International Journal of Science and Research (IJSR)
ISSN: 2319-7064
ResearchGate Impact Factor (2018): 0.28 | SJIF (2018): 7.426

Semiprime Rings and it's Dependent Elements

A. R. Gotmare

GDM Arts KRN Commerce and MD Science College Jamner, India

Abstract: ***In this paper we study and investigate concerning dependent elements of semi prime rings and prime rings R by using generalized derivation and derivation, when R admit to satisfy some conditions, we give some results about that.***

1. Introduction and Preliminaries

Some researchers have studied the notion of free action on operator algebras, Murray and von Neumann [13] and von Neumann [14] introduced the notion of free action on abelian von Neumann algebras and used it for the construction of certain factors (see M.A. Chaudhry and M. S. Samman[5], F. Ali and M. A. Chaudhry [2] and Dixmier [8]. Kallman [11] generalized the notion of free action of auto orphisms of von Neumann algebras, not necessarily abelian, by using implicitly the dependent elements of an automorphism. Choda, Kashahara and Nakamoto [6] generalized the concept of freely acting automorphisms to C – algebras by introducing dependent elements associated to auto orphisms, where C* -algebra is a Banach algebra with an antiautomorphic involution * which satisfies (i) (x*)* = x, (ii) x*y* = (yx)*, (iii) x* + y* = (x + y)* (iv) (cx)* = $\bar{c}\,x^*$ where $\bar{c}$ is the complex conjugate of c and whose norm satisfies $(v)\left\|xx^*\right\| = \left\|x\right\|^2$. Several other authors have studied dependent elements Abrief on operator algebras. account of dependent elements in W^* -algebras has also appeared in the book of Stratila [15]. It is well-known that all C^* - algebras and von Neumann algebras are semiprime rings; in particular, a von Neumann algebra is prime if and only if its center consists of scalar multiples of identity. [Thus a natural extension of the notions of dependent elements of mappings and free actions on C^* -algebras and von Neumann algebras is the study of these notions in the context of semiprime rings and prime rings. Laradji and Thaheem [12] initiated a study of dependent elements of endomorphisms of semiprime rings and generalized a number of results of H. Choda, I. Kasahara, R. Nakamoto [6] to semiprime rings. Vukman and Kosi-Ulbl [16] and Vukman [17] have made further study of dependent elements of various mappings related to auto orphisms, derivations (α,β) -derivations and generalized derivations of semiprime rings. The main focus of the authors of J. Vukman, I.kosi-Ulbl [16] and [17] has been to identify various freely acting mappings related to these mappings, on semiprime and prime rings. The theory of centralizers (also called multipliers) of C^* -algebras and Banach algebras is well established (see C. A. Akemann, G. K. Pedersen, J. Tomiyama [1] and P. Ara, M. Mathieu [3]. Zalar [19] and Vukman and Kosi-Ulbl [18] have studied centralizers in the general framework of semiprime rings. Throughout, R will stand for associative ring with centre Z (R). As usual, the commutator xy-yx will be denoted by [x, y]. We shall use the basic commutator identities [xy, z] = [x, z] y + x [y, z] and [x, yz] = [x, y] z + y [x, z]. A ring R is said to be n-torsion free, where n ≠ o is an integer, if whenever nx = 0, with $x^{\in}$ R, then x=. Recall that a ring R is prime if a Rb = (0) implies that either a = 0 or b = 0, and is semiprime if aRa = (0) implies a = 0. A prime ring is semiprime but the converse is not true in general. An additive mapping d:R → R is called a derivation provided d(xy) = d(x) y + xd (y) holds for all pairs x,y ∈R. An additive mapping d:R →R is called centralizing (commuting) if [d(x),x] ∈ Z(R) ([d(x),x] = 0) for all x∈R. By Zalar [19], an additive mapping T:R → R is called a left (right) centralizer if T (xy) = T(x)y (T(xy)=xT(y)) for all x, y ∈R. If a ∈ R, then La(x) = ax and Ra(x) = xa (x∈R) define a left centralizer and a right centralizer of R, respectively. As additive mapping T:R → R is called a centralizer if T (xy) = T(x)y = xT(y) for all x, y ∈R. Let β be an automorphism of a ring R.An additive mapping d:R → R is called an β -derivation if d(xy) = d(x) y + β(x) d (y) holds for all x, y∈ R. Note that the mapping, d = β - I, where I denotes the identity mapping on R, is an β -derivation. Of course, the concept of an β -derivation generalizes the concept of a derivation, since any I-derivation is a derivation. β -derivations are further generalized as (α, β) – derivations. Let α, β be automorphisms of R, then an additive mapping d:R → R is called as (α, β) derivation if d (xy) = d (x) α (y) + β (x) d(y) holds for all pairs x, y∈ R. β - derivations and (α, β) – derivations have been applied in various situations, in particular, in the solution of some functional equations. An additive mapping T of a ring R into itself is called a generalized derivation, with the associated derivation d, if there exists a derivation d of R such that T(xy) = T(x) y + xd(y) for all x, y ∈ R. The concept of a generalized derivation covers both the concepts of a derivation and of a left centralizer provided T = d and d = 0, respectively (see B. Hvala [10]). Following A. Laradji, A. B. Thaheem [12], an element a ∈R is called a dependent element of a mapping T:R → R if T(x) a = ax holds for all x ∈ R. A mapping T:R → R is called a free action or (act freely) on R if zero is the only dependent element of T. It is shown in [12] that in a semiprime ring R there are no non zero nilpotent dependent elements of a mapping T:R → R. For a mapping T:R →R,D(F) denotes the collection of all dependent elements of F.

Lemma 1:
Let R be a 2 – torsion free semi prime ring and let a, b∈R. If for all x ∈ R, the relation axb + bxa = 0 holds, then axb = bxa = 0 is fulfilled for all x ∈ R.

2. The Main Results

Theorem 2.1

Let R be a semi prime ring and let D and G be derivations of R into itself, then the mapping $x \to D(x) + G^2(x)$ for all $x \in R$ is a free action.

Proof: We have

$F(x)\, a = ax$ for all $x \in R$.

Where F(s) stands for $D(x) + G^2(x)$ (1)

Replacing x by by with some routine calculation, we obtain

$F(xy) = F(x)\, y + xF(y) + 2D(x)\, D(y)$ for all $x, y \in R$. (2)

In (1) putting xa for x with using (2), we get

$F(x)\, a^2 + xF(a)\, a + 2D(x)\, D(a)\, a = axa$ for all $x \in R$. (3)

According to (1), we reduced (3) to

$2D(x)\, D(a)\, a + xa^2 + xa^2 = 0$ for all $x \in R$ (4)

Replacing x by yx in (4) with using (4), we obtain

$2D(y)\, x\, D(a)\, a = 0$ for all $x, y \in R$. (5)

Left-multiplying (4) by D (y) and applying (5), we obtain $D(y)\, xa^2 = 0$ for all $x, y \in R$.

Replacing y by D (a) and y by a, we get

$D(a)^2\, a^2 = 0$. (6)

Right-multiplying (4) by a with replacing x by a and using (6), we obtain

$a^4 = 0$. Which means that also $a = 0$. Thus our mapping is free action.

Theorem 2.2:

Let R be a prime ring, $\psi: R \to R$ be a generalized derivation and $a \in R$ be an element dependent on ψ then either $a \in Z(R)$ or $\psi(x) = x$ for all $x \in R$.

Proof: We have the relation

$\psi(x)\, a = ax$ for all $x \in R$. (7)

Replacing x by xy in (7), we obtain

$(\psi(x)\, y + xd(y))\, a = axy$ for all $x, y \in R$. (8)

According to the fact that ψ can be written in form $\psi = d + T$, where T is a left centralizer, replacing d(y) a by $\psi(y)\, a - T(y)\, a$ in (8) which gives according to (7).

$\psi(x)\, ya\, [x, a]\, y - xT(y)\, a = 0$ for all $x, y \in R$. (9)

Replacing y by $y\psi(n)$ in (9), we obtain

$\psi(x) y\psi(x)\, a + [x, a] y\psi(x) - xT(y\psi(x))\, a = 0$ for all $x,y \in R$. (10)

Again since T is left centralizer, then (10) become

$\psi(x)\, y\, \psi(x) a + [x, a]\, y\psi(x) - xT(y)\, \psi(x)\, a = 0$ for all $x,y \in R$. (11)

According to (7), (11) reduces to

$\psi(x)\, yax + [x, a]\, y\, \psi(x) - xT(y)\, ax = 0$ for all $x, y \in R$. (12)

Right – multiplying (9) by x gives

$\psi(x)\, yax + [x, a]\, yx - xT(y)\, ax = 0$ for all $x \in R$. (13)

Subtracting (12) and (13) we obtain

$[x, a]\, y\, (\psi(x) - x) = 0$ for all $x, y \in R$. then

$[x, a]\, R\, (\psi(x) - (x)) = 0$. Since R is prime ring, we obtain either $[x, a] = 0$ for all $x \in R$, which leads to $a \in Z(R)$ or $\psi(x) = x$ for all $x \in R$.

Proposition 2.3

Let R be a 2-torsion semiprime ring and let $a, b \in R$ be fixed elements. Suppose that $c \in R$ is an element dependent on the mapping $x \to xa + bx$ then $ac = ca$.

Proof: We will assume that $a \neq 0$ since there is nothing to prove in case $a = 0$ and $b = 0$ we have $(xa + bx)\, c = c\, x$ for all $x \in R$. (14)

Replacing x by xy, we obtain

$(xya + bxy)\, c = cxy$ for all $x, y \in R$. (15)

According to (14) the (15) reduces to

$(xya + bxy)\, c = (xa + bx)\, cy$ for all $x, y \in R$. then

$x(yac - acy) + bx\, (yc - cy) = 0$ for all $x, y \in R$. then

$xa\, [y, c] + x[y,a]\, c + bx\, [y, c] = 0$ for all $x,y \in R$.

Replacing y by c we get

$x[c, a]\, c = 0$ for all $x \in R$. Then

$R[c,a]\, c = 0$ Since R is semiprime, we get

$[c, a]\, c = 0$. Then (16)

$[c, a]\, [c, r] + [[c, a]r]\, c = 0$ for all $r \in R$.

$[c, a]\, [c, r] + [c, a]\, rc = 0$ for all $r \in R$. (17)

Right – multiplying (16) by r, we obtain

$[c, a]\, cr = 0$ for all $r \in R$. (18)

Subtracting (17) and (18) we get

$[c, a]\, [c, r] + [c, a]\, [c, r] = 0$ for all $r \in R$. Since R is 2-torsion free with replacing r by ra, we obtain $[c, a]\, r\, [c, a] = 0$ for all $r \in R$. Then

$[c, a]\, R\, [c, a] = 0$. Since R s semi prime ring. Then $ca = ac$. The proof of the theorem is complete.

Theorem 2. 4

Let R be a prime ring and let $a, b \in R$ be fixed elements. Suppose that $c \in R$ is an element dependent on the mapping $x \to axb$, then $ac \in Z(R)$ or $bc \in Z(R)$.

Proof: We will assume that $a \neq 0$ and $b \neq 0$, since there is nothing to prove in case $a = 0$ or $b = 0$. We have $(axb)\, c = cx$ for all $x \in R$. (19)

Let x be x y in (19) we obtain

$(axyb)\, c = cxy$ for all $x,y \in R$. (20)

According to (19) one can replace cx by (axb) in (20), we get

$Ax\, [bc,y] = 0$ for all $xy \in R$. (21)

Replacing x by cyx in the above relation, then we have

$acyx\, [bc, y] = 0$ for all $x,y \in R$. (22)

Again in (21) replacing x by cx with left – multiplying by y, we get

$yacx\, [bc, y] = 0$ for all $x,y \in R$.

Subtracting (22) and (23) we obtain

$[ac, y]\, x[bc, y] = 0$ for all $x,y \in R$. Then

$[ac, y]\, R\, [bc, y] = 0$. Since R is prime, we get.

either $ac \in Z(R)$ or $bc \in Z(R)$, the proof of the theorem is complete.

Theorem 2.5

Let R be a noncommutative 2-torsion free semiprime ring with cancellation property and $a, b \in R$ be fixed elements. Suppose that $c \in Z(R)$, is an element dependent on the mapping $x \to axb + bxa$ then $a \in Z(R)$.

Proof: Similarly, in Theorem 2.4, we will assume that $a \neq 0$ and $b \neq 0$. We have the relation

$(axb + bxa)\, c = cx$ for all $x \in R$. (24)

Replacing x by x y in (24), we get

$(axyb + bxya)\, c = cxy$ for all $x, y \in R$. (25)

Right – multiplying (24) by y, we get

$(axb + bxa)\, cy = cxy$ for all $x, y \in R$. (26)
Subtracting (26) from (25), we obtain
$ax[y,bc] + bx\,[y, ac] = 0$ for all $x, y \in R$. (27)
Replacing x by cx in above relation, we get
$acx\,[y, bc] + bcx\,[y, ac] = 0$ for all $x, y \in R$. (28)
Left-multiplying by y with replacing, by y x, we obtain
$yacyx[y, bc] + ybcyx\,[y, ac] = 0$ for all $x, y \in R$. (29)
Subtracting (29) and (28), we get
$[y, ac]\, x\, [y, bc] + [y, bc]\, x\, [y, ac] = 0$ for all $x, y \in R$. (30)
Suppose that ac non belong to Z (R), we have $[y, ac] \neq$ for some $y \in R$.

Then from (30) with Lemma 1, we obtain $[y, bc] = 0$, thus (27) reduces to $bx\,[y,ac] = 0$ for all $x,y \in R$ by using the cancellation property on b we obtain that $[y, ac] = 0$, contrary to assumption. We have, therefore, $ac \in Z(R)$.

According to (27), we get $ax\,[y, bc] = 0$ for all $x,y \in R$, whence it follows that $bc \in Z(R)$, now we have $ac \in Z(R)$ and $bc \in Z(R)$, therefore, according to (24), we obtain
$((ab + ba)\, c - c)\, x = 0$ for all $x \in R$.
Right – multiplying (31) by $((ab + ba)\, c - c)$ with using R is semiprime
We get $(ab + ba)\, c = c$. (31)
Then $[(ab + ba)\, c, r] = [c. r]$.
$(ab + ba)\,[c, r] + [(ab + ba), r]\, c = [c, r]$ for all $r \in R$.
Replacing r by c above relation reduces to
$[(ab + ba),c]\, c = 0$. By using the cancellation property on $[(ab + bc),c]$ we obtain, $c \in Z(R)$.
The proof of the theorem is complete.

Theorem 2.6
Let R be a noncommutative semiprime ring with extended centroid C and cancellation property, let $a, b \in R$ be fixed elements the mapping $x \rightarrow a\,x\,b - b\,x\,a$ is a free action.

Proof: We assume that $a \neq 0$ and $b \neq 0$ with that a and b are C, independent, otherwise, the mapping $x \rightarrow axb - bxa$ would be zero. Then, we have the following relation.
$(axb - bxa)\, c = cx$ for all $x \in R$. (33)
Replacing xby xy in the above relation, we obtain
$(axyb - bxya)\, c = cxy$ for all $x,y \in R$. (34)
Right – multiplication of (33) by y, we get
$(axb - bxa)\, cy = cxy$ for all $x, y \in R$. (35)
Subtracting (34) and (35) we obtain
$ax[y,bc] - bx[y, ac] = 0$ for all $x, y \in R$ (36)
Replacing x by cx, we get
$acx\,[y, bc] - bcx\,[y, ac] = 0$ for all $x,y \in R$. (37)
Left – multiplying (37) by y, we get
$yacy\,[y, bc] - ybcx\,[y, ac] = 0$ for all $x, y \in R$. (38)
In (37) replacing x by yx we obtain
$acyx\,[y, bc] - bcyx[y, ac] = 0$ for all $x, y \in R$. (39)
Subtracting (39) and (38), we obtain
$[y, bc] = \lambda y\,[y, ac]$ for all $y \in R$. (40)
Holds for some $\lambda y \in C$. According to (40) one can replace $[y,bc]$ by $\lambda y\,[y, ac]$ in (36) we obtain $(b - \lambda ya)\, x\, [y, ac] = 0$ for all $y \in R$.

Replacing x by cxc, we obtain $(b - \lambda ya)\, cxc\, [y, ac] = 0$ for all $x, y \in R$.

Using the cancellation property on $[y, ac]$ in (41), we obtain $(b - \lambda ya)\, cxc = 0$ for all $x, y \in R$.

Again using the cancellation property on $(b - \lambda ya)$ in (42) with using R is semiprime, we obtain $c = 0$, which completes the proof of the theorem.

Theorem 2.7
Let R be a prime ring and let $\psi: R \rightarrow R$ be a non – zero (σ, β) – derivation, then ψ is a free action.

Proof:
We have the relation $\psi(x)\, a = ax$ for all $x \in R$. (43)
Replacing x by xy, we obtain
$\psi(x)\, \sigma(y)\, a + \beta(r)\, \psi(y)\, a = axy$ for all $x, y \in R$. (44)
According to (49) one can replace $\psi(y)\, a$ by ay above relation, which gives
$\psi(x)\, \sigma(y)\, a + (\beta(x)\, a - ax)\, y = 0$ for all $x, y \in R$. (45)
Replacing y by yz in (50) we obtain
$\psi(x)\, \sigma(y)\, \sigma(z)\, a + \beta(x)\, a - ax)\, yz = 0$ for all $x, y, z \in R$.
Right multiplying (50) by z, we get
$\psi(x)\, \sigma(y)\, (az) + (Bx)\, a - ax)\, yz = 0$ for all $x, y, z \in R$. (52)
Subtracting (52) from (51), we get
$\psi(x)\, \sigma(y)\, (\sigma(z)\, a - az) = 0$ for all $x, y, z \in R$. In other words, we have
$\psi(x)\, y(\sigma(z)\, a - az) = 0$ for all $x, y, z \in R$. Then
$\psi(x)\, R(\sigma(z)\, a - az) = 0$. Since R is prime and ψ is non-zero, we obtain
$\sigma(z)\, a = az$ for all $z \in R$. (53)
Since σ is automorphism of R, then by other words from (53) we have
$za = az$ for all $z \in R$. (54)
Also, since B is automorphism of R, then from (50) we obtain
$\psi(x)\, \sigma(y)\, a + (xa - ax) = 0$ for all $x, y \in R$. (55)
Apply (54) in above relation, we obtain
$\psi(x)\, \sigma(y)\, a = 0$ for all $x, y \in R$. By other words we have
$\psi(x)\, ya = 0$ for all $x, y \in R$, then
$\psi(x)\, Ra = 0$. By the primeness of R and ψ is non-zero of R, we obtain.
$a = 0$, the proof of the theorem is complete.

References

[1] C.A. Akemann, G. K. Pedersen, J. Tomiyama: Multipliers of C^* - algebras J. Funct. Anal 13 (1973), 277 – 301.
[2] F. Ali and M. A. Chaudhry, Dependent elements of derivations on semiprime rings, International Journal of Mathematics and Mathematical Sciences, Vol.2009 Article ID 696737 pages, doi:10.155/2009/696737.
[3] P. Ara, M. Mathieu: An application of local multipliers to centralizing mappings of C^* - algebras. Quart J. Math Oxford 44 (1993), 129 – 138.
[4] M. Bresar Jordan derivations on semiprime rings, Proc.Amer. Math Soc. 104(1988), no. 4,1003 1006. Z
[5] M. A. Chaudhry and M. S. Samman, Free actions on semiprime rings, Mathematica Bohemica, No. 2, 133 (2008), 197-208 zbl
[6] H. Choda I. Kasahara, R. Nakamoto: Dependent elements of auromorphisms of a C^* - algebra. Proc. Japan Acad. 48 (1972), 561-565. Abl

[7] J. Dixmier: Les Algebres d' Operateurs dans I'Espace Hilbertien, Gauthier – Villars, Parts 1957. Zbl
[8] I. N. Herstein: Rings with involution. Univ. Chicago Press, Chicago, 1976. zbl
[9] B. Hvala: Generalized derivations in rings, Comm. Algebra 26 (1998), 1147 – 1166.
[10] R. R. Kallman: A generalization of free action. Duke Math. J. 36 (1969), 781 – 789. Zbl