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Convergence via Filter in Locally Solid Riesz Spaces

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Abstract: ***Let (E,τ) be a locally solid vector lattice. A filter F on the set E is said to be convergent to a vector x∈E if, each zero neighborhood set U containing x, U belongs to F. In this paper, we introduce the notion of filter convergence with respect to locally solid topology and study the concept of this convergence and give some basic properties of it..***

Keywords: filter convergence, locally solid Riesz space.

1. Introduction and Main Results

The filter convergence is defined on the natural topological spaces with respect to the neighborhood of limit points; see for example Definition 3.7 (Joshi, 1983). In this paper, we introduce the filter convergence with respect to the locally solid topology. The aim of this paper is to study the concept of this convergence.

Let's give some basic notations and terminologies that will be used in this paper. Let E be real vector space. Then E is called ordered vector space if it has an order relation $\leq$, that means it is reflexive, antisymmetric and transitive, and also it satisfies the axiom: $x \leq y$ implies $x+z \leq y+z$ and $\lambda x \leq \lambda y$ for all $z \in E$ and for every positive real number $\lambda > 0$. An ordered vector space E is said to be vector lattice (or, Riesz space) if, for each pair of vectors $x, y \in E$, the supremum $x \vee y = \sup\{x, y\}$ and the infimum $x \wedge y = \inf\{x, y\}$both exist in E. Also, $x^+ := x \vee 0$,$x^- := (-x) \vee 0$, and $|x| := (-x) \vee x$ are called the positive part, the negative part, and the absolute value of $x$in vector lattice E, respectively. In a vector lattice E, a subset A is called solid if, for $y \in A$ and $x \in E$ with $|x| \leq |y|$, we have $x \in A$. Moreover, two vector x, yin a vector lattice is said to be disjointwhenever $|x| \wedge |y| = 0$; for more details (Aliprantis and Burkinshaw, 2003 and 2006).

By a linear topology τ on a vector space E, we mean a topology τ on E that makes the addition and the scalar multiplication are continuous. Let E be vector lattice E and τ be a linear topology on it. Then (E, τ) is said to be a *locally solid vector lattice* (or, *locally solid Riesz space*) if τ has a base which consists of solid sets. We refer the reader for detail information about these notions; see for example (Aliprantis and Burkinshaw, 1999, 2003 and 2006). As we shall see, this concept is a natural topological condition relating to the vector order. A linear topology τ on a vector space E has a base $\aleph$for the zero neighborhoods satisfying the following properties. For each $V \in \aleph$, we have $\lambda V \in \aleph$ for all scalar $\lambda \leq 1$. For any $V_1, V_2 \in \aleph$ there is another $V \in \aleph$such that $V \subseteq V_1 \cap V_2$. For each $V \in \aleph$there exists another $U \in \aleph$with $U + U \subseteq V$. For any scalar λ and for each $V \in \aleph$, the set λV is also in $\aleph$; see for example (Aliprantis and Burkinshaw, 2003; Aydın, 2018a and b; Troitsky, 2001). So, every locally solid vector lattice satisfies these properties. Also, it follows from Theorem 2.28 (Aliprantis and Burkinshaw, 2003) that a linear topology τ on a vector lattice E is a locally solid iff it is generated by a family of Riesz pseudonorms $\{\rho_j\}_{j \in J}$. A *Rieszpseudonorm* is a real valued map ρ on a vector space E if it satisfies the following conditions; $\rho(x) \geq 0$ for all $x \in E$; if $x = 0$ then $\rho(x) = 0$; $\rho(x+y) \leq \rho(x) + \rho(y)$ for all $x, y \in E$; if $\lim_{n\to\infty} \lambda_n = 0$ in $\mathbb{R}$ then $\rho(\lambda_n x)$ converges to zero in $\mathbb{R}$ for all $x \in E$; if $|x| \leq |y|$ then $\rho(x) \leq \rho(y)$. Moreover, if a family of Riesz pseudonorms generates a locally solid topology τ on a vector lattice E then, for a net (x_α), we have$x_\alpha \xrightarrow{\tau} x$ iff $\rho(x_\alpha - x) \to 0$in $\mathbb{R}$ for each $i \in J$. In this paper, unless otherwise, the pair (E, τ) refers to as a locally solid vector lattice, and the topologies in locally solid vector lattices are generated by families of Rieszpseudonorms$\{\rho_j\}_{j \in J}$. Also, when we mention a zero neighborhood, it means that it belongs to a base that holds the above properties. Recently, there are some studies on locally solid Riesz spaces; see for example (Aydın, 2018a and b; Hong, 2016).

Next, we give the concept of the filter. Let X be a set. A subset $\mathcal{F}$ of the power set of X is said to be *filter* on X if the following properties hold; $\emptyset \notin \mathcal{F}$; if $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$; $\mathcal{F}$ is closed under finite intersections; see for example (Aliprantis and Burkinshaw, 1999). The second condition says that the set X belongs to the filter on it. A filter can be defined thanks to its base. A base $\mathcal{B}$ satisfies the following properties; $\mathcal{B}$ is nonempty; each $B \in \mathcal{B}$ is nonempty; for each $B_1, B_2 \in \mathcal{B}$, there is another $B \in \mathcal{B}$ such that$B \subseteq B_1 \cap B_2$. A nonempty subset $\mathcal{B} \subseteq \mathcal{F}$ is called a *filter base* for a filter $\mathcal{F}$if $\mathcal{F} = \{F \subseteq X : \exists B \in \mathcal{B}, B \subseteq F\}$.

Definition 1. Let (E, τ) be a locally solid vector lattice and $\mathcal{F}$ be a filter on the set E. A vector $x \in E$ is called τ*-limit of* $\mathcal{F}$ (or, $\mathcal{F}\tau$*-converges*to x) if each τ-bounded τ-neighborhood of zero set containing x belongs to $\mathcal{F}$, abbreviated as $\mathcal{F} \xrightarrow{\tau} x$. Moreover, a vector $x \in E$ is said to be a *cluster point* of $\mathcal{F}$ if each τ-bounded zero neighborhood set containing x intersects with every member of $\mathcal{F}$.

One can also give the following definition: x is said to be o-limit of $\mathcal{F}$ (or, $\mathcal{F}$*order converges* to x) if each order bounded τ-neighborhood of zero set containing x belongs to $\mathcal{F}$, abbreviated as $\mathcal{F} \xrightarrow{o} x$.

Remark 2. By Theorem 2.19 (Aliprantis and Burkinshaw, 2003), it can be seen that every order bounded subset is τ-bounded in any locally solid vector lattice. So, the order convergence of filter implies the τ-convergence. For the converse, consider Theorem 2.2 (Hong, 2016), in a locally solid vector lattice(E,τ) with E having an order bounded τ-neighborhood of zero, every τ-bounded subset is order bounded. Thus, the τ-convergence of a filter in such spaces implies the order convergence.

By applying the solid property, we can get some results that not exist in the general filter convergence. For example, if a filter $\mathcal{F} \xrightarrow{o} e$ (respectively, $\mathcal{F} \xrightarrow{\tau} e$) in a locally solid vector lattice (E,τ) then $\mathcal{F} \xrightarrow{o} x$ (respectively, $\mathcal{F} \xrightarrow{\tau} x$) for all $x \in E$ with $|x| \leq |e|$.

After then, we only focus on the τ-convergence of filters because the results for the o-convergence can be obtained in the same way.

Remark 3. Let (E,τ) be a locally solid vector lattice and $x \in E$. Then each neighborhood of zero containing x contains a closed neighborhood of zero containing x; see for example Theorem 1.3 (Schaefer, 1999). So, the τ-closedness can be removed from the definition of the τ-convergence, and so we will use the definition without τ-closedness.

If (E,τ) has the Hausdorff property then the τ-limit is unique. Indeed, suppose $\mathcal{F} \xrightarrow{\tau} x$ and $\mathcal{F} \xrightarrow{\tau} y$with $x \neq y$. So, there exist zero τ-neighbourhoods U of x and V of y such that $U \cap V = \emptyset$. Thus, $\mathcal{F}$ does not contain both $U$and V, and so it can not converge to both x and y. On the other hand, we give some results on the lattice operations in the following proposition.

Theorem 4. Let (E,τ) be a locally solid vector lattice, and $\mathcal{F}$ be a filter on E. Then the followings hold;

1) If $\mathscr{F} \xrightarrow{\tau} e$ for some $e \in E$ then $\mathscr{F} \xrightarrow{\tau} e + x$ for all $x \in E$ whenever e and x are positive, or disjoint;
2) $\mathscr{F} \xrightarrow{\tau} e$ for $e \in E$ implies $\mathscr{F} \xrightarrow{\tau} |e|$;
3) If$\mathscr{F} \xrightarrow{\tau} e^+$ or $\mathscr{F} \xrightarrow{\tau} e^-$ for some $e \in E$ then $\mathscr{F} \xrightarrow{\tau} e$.

Proof:(i) Let U be a τ-bounded zero neighborhood containing $e + x$. So, we show $U \in \mathcal{F}$. Under the condition of positivity of e and x, we have $e, x \in U$ because U is a solid set and $e, x \leq e + x$. Therefore, by using the convergence of $\mathcal{F}$ to e, we get $U \in \mathcal{F}$. On the other hand, if e and x are disjoint then we have $|e + x| = |e| + |x|$; see Theorem 1.7 (Aliprantis and Burkinshaw, 2006). So, by solidness of U, we get $e, x \in U$. Therefore, $U \in \mathcal{F}$ because of $\mathcal{F} \xrightarrow{\tau} e$.

(ii) Suppose $\mathcal{F} \xrightarrow{\tau} e$ and U is a zero neighborhood containing $|e|$. By the formula $|e| \leq \big||e|\big| = |e|$and by the solidness of U, we have $e \in U$. Therefore, $U \in \mathcal{F}$ because of $\mathcal{F} \xrightarrow{\tau} e$.

(iii) Assume $\mathcal{F} \xrightarrow{\tau} e^+$and U is a zero neighborhood which containse$e$. By the formula $|e| = e^+ + e^-$; see Theorem 1.5(2) (Aliprantis and Burkinshaw, 2006), and by using the solidness of U, we get $e^+ \in U$. Therefore, we have $U \in \mathcal{F}$.

Remark 5. Let (E,τ) be a locally solid vector lattice.If a filter $\mathcal{F} \xrightarrow{\tau} e$ then e is a cluster point of $\mathcal{F}$. Indeed, let U be zero neighborhood set and it contains e, and $F \in \mathcal{F}$. Since U and Fin $\mathcal{F}$, we have $U \cap F \in \mathcal{F}$. Thus, we get $U \cap F \neq \emptyset$ because emptyset is not in the filter $\mathcal{F}$.

Let $\mathcal{F}_1, \mathcal{F}_2$ be filters on E with $\mathcal{F}_1 \subseteq \mathcal{F}_2$. So, if $\mathcal{F}_1 \xrightarrow{\tau} e$ then $\mathcal{F}_2 \xrightarrow{\tau} e$for $e \in E$. Indeed, every zero neighborhood containing e is in $\mathcal{F}_1$, and so is in $\mathcal{F}_2$.

The following result gives a relation between filter the convergence and the cluster point.

Theorem 5. Let (E,τ) be a locally solid vector lattice and $\mathcal{F}$ be a filter on the set E. Then a vector $e \in E$ is cluster point of $\mathcal{F}$ iff there exists another filter $\mathcal{F}_1$ containing $\mathcal{F}$ such that $\mathcal{F}_1 \xrightarrow{\tau} e$.

Proof: Suppose $e \in E$ is a cluster point of $\mathcal{F}$. Then the set

$$\mathcal{B} = \{U \cap F : F \in \mathcal{F}, \quad U is zero neighborhood and contains e\}$$

is a filter base. Indeed, we show the properties of filter base. The set $\mathcal{B}$ is non empty because each zero neighborhood U containing e intersects with every member of $\mathcal{F}$. For $U_1 \cap F_1$ and $U_2 \cap F_2$in $\mathcal{B}$, we can take $B = (U_1 \cap U_2) \cap (F_1 \cap F_2) \in \mathcal{B}$. So, we assume that it generates the filter $\mathcal{F}_1$. Therefore, for given a zero neighborhood Ucontaining e, we have $U \in \mathcal{B}$, and so we get $\mathcal{F}_1 \xrightarrow{\tau} e$.

Conversely, assume such filter $\mathcal{F}_1$exists. That means $\mathcal{F}_1$is a filter on the set E with $\mathcal{F} \subseteq \mathcal{F}_1$and $\mathcal{F}_1 \xrightarrow{\tau} e$. So, all zero neighborhoods containing e is in $\mathcal{F}_1$. By the definition of the filter convergence, each zero neighborhood containing e intersects with the members of $\mathcal{F}_1$, otherwise emptyset is in $\mathcal{F}_1$. Therefore, in particular, intersects each set in $\mathcal{F}$, and so we get e is a cluster point of $\mathcal{F}$.

Now, we use nets to define a filter on locally solid vector lattice. Let $(x_\alpha)_{\alpha \in I}$ be a net in the locally solid vector lattice (E,τ). Then we define its associated filter$\mathcal{F}$ on the set E as follow; consider the tail $\check{x}_\beta = \{x_\alpha : \alpha \in I and \alpha \geq \beta\}$and $\mathcal{B} = \{\check{x}_\beta : \beta \in I\}$. So, $\mathcal{B}$ is a filter base. Indeed, $\mathcal{B}$ is not empty and every $\check{x}_\beta \in \mathcal{B}$is not empty since I is directed set. Also, for any $\check{x}_{\beta_1}\check{x}_{\beta_2}$in $\mathcal{B}$, consider the index $\beta = \max\{\beta_1, \beta_2\}$so that $\check{x}_\beta \subseteq \check{x}_{\beta_1} \cap \check{x}_{\beta_2}$ and $\check{x}_\beta$ in $\mathcal{B}$. Thus, the filter which is generated by $\mathcal{B}$ is the associated filter of $(x_\alpha)_{\alpha \in I}$.

Theorem 6. Let (E,τ) be a locally solid vector lattice and $(x_\alpha)_{\alpha \in I}$ be a net in E. Assume $\mathcal{F}$ is the associated filter of $(x_\alpha)_{\alpha \in I}$and $e \in E$. Then $x_\alpha \xrightarrow{\tau} e$as a net iff$\mathcal{F} \xrightarrow{\tau} e$ as a filter. Moreover, e is a cluster point of $(x_\alpha)_{\alpha \in I}$ iff e is a cluster point of $\mathcal{F}$.

Proof: We show only the convergence part of proof, the cluster point case is analogous. Suppose $x_\alpha \xrightarrow{\tau} e$ as a net. Since every locally solid vector lattice has a base of zero neighborhoods, we can consider any zero neighborhood U which contains e. Then there exists an index α_0 such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. So, by definition of $\mathcal{F}$, we get $U \in \mathcal{F}$. Thus, we have $\mathcal{F} \xrightarrow{\tau} e$ as a filter.

Conversely, assume the filter $\mathcal{F}$ converges to e. Let V be a zero neighborhood in E and contains e. Then $V \in \mathcal{F}$. By definition of $\mathcal{F}$, there exists an index α_1 such that $\check{x}_{\alpha_1} \subseteq V$. Therefore, we get $x_\alpha \in V$ for every $\alpha \geq \alpha_1$, and so $x_\alpha \xrightarrow{\tau} e$.

Corollary 7: Let (E, τ) be a locally solid vector lattice which is generated by a family of Riesz pseudonorms $\{\rho_j\}_{j\in J}$, $(x_\alpha)_{\alpha\in I}$ be a net in E and $\mathcal{F}$ be associated filter of it, and $e \in E$. Then $\mathcal{F} \xrightarrow{\tau} e$ iff $\rho_j(x_\alpha - e) \to 0$ for all $j \in J$.

Proof: It follows from Theorem 2.28 (Aliprantis and Burkinshaw, 2003) and Theorem 6.

We give convergence of filters with respect to continuous function. Let (E, τ) and (F, τ)be locally solid vector lattices, and $\mathcal{F}$ be a filter on E. For a function $f: E \to F$, the set

$$f'_{\mathcal{F}} := \{B \subseteq F: f^{-1}(B) \in \mathcal{F}\}$$

is a filter on E which is generated by $\{f(A): A \subseteq \mathcal{F}\}$.

Proposition 8. Suppose (E, τ) and (F, τ) are locally solid vector lattices. Then a function $f: E \to F$ is continuous iff $f'_{\mathcal{F}} \xrightarrow{\tau'} f(e)$ in F for each $\mathcal{F} \xrightarrow{\tau} e$in E.

Proof: Suppose f is continuous. For fixed $e \in E$, consider the following set

$$N_e = \{A \subseteq E: \exists U \text{ zero neighborhood such that } e \in U \subseteq A\}.$$

Thus, we get $N_e \xrightarrow{\tau} e$. Note also that a filter $\mathcal{F}$ converges to $e$iff $N_e \subseteq \mathcal{F}$. Take a filter $\mathcal{F}$ such that $\mathcal{F} \xrightarrow{\tau} e$. The continuity of $f$implies that $N_{f(e)} \subseteq f'_{\mathcal{F}} N_e$. Therefore, if $\mathcal{F} \xrightarrow{\tau} e$ then $N_e \subseteq \mathcal{F}$and so $N_{f(e)} \subseteq f'_{\mathcal{F}} N_e \subseteq f'_{\mathcal{F}}$so that $f'_{\mathcal{F}} \xrightarrow{\tau'} f(e)$.Assume $f'_{\mathcal{F}} N_e$ is the set of all subsets of E whose preimage is a neighborhood of e. Since $N_e \xrightarrow{\tau} e$, we conclude that the preimage of any neighborhood of $f(e)$ is a neighborhood of e. Hence, f is continuous.

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