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A Study on Charpit's Method for Finding the Solution of Nonlinear Partial Differential Equations of First Order with Three Independent Variables

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Abstract: ***In this paper, we have studied the non linear differential equation of first order with three variables. Also we have studied the necessary and sufficient condition that the surface be integral surface of a partial differential equation is that at each point its tangent element should touch the elementary cone of the equation. It is examined that along every characteristic strip of the partial differential equation*** $\phi(x_1, x_2, x_3, z, p_1, p_2, p_3) = 0$, ***the function*** $\phi(x_1, x_2, x_3, z, p_1, p_2, p_3)$ ***is a constant.***

Keywords: Cauchy method, Non-linear partial differential equations, Plaff's equations, Compatibility, Charpit's method

1. Introduction

We consider first order of nonlinear partial differential to present some ideas particularly in nonlinear equations. Nonlinear partial differential is a type of partial differential equations which deals with nonlinear systems. The solutions nonlinear partial differential is essentially used to describe many physical phenomena and also for modeling nonlinear dynamics of different objects in spaces. The concept of Charpit's method often appears in different literature, which is used to study nonlinear partial differential equations in a classical way. In general term, this Charpit's method is used for finding the general solution of a nonlinear partial differential equations and is given in the form of $F(x, y, z, p, q) = 0$ In this work, we have discussed about nonlinear partial differential equation of first degree in three independent variables as $\phi(x_1, x_2, x_3, z, p_1, p_2, p_3) = 0$ where $p_j^i = i^{th}$ order partial derivative of dependent variable Z with respect to independent variable x_j^i for i=1 and j=3. We study about nonlinear partial differential equation of three independent variables x_1, x_2, x_3 with dependent variable Z in which p_1, p_2, p_3 are nonlinear in nature. This paper is mainly concerned on complete integral and the characterization of Cauchy's method for solving nonlinear partial differential equations which indicates problems without uniqueness, compatibility or transversality cannot be true. We discussed two important theorems to prove a complicated geometrical proof for existence and uniqueness of solutions of a Cauchy problem and its compatibility for the particular given equations.

Generalization of Cauchy's method for solving non-linear partial differential equation

Let us take a nonlinear partial differential equation of first order of 7-variables as

$$\phi(x_1, x_2, x_3, z, \frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \frac{\partial z}{\partial x_3}) = 0$$

$$\text{i.e } \phi(x_1, x_2, x_3, z, p_1, p_2, p_3) = 0) \quad \ldots\ldots\ldots(1)$$

Where z is a function of independent variables x_1, x_2, x_3 and $p_i = \frac{\partial z}{\partial x_i}$ for i=1.

A hyper plane passing through the points $P(x_{1_0}, x_{2_0}, x_{3_0}, z_0)$ with its normal parallel to the direction n determined by the direction ratios $(p_{1_0}, p_{2_0}, p_{3_0}, -1)$ is uniquely stated by the set of seven numbers $H(x_{1_0}, x_{2_0} x_{3_0}, z_0, p_{1_0}, p_{2_0}, p_{3_0})$ and this type of set defines a plane element, in which its element satisfies the equation (1) of

$$\phi(x_1, x_2, x_3, z, p_1, p_2, p_3) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

This is termed as integral element of equation (1) at the point $P(x_{1_0}, x_{2_0}, x_{3_0}, z_0)$.

We solve this equation (2) in order to obtain P_3, which is given as

$$p_3 = G(x_1, x_2, x_3, z, p_1, p_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

We obtained the value of p_3 when the values of $x_1, x_2, x_3, z, p_1, p_2$ are known.

If we consider a condition in a certain state that when $x_{1_0}, x_{2_0}, x_{3_0}, z_0, p_{1_0}, p_{2_0}$ are all kept fixed and varying p_3, then we shall come about at the set of plane elements $\{x_{1_0}, x_{2_0}, x_{3_0}, z_0, p_{1_0}, p_{2_0} G(x_{1_0}, x_{2_0}, x_{3_0}, z_0, p_{1_0}, p_{2_0}\}$ which depends only on a single parameter p_3. In a same way as p_3 varies, we determined a set of plane elements all of which passes through the point p_3 and thereby create an envelope of a cone with vertex P. As a result, the cone produced is termed as elementary cone of equation (2).

We assume an equation of surface S as,

$$z = \psi(x_1, x_2, x_3) \quad\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

The function $\psi(x_1, x_2, x_3)$ and its first partial derivatives $\psi_{x_1}(x_1, x_2, x_3), \psi_{x_2}(x_1, x_2, x_3), \psi_{x_3}(x_1, x_2, x_3)$ are continuous in a certain region R of the hyper plane x_1, x_2, x_3, then a tangent element of the surface S is established in the form of $\{x_{1_0}, x_{2_0}, x_{3_0}, \psi(x_{1_0}, x_{2_0}, x_{3_0}), \psi_{x_1}(x_1, x_2, x_3), \psi_{x_2}(x_1, x_2, x_3), \psi_{x_3}(x_1, x_2, x_3)\}$ at the point $\{x_{1_0}, x_{2_0}, x_{3_0}, \psi(x_{1_0}, x_{2_0}, x_{3_0})\}$ when the tangent plane at each point of S determines a plane element .

Theorem I: A necessary and sufficient condition that the surface be integral surface of a partial differential equation is that at each point its tangent element should touch the elementary cone of the equation.

Proof: We assume a parametric equation of curve Ω as follows

$$x_1 = x_1(t), x_2 = x_2(t), x_3 = x_3(t), z = z(t) \quad\ldots\ldots(6)$$

Which lies on the surface (4) of $z(t) = \psi\{x_1(t), x_2(t), x_3(t)\}$ for entire values of t in an proper interval I. Suppose p_0 is a point on this curve defined by the parameter at $t = t_0$, then the direction ratios of the tangent line $p_0 p_1$ are $x_1'(t_0), x_2'(t_0), x_3'(t_0)$, where $x_i'(t_0)$ denote the value of $\frac{dx_i}{dt}$ for $t = t_0$. This direction is perpendicular to the direction ratios of $(p_{1_0}, p_{2_0}, p_{3_0}, -1)$ if

$$p_{1_0} x_1'(t_0) + p_{2_0} x_2'(t_0) + p_{3_0} x_3'(t_0) + (-1)z'(t_0) = 0$$

or

$$z'(t_0) = p_{1_0} x_1'(t_0) + p_{2_0} x_2'(t_0) + p_{3_0} x_3'(t_0).$$

For this purpose, we say that the set

$$\{x_1(t), x_2(t), x_3(t), z(t), p_1(t), p_2(t), p_3(t)\} \ldots (7)$$

of seven real functions satisfy the condition

$$z'(t) = p_1 x_1'(t) + p_2 x_2'(t) + p_3 x_3'(t) \quad\ldots\ldots(8)$$

Determined a strip at the point (x_1, x_2, x_3, z) of the curve Ω. Suppose this type of strip is an integral element of equation (2), then we called it as an integral strip of equation (2). The set of functions (7) is an integral strip of equation (2) produced that these satisfy condition (8) and another condition

$$\phi\{x_1(t), x_2(t), x_3(t), z(t), p_1(t), p_2(t), p_3(t)\} = 0 \quad (9)$$

For all t in I.

If at each point of curve (6), touches a generator of the elementary cone, then, we consider the corresponding strip as a characteristic strip. We determined this characteristic strip by deriving an equation. Obviously, the point $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3 z + dz)$ lies on the tangent plane to the elementary cone at P if the Pfaff's equation

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3 \quad\ldots\ldots\ldots\ldots(10)$$

where p_1, p_2, p_3 satisfies the condition (2) and that allow us to produce an integrable Plaff's equation.

So by differentiating (10) w.r.t. p_1 we have,

$$0 = dx_1 + \frac{dp_2}{dp_1} dx_2 + \frac{dp_3}{dp_1} dx_3 \quad\ldots\ldots\ldots(11)$$

Also, differentiating (2) w.r.t. p_1 we get,

$$\frac{\partial\phi}{\partial p_1} + \frac{\partial\phi}{\partial p_2}\frac{dp_2}{dp_1} + \frac{\partial\phi}{\partial p_3}\frac{dp_3}{dp_1} = 0 \ldots\ldots\ldots \quad (12)$$

Again differentiating (2) w.r.t. p_2 we get,

$$\frac{\partial\phi}{\partial p_2} + \frac{\partial\phi}{\partial p_1}\frac{dp_1}{dp_2} + \frac{\partial\phi}{\partial p_3}\frac{dp_3}{dp_2} = 0 \ldots\ldots\ldots \quad (13)$$

By solving the above four equations in four unknowns for the following the ratios of dx_1, dx_2, dx_3, dz are determined as

$$\frac{dx_1}{\phi_{p_1}} = \frac{dx_2}{\phi_{p_2}} = \frac{dx_3}{\phi_{p_3}} = \frac{dz}{p_1\phi_{p_1} + p_2\phi_{p_2} + p_3\phi_{p_3}} \quad (14)$$

Therefore, along the characteristic strip $x_1'(t), x_2'(t), x_3'(t), z(t)$ must be proportional to ϕ_{p_1}, ϕ_{p_2}, ϕ_{p_3}, $p_1\phi_{p_1} + p_2\phi_{p_2} + p_3\phi_{p_3}$ respectively.

When the parameter t satisfying the condition

$$x_1'(t) = \phi_{p_1}, x_2'(t) = \phi_{p_2}, x_3'(t) = \phi_{p_3} \quad\ldots\ldots \quad (15)$$

Then, we get $z'(t) = p_1\phi_{p_1} + p_2\phi_{p_2} + p_3\phi_{p_3}$ (16)

As p_1 is a function of t along a characteristic strip, thus

$$p_1'(t) = \frac{dp_1}{dx_1}x_1'(t) + \frac{dp_2}{dx_2}x_2'(t) + \frac{dp_3}{dx_3}x_3'(t) = \frac{dp_1}{dx_1}\phi_{p_1} + \frac{dp_2}{dx_2}\phi_{p_2} + \frac{dp_3}{dx_3}\phi_{p_3}$$

Since $\frac{\partial p_i}{\partial x_j} = \frac{\partial p_j}{\partial x_i}$ for each i,j=1,2,3.

$$p_1'(t) = \frac{dp_1}{dx_1}\phi_{p_1} + \frac{dp_2}{dx_2}\phi_{p_2} + \frac{dp_3}{dx_3}\phi_{p_3}$$

Again differentiating (2) with respect to x_1, we get,

$$\frac{\partial\phi}{\partial x_1} + \frac{\partial\phi}{\partial z}p_1 + \frac{\partial\phi}{\partial p_1}\frac{dp_1}{dx_1} + \frac{\partial\phi}{\partial p_2}\frac{dp_2}{dx_1} + \frac{\partial\phi}{\partial p_3}\frac{dp_3}{dx_1} = 0$$

Therefore, the characteristic strip is

$$p_1'(t) = -(\phi_{p_1} + p_1\phi_z),\ p_2'(t) = -(\phi_{p_2} + p_2\phi_z),\ p_3'(t) = -(\phi_{p_3} + p_3\phi_z) \quad \ldots\ldots\ldots (17)$$

From the above equations (15), (16) and (17), we obtained the following system of seven ordinary differential equations for the determination of characteristic strip

$$x_1'(t) = \phi_{p_1},\ x_2'(t) = \phi_{p_2},\ x_3'(t) = \phi_{p_3}$$

$$z'(t) = p_1\phi_{p_1} + p_2\phi_{p_2} + p_3\phi_{p_3}$$

$$p_1'(t) = -(\phi_{p_1} + p_1\phi_z),\ p_2'(t) = -(\phi_{p_2} + p_2\phi_z),\ p_3'(t) = -(\phi_{p_3} + p_3\phi_z)$$

$$\ldots\ldots\ldots\ldots\ (18)$$

The above following equations are known as the Cauchy's characteristic equation of equation (2). Consequently, when the functions present in equation (18) satisfy a Lipschitz condition there exist a unique solution of equation (18) for a specify set of initial values of variables. Thus, the characteristics of strip is obtained uniquely by initial element $(x_{1_0}, x_{2_0}, x_{3_0}, z_0, p_{1_0}, p_{2_0}, p_{3_0})$ at initial value t_o of t.

Theorem II: Along every characteristic strip of the partial differential equation $\phi(x_1, x_2, x_3, z, p_1, p_2, p_3) = 0$, the function $\phi(x_1, x_2, x_3, z, p_1, p_2, p_3)$ is a constant.

Proof:- Along the characteristics strip, we have

$$\frac{d}{dt}\{\phi(x_1(t), x_2(t), x_3(t), z(t), p_1(t), p_2(t), p_3(t))\}$$
$$= \phi_{x_1}.x_1'(t) + \phi_{x_2}.x_2'(t) + \phi_{x_3}.x_3'(t) + \phi_z.z'(t) + \phi_{p_1}p_1'(t) + \phi_{p_2}p_2'(t) + \phi_{p_3}p_3'(t)$$
$$= \phi_{x_1}.\phi_{p_1} + \phi_{x_2}.\phi_{p_2} + \phi_{x_3}.\phi_{p_3} + \phi_z.(p_1\phi_{p_1} + p_2\phi_{p_2} + p_3\phi_{p_3})$$
$$-\phi_{p_1}(\phi_{x_1} + p_1\phi_z) - \phi_{p_2}(\phi_{x_2} + p_2\phi_z) - \phi_{p_3}(\phi_{x_3} + p_3\phi_z)$$
$$= 0$$

Proved that $\phi(x_1, x_2, x_3, z, p_1, p_2, p_3) = 0$ is constant along the strip.

Compatible system of first order nonlinear partial differential equations

If every solution of the first order nonlinear partial differential equation

$$\phi(x_1, x_2, x_3, z, p_1, p_2, p_3) = 0 \quad \ldots\ldots\ldots (1)$$

Is also the solution of the equation

$$\psi(x_1, x_2, x_3, z, p_1, p_2, p_3) = 0 \quad \ldots\ldots\ldots\ldots(2)$$

Then we say ϕ and ψ are compatible.

From equation (18) each ratio becomes

$$\frac{dx_1}{\phi_{p_1}} = \frac{dx_2}{\phi_{p_2}} = \frac{dx_3}{\phi_{p_3}} = \frac{dz}{p_1\phi_{p_1} + p_2\phi_{p_2} + p_3\phi_{p_3}}$$
$$= \frac{dp_1}{-(\phi_{x_1} + p_1\phi_z)} = \frac{dp_2}{-(\phi_{x_2} + p_2\phi_z)} = \frac{dp_3}{-(\phi_{x_3} + p_3\phi_z)},$$

these equations are known as Charpit's method of nonlinear partial differential equations of first order.

2. Conclusion

The modeled of complicated physical phenomena and dynamical processes represented by the solutions of nonlinear partial differential can be effectively better understood. One of the most exciting fact is that with the advancement of nonlinear science, scientists, researchers, physicists and mathematicians were developed more methods to find out the exact solutions of nonlinear partial differential as nonlinear physical science can provide a large amount of physical aspects of the problem which cause to further applications. In this paper, we solved the solutions of nonlinear partial differential of first order in seven variables and also proved the derivation of compatibility systems along with establishment of Charpit's method for three dimensional space. In addition, this work can also be carry out by Jacobi's method for further research purpose with three or more independent variables. The main idea of Jacobi's method is almost similar to that of Charpit's method but the results system can approach more easily with suitable adjustment for the case of four independent variables and so on.

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