convergent to $T^{n_0-1}x_0$ and proof is all. Now suppose that for each $n \in \mathbb{N}$, $T^n x_0 \neq T^{n-1}x_0$, then we indicate that $c \in (0,1)$ exists in a way which for each $n = 0,1,2, \ldots$ we have:

$$\alpha(D(T^{n-1}x_0, T^n x_0)) < c. \qquad (2\text{-}3)$$

Suppose the posterior argument for a subsequence $\{n_k\}$ from cardinal number

$$\lim_{k\to\infty} \propto \left(D(T^{n_k-1}x_0, T^{n_k}x_0)\right) = 1,$$

For a subsequence of $\{\alpha\left(D(T^{n_k-1}x_\circ, T^{n_k}x_\circ)\right)\} \subset \{\alpha\left(D(T^{n-1}x_\circ, T^n x_\circ)\right)\}$ whereas the $\{\alpha\left(D(T^{n-1}x_\circ, T^n x_\circ)\right)\}$ sequence is boundary, $\{\alpha\left(D(T^{n_k-1}x_\circ, T^{n_k}x_\circ)\right)\}$ is also a boundary sequence, therefore by passing a subsequence we can assume that it is convergent sequence. Suppose we have:

$$r_0 = \lim_{k\to\infty} D\,(T^{n_k-1}x_0, T^{n_k}x_0)$$

Hence, $\lim_{k\to r_0} \sup \alpha(t) = 1$ which it is contradiction therefore, the (2-3) condition is established. Now we show the $\{T^n x_0\}$ sequence is Cauchy. To prove this claim, first by analysis we indicate which for $n \geq 2$:

$$D(T^{n-1}x_0, T^n x_0) \leq kc^{n-1} \qquad (2\text{-}4)$$

In which k is a boundary for $\{D(x_0, T^{n_k}x_0)\}$ sequence. If $n = 2$, hence we have:

$$D(Tx_0, T^2x_0) \leq \alpha\left(D(x_0, Tx_0)\right) \max\{D(x_0, Tx_0), (Tx_0, T^2x_0), D(x_0, T^2x_0)\}$$
$$= \alpha\left(D(x_0, Tx_0)\right) \max\{D(x_0, Tx_0), D(x_0, T^2x_0)\} \leq kc.$$

Therefore, (2-4) condition satisfies for $n = 2$.

Suppose (2-4) condition is applied on $k < n$. We indicate that it is applied on $n < k$ whereas T is a generalized Ciric quasi-contraction mapping, we have:

$$D(T^{n-1}x_0, T^n x_0) \leq \alpha(D(T^{n-2}x_0, T^{n-1}x_0)u \leq cu$$

In which $u \in \{D(T^{n-2}x_0, T^{n-1}x_0), D(T^{n-2}x_0, T^n x_0)\}$.
If $u = D(T^{n-2}x_0, T^{n-1}x_0)$, (2-4) condition is clear.
Now, suppose $u = D(T^{n-2}x_0, T^{n-1}x_0)$.

In this case, we have:

$$D(T^{n-2}x_0, T^n x_0) \leq cu_1$$

In which

$$u_1 \in \left\{ \begin{matrix} D(T^{n-3}x_0, T^{n-1}x_0), D(T^{n-2}x_0, T^{n-1}x_0), D(T^{n-3}x_0, T^{n-2}x_0), \\ D(T^{n-3}x_0, T^n x_0) D(T^{n-1}x_0, T^n x_0) \end{matrix} \right\}.$$

Again, if $u_1 = D(T^{n-1}x_0, T^n x_0)\ or\ u_1 = D(T^{n-3}x_0, T^{n-2}x_0)$, hence, (2-4) condition is applied. If $u_1 = D(T^{n-2}x_0, T^{n-1}x_0)$ so we have:

$$D(T^{n-1}x_0, T^n x_0) \leq c^2 D(T^{n-2}x_0, T^{n-1}x_0).$$

By analysis, we have:

$$D(T^{n-2}x_0, T^{n-1}x_0) \leq kc^{n-2}.$$

Therefore,

$$D(T^{n-1}x_\circ, T^n x_\circ) \leq kc^n \leq kc^{n-1}.$$

If $u_1 = D(T^{n-3}x_0, T^{n-1}x_0)$ then we have:

$$D(T^{n-1}x_0, T^n x_0) \leq c^2 D(T^{n-3}x_0, T^{n-1}x_0).$$

And if $u_1 = D(T^{n-3}x_0, T^n x_0)$, hence

$$D(T^{n-1}x_0, T^n x_0) \leq c^2 D(T^{n-3}x_0, T^n x_0).$$

Therefore, by the continuous of this process we see that for each $n \geq 2$ the (2-4) condition is satisfied. We result from the (2-4) condition that $\{T^n x_0\}$ is a Cauchy sequence. Whereas (X, D) is complete; there is a $x' \in X$ so that

$$\lim_{k\to\infty} T^n x_0 = x'.$$

Now we denote that x' is a fixed point of T. For indication of this claim, first we denote there is $k \in (0,1)$ so that for each $n \in \mathbb{N}$, we have:

$$\alpha\left(D(x', T^n x_0)\right) < k.$$

For getting to contradiction, suppose for a subsequence n_j,

$$\lim_{j\to\infty} \alpha\left(D(x', T^{nj}x_0)\right) = 1.$$

Whereas $\lim_{j\to\infty} \alpha\left(D(x', T^{n_j}x_0)\right) = 0$ it obtains:

$$\limsup_{t\to 0^+} \alpha(t) = 1.$$

Which it is a contradiction. Whereas T is a generalized Ciric quasi-contraction, we have:

$$\begin{aligned} &D(Tx', T^{n+1}x_0) \\ &\leq \alpha\left(D(x', T^n x_0)\right) \max\{D(x', T^n x_0), D(x', Tx'), \\ &\quad D(T^n x_0, T^{n+1}x_0), D(x', T^{n+1}x_0), D(T^n x_0, Tx')\} \\ &\leq k \max\{D(x', T^n x_0), D(x', Tx'), D(T^n x_0, T^{n+1}x_0), \\ &\quad D(x', T^{n+1}x_\circ), D(T^n x_\circ, Tx')\}. \end{aligned}$$

Hence, we have:

$$D(Tx', x') = \limsup_{n\to\infty} D(Tx', T^{n+1}x_0) \leq k \limsup_{n\to\infty} D(Tx', T^n x_0) = kD(Tx', x'),$$

Which results $D(Tx', x') = 0$, therefore $Tx' = x'$. Now, suppose that $y'\ and\ x'$ are two fixed points of T so that $D(x', y') < \infty$, then we have:

$$\begin{aligned} &D(x', y') = D(Tx', Ty') \\ &\leq \alpha\left(D(x', y')\right) \max\{D(x', y'), D(x', Ty'), D(y', Tx')\} \\ &= \alpha\left(D(x', y')\right) D(x', y') \end{aligned}$$

Therefore, $x' = y'$. Notice that $\alpha(D(x', y')) < 1$.
The following example shows that in 2.9. theorem the $D(x, Tx) < \infty$ condition is necessary for $x \in X$.

2.10. Example

Suppose $D(0,0) = D(\infty,\infty) = 0, X = \{0,\infty\}$ and $D(0,\infty) = \infty$. Also suppose $T: X \to X$ is defined by $T0 = \infty$ and $T\infty = 0$ criterions, hence, we have:

$D(Tx, Ty) \leq \frac{1}{2} D(x,y)$

$\leq \frac{1}{2} \max\{D(x,y), D(x,Tx), D(y,Ty), D(x,Ty), D(y,Tx)\}$.

But T does not have a fixed point for each $x, y \in X$.

2.11. Example

Suppose $D(x,y) = |x - y|, X = [0,\infty]$, for each $x, y \in [0,\infty)$ and $D(x,\infty) = \infty$. For each $x \in [0,\infty)$ and suppose $D(\infty,\infty)$, hence (X, D) is a complete generalized metric space.

Suppose $T: X \to X$ which $Tx = 2x$. For each $x \in [0,\infty)$ and $T\infty = \infty$. The $\alpha: [0,\infty] \to [0,1)$ is defined by $\alpha(t) = \frac{2t}{1+2t}$ criterion for $t \in [0,\infty)$ and $\alpha(\infty) = \frac{1}{2}$. Hence, $|Tx - Ty| \leq \alpha(|x - y|) \max\{|x - y|, |x - Tx|, |y - Ty|, |x - Ty|, |y - Tx|$,

For each $x, y \in X$ and $D(x, Tx) < \infty$. Therefore, all the supposed theorem of 2.9. are applied and T is a unique fixed point. In fact, $x = \infty$ is the only fixed point of T.

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