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Harmonic Forms and Killing Tensor Fields on Almost Para Complex Manifold

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In this paper we have studied different aspect of para complex and almost para complex manifold which is similar to almost para complex manifold.

1. Introduction:

An almost para complex structure F is intregrable if an only if N_F=0.

Proof:

Consider the two projections $\pi \pm : Tm \to T \pm M,$

$$\pi \pm := \frac{1}{2}\left(Fd \pm F\right)$$

Then by the Frobeneus theorem, the integrability of T+M and T-M is equivalent to respectively

$$\left.\begin{aligned} &\pi - \left[\pi + X, \pi + Y\right] = 0 \;\; and \\ &\pi + \left[\pi - X, \pi - Y\right] = 0 \end{aligned}\right\}$$

For all vector fields X and Y. The sum and the difference of these expressions are proportional to $N_F\left(X,Y\right)$ and $FN_F\left(X,Y\right)$

Ex.1Any para complex vector space $\left(V,F\right)$ can be considered as a Para complex manifold, with constant Para-complex structure.

Ex.2The certesian product MXN of Two Para-complex manifolds $\left(M,F_M\right)$ and $\left(N,F_N\right)$ is a Para complex manifold with the P

Ex.3Let $M = M_+ X \; M_-$ be the certesian product of two smooth manifolds M_+ and M_- of same dimension. We can identify $T_{(}$

Eigen value of *F* on Para:

F has *M* eigen values +*i* and *M* eigen value -*i*.

Solution:

I be a eigen values of F and the corresponding eigen value vector P then

$$\overline{P} = IP$$

Conversely

$$\begin{aligned} -P &= \overline{\overline{P}} \\ &= I\overline{P} \\ &= I^2 P \\ \because I^2 &= -1 \end{aligned}$$

Since I is a real and of rank 2m.Then M pairs of complex conjugate eigen value (i, $-i$)

Theorem 1.1:

The necessary and sufficient condition that $-V_n$ be a almost para complex manifold is that it contains a tangent bundle π_M of dimension M and a tangent bundle $\tilde{\pi}_M$ conjugate to $\pi_M S \in \tilde{\pi}_M \cap \pi_M = \phi$ .And they span together a tangent bundle of dimension 2m, projections on π_M and $\tilde{\pi}_M$ being L and$M$ given by

(3.1)a, $2L \; \underline{\underline{def}} \; I_n - I_R$

b) $2M \; \underline{\underline{def}} \; I_n + IF$

$$L = \frac{I-F}{2}$$

$$M = \frac{I+F}{2}$$

Solution:

$$\begin{aligned} L^2 &= \frac{\left(I-F\right)^2}{4} \\ &= \frac{I^2 + F^2 - 2iF}{4} \\ &= \frac{I - 2F + I}{4} \\ &= \frac{2I-2F}{4} \; \frac{I-F}{4} \; m^2 = \frac{\left(I+M\right)^2}{4} \Rightarrow \frac{I^2+M^2+2iF}{4} \end{aligned}$$

$$\begin{aligned} M^2 &= \frac{\left(I+M\right)^2}{4} \Rightarrow \frac{I^2+M^2+2iF}{4} \\ &= \frac{I^2+M^2+2iF}{4} \Rightarrow \frac{I+I+2F}{4} \Rightarrow \frac{2I+2F}{4} \\ &= \frac{I+F}{2} \end{aligned}$$

$$\because LM = ML \Rightarrow \frac{I^2-F^2}{4} \Rightarrow \frac{I^2-F2}{4} = \frac{I-I}{4} 0$$

$\because$ LM is complementary projection on π_M

$$a^x P_x = 0 \;\Rightarrow a^x = 0 \quad \forall x$$

$$b^x Q_x = 0 \Rightarrow b^x = 0$$

$$c^x P_x + d^x Q_x = 0 \qquad \text{(i)}$$

$$cF^x P_x + d^x F \; Q_x = 0$$

$$c^x P_x - d^x Q_x = 0 \qquad \text{(ii)}$$

Adding 1 + 2

$$2c^x P_x = 0 \quad \Rightarrow \; c^x, \; d^x = 0$$

$\therefore P_x, Q_x$ is linearly independent

$$LP_x = P_x$$

$$L_{Px} = \frac{1}{2}(I+F)P_x$$

$$= \frac{1}{2}(P_x + P_x)$$

$$= P_x$$

$$LQ_x = \frac{1}{2}(I+F)Q_x$$

$$= \frac{1}{2}(Q_x - Q_x) = 0$$

$$MP_x \Rightarrow \frac{1}{2}(I-F)P_x \Rightarrow \frac{1}{2}(P_x - P_x) = 0$$

$$MQ_x \Rightarrow \frac{1}{2}(I-F)Q_x$$

$$\Rightarrow Q_x$$

L M

π_M $\cap \tilde{\pi}_M = \phi$

$(P_x\ Q_x)^{-1}$ $(P^x\ Q^x)$

$$P_x P_x = 0$$

$$P_x Q^x = 0$$

Similarity –

$$Q_x P^x = 0$$

$$Q_x Q^x = 1$$

$$I = P^x \otimes P_x + q^x \otimes Q_x$$

$$= P_x P_y = \delta_4^x$$

and

$$q^x Q_y = \delta_y^x$$

$$P^x Q_y = 0$$

$$F = 1\{P^x \otimes P_x - q^x \otimes Q_x\}$$

$$F^2 = FF = \{P^x \otimes P_x - q^x \otimes OF \otimes Q_x\}$$

$$= P^x O\ F \otimes P_x - q^x\ O\ F \otimes Q_x$$

$$= P_x P_x - q^x Q_x$$ Proved.

Definition 2.1:

A vector field V is said to be contravariant almost para if it satisfies.

$$L_V F = 0$$

A vector field V said to be strictly contravariant almost para and if both V and $\bar{V}$ are contravariant almost para analytic.

$$L_x Y = [X, Y]$$

$$(L_x F)(Y) = L_x(FY) - FL_x Y$$

$$(L_x F)(Y) = L_x(FY) - FL_x Y$$

$$(L_x F)(Y) = L_x \bar{Y} - FL_x Y$$

$$(L_x F)(Y) = [X, \bar{Y}] - F[X, Y]$$

$$(L_x F)(Y) = [X, \bar{Y}] - \overline{[X, Y]} \quad (1)$$

If

$$(L_x F)(Y) = 0$$

$$[X, \bar{Y}] - [X, Y] = 0$$

Barring $X = V$ in (1)

$$(L_V F)(Y) = [V, \bar{Y}] - \overline{[V, Y]}$$

$$(V, \bar{Y}) - \overline{[V, Y]} = 0$$

$\because V$ is contravariant para complex.

$$(L_V F)(X) = [\bar{V}, \bar{X}] - [\bar{V}, X] \text{ (A)}$$

$$\overline{(L_{\bar{V}} F)(X)} = \overline{[\bar{V}, \bar{X}]} - \overline{\overline{[V, X]}} \text{ (B)}$$

(A) – (B)

$$(L_V F)(X) - \overline{(L_V F)(X)} = [\bar{V}, \bar{X}] - \overline{[\bar{V}, X]} - [V, \bar{X}] + \overline{\overline{[V, Y]}}$$

$$= [\bar{V}, \bar{X}] - [\bar{V}, X] - [V, \bar{X}] + [V, X]$$

$$= N[V, X]$$

$$(L_V F)(X) - \overline{(L_V F)(X)} = N(V, Y)$$

$$(L_V F)(X) = \overline{(L_V F)}(X) + N(V, Y)$$

$$\overline{(L_V F)(X)} = \overline{\overline{[L_v, F]}}(X) + N\overline{(V, Y)}$$

$$\overline{(L_V F)(X)} = [L_v, F](X) + N(V, Y)$$

$$\overline{(L_V F)(X)} - (L_V F)(X) = N(V, X)$$

Necessary and sufficient condition on Para complex manifold:

$$N[V, X] = 0$$

$$\overline{(L_V F)(X)} = (L_V F)(X)$$

$$\overline{\overline{(L_V F)}}(X) = \overline{(L_V F)}(X) \Rightarrow \boxed{(L_{\bar{V}} F)(X) = \overline{(L_V F)}(X)}$$

$$(L_{\bar{V}} F)(X) = \overline{(L_{\bar{V}} F)(X)}$$

$$(L_V F)(X) = \overline{(L_{\bar{V}} F)(X)}$$

$$(L_V F)(X) - (L_{\bar{V}} F)(X) = 0 \quad \text{(i)}$$

$$(L_{\bar{V}} F)(X) - \overline{(L_{\bar{V}} F)(X)} = 0$$

$$(L_{\bar{V}} F)(X) - \overline{(L_V F)(X)} = 0 \quad \text{(ii)}$$

Adding (1) and (2) we get

$$(L_V F)(X) = 0$$

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Theorem-2.3:
A necessary and sufficient condition that a vector field V on an almost para complex manifold be contra variant almost analytic.

(a) $L_V \bar{X} = \overline{L_V X}$

$$\left[V, \bar{X}\right] = \overline{\left[V, X\right]}$$

(b) $\overline{L_V X} + L_V X = 0$

$$\overline{\left[V, X\right]} + \left[V, X\right] = 0$$

From (a)

$$\left(L_V F\right)\left(X\right) = \left[V, \bar{X}\right] - \left[V, X\right]$$

$$\left(L_V F\right)\left(X\right) = \left[L_V, \bar{X}\right] - \overline{L_V X}$$

If $\left(L_V F\right)\left(X\right) = 0$ then

$$\boxed{\begin{aligned} L_V\left(FY\right) &= \left(L_V F\right)\left(X\right) + FL_V Y \\ D_x\left(FY\right) &= \left(D_x F\right)\left(Y\right) + FD_x Y \end{aligned}}$$

Theorem 2.4:
Nijenhuis tensor w.r.t. a contravariant Almost Para analytic vector V is Lie constant i.e. Lee derivative of Nijenhuis tensor with r to V vanishes.

Proof:

$$N\left[X,Y\right] = \left[\bar{X},\bar{Y}\right] + \left[X,Y\right] - \overline{\left[\bar{X},Y\right]} - \overline{\left[X,\bar{Y}\right]}$$

$$L_V N\left[X,y\right] = \left[L_V N\right]\left(X,Y\right) + N\left(L_V X,Y\right) + N\left(X,L_V Y\right)$$

Where v is contravariant almost para analytic

$$L_V\left\{\left[\bar{X},\bar{Y}\right] + \left[X,Y\right] - \left[\bar{X},Y\right] - \left[X,\bar{Y}\right]\right\} = \left(L_V N\right)\left(X,Y\right) + N\left(\left(V,X\right)Y\right)$$
$$+ N\left(X,\left(V,Y\right)\right)$$

$$\left[V,\left(\bar{X},\bar{Y}\right)\right] + \left[V,\left[X,Y\right]\right] - \left[V,\overline{\left[\bar{X},Y\right]}\right] - \left[V_1\overline{\left[X,\bar{Y}\right]}\right]$$

$$= \left(L_V N\right)\left[X,Y\right] + N\left[\left[V,X\right]Y\right] + N\left[X,\left[V,Y\right]\right]$$

$$\left[V,\left[\bar{X},\bar{Y}\right]\right] + \left[V,\left[X,Y\right]\right] - \left[V,\overline{\left[\bar{X},\bar{Y}\right]}\right] - \left[V,\overline{\left[X,\bar{Y}\right]}\right]$$

$$= L_V N\left(X,Y\right)\Big] + \left[\overline{\left[V,X\right]},\bar{Y}\right] + \left[\left(V,X\right),Y\right] - \left[\overline{\left(V,X\right)},Y\right]$$

$$- \left[\left(V,X\right),\bar{Y}\right] + \left[\bar{X},\overline{\left(V,Y\right)}\right] + \left[X,\left(V,Y\right)\right]$$

$$- \left[\bar{X},\left(V,Y\right)\right] - \left[X,\overline{\left(V,Y\right)}\right]$$

$$L_V N\left(X,Y\right) + \left[\left(V,\bar{X}\right),\bar{Y}\right] + \left[\left(V,X\right)Y\right] - \left[\overline{\left(V,X\right)},Y\right] - \overline{\left(V,X\right)},Y +$$

$$\left[X,\left[V,\bar{X}\right] + \left[X,\left[V,Y\right] - \left[\bar{X},\left[V,Y\right] - \left[X,\left(\overline{V,Y}\right)\right]\right]\right]\right] - V\left[\bar{X},\bar{Y}\right] +$$

$$\left[V,\left(X,Y\right)\right] - \left[V,\left(\bar{X},Y\right)\right] - V\overline{\left(X,\bar{Y}\right)} = 0$$

By the Jacobi's identies, this equation assume the from

$$\boxed{L_V N = 0}$$

References

[1] H.B. Pandey & Anil Kumar : Anti - Invariant submanifold of Almost para contact manifold. Prog. of Maths volume 211(1): (1987).

[2] Hasan Shahid, M.: CR-submanifolds of Kahlerian product manifold, Indian J. Pure Appl. Math 23 (12) (992); 873-879.

[3] I Sato and K. Matsumoto : On P- Sasakian manifold satisfying certain conditions. Tensor, 33, p. 173-78 (1979).

[4] K. Matsumoto : On Lorentzian para contact manifolds, Bull. Of Yamagata Univ. Nat. Sci., 12 (1989), 51-156.

[5] K. Yano: On a structure defined by a tensor field f of the type (1,1) satisfying $f^3+f=0$. Tensor N.S., 14 (1963), 99-109.