SOME SPECTRAL PROPERTIES OF THE BANACH ALGEBRA
\( A \times_d B \) WITH THE DIRECT-SUM PRODUCT

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Abstract. Let \( A \) be a commutative Banach algebra and \( B \) be a subalgebra of \( A \). Then \( A \times B \) is a commutative algebra with co-ordinatewise linear operations and the direct-sum product: 
\[
(a,b)(c,d) = (ac + ad + bc, bd) \quad ((a,b), (c,d) \in A \times B).
\]
This algebra will be denoted by \( A \times_d B \). Further, if \( A \) is a Banach algebra and \( B \) is a closed subalgebra of \( A \), then \( A \times_d B \) is a Banach algebra with respect to the norm \( \|(a,b)\|_1 = \|a\| + \|b\| \) \( ((a,b) \in A \times_d B) \). Some basic properties, uniqueness properties, regularity properties, and the Gel’fand theory of the Banach algebra \( A \times_d B \) have been studied in \cite{3}. In this paper, we further explore this Banach algebra to study its some spectral properties. These properties are spectral extension property, topological divisor of zero, multiplicative Hahn-Banach property, Quasi divisor of zero, topological annihilator condition, Ditkin’s condition, and Tauberian condition.

Let \( \sigma_A(a) \) and \( r_A(a) \) denote the spectrum and the spectral radius of \( a \) in \( A \). Let \( \Delta(A) \) denote the set of all non-zero, multiplicative, linear functionals on a commutative Banach algebra \( A \). For \( a \in A \), the map \( \tilde{\sigma} : \Delta(A) \rightarrow \mathbb{C} \) is defined as \( \tilde{\sigma}(\varphi) = \varphi(a) \). The topology on \( \Delta(A) \) is the smallest topology such that \( \tilde{\sigma} \) is continuous for each \( a \in A \). Let \( \varphi \in \Delta(A) \) and \( S \) be a non-empty subset \( A \). Define \( \varphi_o : A \times S \rightarrow \mathbb{C} \) as \( \varphi_o((a,x)) := \varphi(x) \). Now let \( I \) be an ideal in \( A \), let \( \varphi \in \Delta(I) \), and \( u \in I \) such that \( \varphi(u) = 1 \). Define \( \varphi^+ : A \times I \rightarrow \mathbb{C} \) as \( \varphi^+(a, x) := \varphi(ax) + \varphi(x) \). Next, for \( F \subseteq \Delta(A) \), define \( F^+ := \{ \varphi^+ : \varphi \in F \} \) and \( F_o := \{ \varphi_o : \varphi \in F \} \). In the case \( F = \Delta(A) \), we shall write \( \Delta^+(A) \) and \( \Delta_o(A) \) for \( F^+ \) and \( F_o \), respectively. We shall need the following result in proofs.

1. Introduction

Throughout let \( A \) be a commutative algebra and \( B \) be a subalgebra of \( A \). Then \( A \times B \) is a commutative algebra with co-ordinatewise linear operations and the direct-sum product defined as
\[
\sigma_A(a) = \sigma_B(b)
\]
Let \( \sigma_A(a) \) and \( r_A(a) \) denote the spectrum and the spectral radius of \( a \in A \). Let \( \Delta(A) \) denote the set of all non-zero, multiplicative, linear functionals on a commutative Banach algebra \( A \). For \( a \in A \), the map \( \tilde{\sigma} : \Delta(A) \rightarrow \mathbb{C} \) is defined as \( \tilde{\sigma}(\varphi) = \varphi(a) \). The topology on \( \Delta(A) \) is the smallest topology such that \( \tilde{\sigma} \) is continuous for each \( a \in A \). Let \( \varphi \in \Delta(A) \) and \( S \) be a non-empty subset \( A \). Define \( \varphi_o : A \times S \rightarrow \mathbb{C} \) as \( \varphi_o((a,x)) := \varphi(x) \). Now let \( I \) be an ideal in \( A \), let \( \varphi \in \Delta(I) \), and \( u \in I \) such that \( \varphi(u) = 1 \). Define \( \varphi^+ : A \times I \rightarrow \mathbb{C} \) as \( \varphi^+(a, x) := \varphi(ax) + \varphi(x) \). Next, for \( F \subseteq \Delta(A) \), define \( F^+ := \{ \varphi^+ : \varphi \in F \} \) and \( F_o := \{ \varphi_o : \varphi \in F \} \). In the case \( F = \Delta(A) \), we shall write \( \Delta^+(A) \) and \( \Delta_o(A) \) for \( F^+ \) and \( F_o \), respectively. We shall need the following result in proofs.

2010 Mathematics Subject Classification. Primary 46K05; Secondary 46H05.

Key words and phrases. Banach algebras, Direct-sum product, Spectrum, Spectral radius, Spectral extension property, Gel’fand space, and Gel’fand transform.

This research work is supported by the UGC-SAP-DRS-III; The grant number is F.510/1/DRS-III/2015(SAP-I) given to the Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar.
Lemma 1.1. [4, Chapter-3] The Gel’fand space $\Delta(A \times_d B)$ is homeomorphic to $\Delta^+(A) \cup \Delta_0(B)$ equipped with the sum-topology. Moreover, the Shilov boundary $\partial(A \times_d B)$ is homeomorphic to $\partial^+(A) \cup \partial_0(B)$ equipped with the sum-topology.

2. Spectral Properties in Commutative Banach Algebras

Definition 2.1. [7] A norm $|\cdot|$ on $A$ is a spectral norm if $r_A(a) \leq |a|$ ($a \in A$). The Banach algebra $A$ has spectral extension property (SEP) if every norm on $A$ is a spectral norm.

Theorem 2.2. If $A \times_d B$ has SEP, then $A$ and $B$ have SEP.

Proof. Let $|\cdot|$ be a norm on $A$. Define $|(a, b)| = |a| + |b|$. Then $|\cdot|$ is an algebra norm on $A \times_d B$. Since $A \times_d B$ has SEP, we have

$$r_A(a) = r_{A \times_d B}(a, 0) \leq |(a, 0)| = |a| \quad (a \in A).$$

Thus $|\cdot|$ is a spectral norm on $A$, and so $A$ has SEP. Next suppose that $|\cdot|$ is a norm on $B$. Define $|(a, b)| = \|a + b\| + |b|$ on $A \times_d B$, where $\|\cdot\|$ is the Banach algebra norm on $A$. Then, by [4, Lemma 3.2.2], $|\cdot|$ is an algebra norm on $A \times_d B$. Since $A \times_d B$ has SEP,

$$r_B(b) = r_{A \times_d B}(-b, b) \leq |(-b, b)| = |b| \quad (b \in B).$$

Hence, $|\cdot|$ is a spectral norm on $B$. Therefore, $B$ has SEP.

Definition 2.3. [4, Definition 1.4.18] A non-zero element $a \in A$ is a topological divisor of zero (TDZ) if there is a sequence $(a_n)$ in $A$ such that $|a_n| = 1 \quad (n \in \mathbb{N})$ and either $a_n a \to 0$ as $n \to \infty$. The Banach algebra $A$ has topological divisor of zero (TDZ) property if every element of $A$ is a topological divisor of zero.

Theorem 2.4. If $A$ and $B$ have TDZ property, then $A \times_d B$ has TDZ property.

Proof. Suppose that $A$ and $B$ have TDZ property. Let $(a, b) \in A \times_d B$. Then $a + b \in A$. Suppose that $a + b \neq 0$. Since $A$ has TDZ property, there exists a sequence $(a_n) \subset A$ such that $|a_n| = 1 \quad (n \in \mathbb{N})$ and $a_n(a + b) \to 0$ as $n \to \infty$. Then $(a_n, 0)$ is a sequence in $A \times_d B$ such that $|\{(a_n, 0)\}| = |a_n| = 1 \quad (n \in \mathbb{N})$ and $(a_n, 0)(a, b) = (a_n a + a_n b, 0) \to (0, 0)$ as $n \to \infty$. Therefore, $(a, b)$ is a TDZ. If $a = b = 0$, then $a = 0 \neq 0$. Since $B$ has TDZ property, there exists a sequence $(b_n)$ in $B$ such that $|b_n| = 1$ and $b_n b \to 0$ as $n \to \infty$. In this case, $(0, b_n)$ is a sequence in $A \times_d B$ such that $|\{(0, b_n)\}| = |b_n| = 1$ and $(0, b_n)(a, b) = (-b_n b, b_n b) \to (0, 0)$ as $n \to \infty$. Thus, in all cases, $(a, b)$ is a TDZ in $A \times_d B$. Hence $A \times_d B$ has TDZ property.

Definition 2.5. [7] A commutative Banach algebra $A$ has Multiplicative Hahn-Banach Property (MHBP) if, for every commutative extension $B$ of $A$, every $\varphi \in \Delta(A)$ can be extended to some element of $\Delta(B)$.

Theorem 2.6. $A \times_d B$ has MHBP if and only if both $A$ and $B$ have MHBP.

Proof. Let $A \times_d B$ have MHBP. Let $C$ be a commutative extension of $A$, then $C \times_d B$ is a commutative extension of $A \times_d B$. Let $\varphi \in \Delta(A)$. Then $\varphi^+ \in \Delta^+(A) \cup \Delta_0(B)$. Since $A \times_d B$ has MHBP, there exists $\tilde{\eta} \in \Delta(C \times_d B) = \Delta^+(C) \cup \Delta_0(B)$ such that $\tilde{\eta} = \varphi^+$ on $A \times_d B$. Now, if $\tilde{\eta} \in \Delta_0(B)$, then we get $\varphi(a) = \varphi^+((a, 0)) = \tilde{\eta}(a, 0) = 0$ on $A$. This is not possible. Hence, $\tilde{\eta}$ must be in $\Delta^+(C)$. Therefore, there exists $\tilde{\varphi} \in \Delta(C)$ such that $\tilde{\eta} = \tilde{\varphi}^+$ on $C \times_d B$. Also, $\tilde{\varphi} = (\tilde{\varphi})^+ = \varphi^+$ on $A \times_d B$, implies
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Let $\tilde{\varphi} = \varphi$ on $A$. Thus $\tilde{\varphi}$ is an extension of $\varphi$. Hence $A$ has MHBP. Similarly, it can be proved that $B$ has MHBP.

Conversely, assume that $A$ and $B$ have MHBP. Let $C$ be any extension of $A \times_d B$. Then $C$ is an extension of both $A \times_d \{0\}$ and $\{0\} \times_d B$. Let $\tilde{\eta} \in \Delta^+(A) \cup \Delta_0(B)$. Then either $\tilde{\eta} \in \Delta^+(A)$ or $\tilde{\eta} \in \Delta_0(B)$. Suppose that $\tilde{\eta} \in \Delta^+(A)$. Then $\tilde{\eta} = \varphi^+$ for some $\varphi \in \Delta(A)$.

By similar arguments, it follows that $V$ is a non-empty open set.

Let $\tilde{\eta} \in \Delta^+(A) \cup \Delta_0(B)$.

Since $\tilde{\eta}$ is an extension of both $A \times_d \{0\}$ and $\{0\} \times_d B$, then there exists an open set $\tilde{A}$.

Therefore, either $\tilde{\eta} = \varphi^+$ in $\Delta(A)$ and $\tilde{\eta} = \tilde{\varphi}^+$ on $A \times_d B$. Similarly, it can be extended to some element $\tilde{\varphi}$ of $\Delta(\tilde{C})$.

Thus $A \times_d B$ has MHBP.

**Definition 2.7.** [6] A commutative Banach algebra $A$ has Quasi Divisor of Zero (QDZ) property if there exists an open subset $G$ of $\Delta(A)$ such that

1. $\partial(A) \subset \overline{G}$;
2. For every open subset $U$ of $G$, there exist $a \in A$ and a non-empty open set $V \subset U$ such that

$$\hat{a}(\varphi) = \begin{cases} 0 & \text{if } \varphi \in U^c; \\ 1 & \text{if } \varphi \in V. \end{cases}$$

**Theorem 2.8.** $A \times_d B$ has QDZ property iff both $A$ and $B$ have QDZ property.

**Proof.** Let $A \times_d B$ has QDZ property. Then there exists an open set $\tilde{G} \subset \Delta(A \times_d B)$ which satisfies the following properties.

1. $\partial^+(A) \cup \partial_0(B) \subset \overline{G}$;
2. For every open subset $\tilde{U}$ of $\tilde{G}$, there exists $(a, b) \in A \times_d B$ and a non-empty open subset $V$ of $\tilde{U}$ such that

$$\hat{(a,b)}(\varphi) = \begin{cases} 0 & \text{if } \varphi \in \tilde{U}^c; \\ 1 & \text{if } \varphi \in V. \end{cases}$$

Let $G_A = \{ \varphi \in \Delta(A) : \varphi^+ \in \tilde{G} \}$ and $G_B = \{ \varphi \in \Delta(B) : \varphi_0 \in \tilde{G} \}$. Then $G_A$ and $G_B$ are open sets in $\Delta(A)$ and $\Delta(B)$, respectively as such that $G_A \cup G_B = \tilde{G}$.

Also, from (1) above, we get $\partial A \subset \overline{G_A}$ and $\partial B \subset \overline{G_B}$. Now, let $U \subset G_A$ be open. Then $U^+$ will be open in $\tilde{G}$. Hence, by (2) above, there exist $(a, b) \in A \times_d B$ and a non-empty open set $V \subset U$ such that $(a, b)^+ = 0$ on $(U^+)^c$ and $(a, b)^+ = 1$ on $V^+$. Now, if $\varphi \in U^c$, then $\varphi^+ \in (U^+)^c$ and $(a+b)^+ = (a+b)^+ = 0$ on $U^c$. If $\varphi \in V$, then $\varphi^+ \in V^+$ and $(a+b)^+ = \varphi^+((a, b)) = 1$. Hence $A$ has QDZ property. By similar arguments, it follows that $B$ has QDZ property.

Conversely, suppose $A$ and $B$ have QDZ property. Then there exist open subsets $G_A \subset \Delta(A)$ and $G_B \subset \Delta(B)$ satisfying the properties in the definition of QDZ. Then $\tilde{G} = G_A \cup G_B$ and

$$\partial(A \times_d B) = \partial^+(A) \cup \partial_0(B) \subset \overline{G_A} \cup \overline{G_B} = \overline{G_A \cup G_B} = \tilde{G}.$$
Definition 2.9. [6] Let $I$ be an ideal of a commutative semisimple Banach algebra $A$. A *separating net* for $I$ is a net $(q_\lambda)_{\lambda \in \Lambda}$ of quasi divisors of zero in $A$ such that

1. $\sup \{r_A(q_\lambda) : \lambda \in \Lambda\} < \infty$;
2. $\lim_{\lambda \to \infty} r_A(q_\lambda) = 0$ (a $I$);
3. There exists an element $b \in A$ such that $q_\lambda b = q_\lambda$ ($\lambda \in \Lambda$).

Definition 2.10. [6] A commutative Banach algebra $A$ satisfies *Topological Anihilator (TAN) property* if there exists a dense set $D \subset \partial A$ such that, for every $\varphi \in D$, the ker $\varphi$ admits a separating net.

Theorem 2.11. $A \times_d B$ has TAN property iff both $A$ and $B$ have TAN property.

Proof. Let $A \times_d B$ has TAN property. Then there exists dense subset $\tilde{D}$ of $\partial(A \times_d B)$ such that ker $\tilde{\eta}$ ($\tilde{\eta} \in \tilde{D}$) admits a separating net. Let $D_A = \{ \varphi \in \Delta(A) : \varphi^+ \in \tilde{D} \}$. Then $D_A$ is a dense subset of $\partial A$. Let $\varphi \in D_A$. Then $\varphi^+ \in \tilde{D}$. Hence ker $\varphi^+$ admits a separating net say $((a_\lambda, b_\lambda))_{\lambda \in \Lambda}$. Then $(a_\lambda + b_\lambda)_{\lambda \in \Lambda}$ is a separating net for ker $\varphi$. Thus $A$ has TAN property. By similar arguments it follows that $B$ has TAN property.

Conversely, assume that $A$ and $B$ have TAN property. Then there exist dense subsets $D_A \subset \partial A$ and $D_B \subset \partial B$ such that ker $\varphi$ ($\varphi \in D_A \cup D_B$) admits a separating net. Let $\tilde{D} = D_A^+ \cup D_B^\circ$. Then $\tilde{D}$ is a dense subset of $\partial^+ (A) \cup \partial^\circ (B)$. Let $\tilde{\eta} \in \tilde{D}$. Then either $\tilde{\eta} = \varphi^+$ for some $\varphi \in D_A$ or $\tilde{\eta} = \psi_0$ for some $\psi \in D_B$. If $\tilde{\eta} = \varphi^+$, then ker $\varphi$ admits a separating net $((a_\lambda))_{\lambda \in \Lambda}$. Hence $((a_\lambda, 0))_{\lambda \in \Lambda}$ is a separating net for ker $\varphi$. Similarly, if $\tilde{\eta} = \psi_0$, then ker $\psi$ admits a separating net $((b_\lambda))_{\lambda \in \Lambda}$. In this case, $((\psi_0 - b_\lambda))_{\lambda \in \Lambda}$ is a separating net for ker $\tilde{\eta}$. Hence $A \times_d B$ has TAN property.

Lemma 2.12. Let $a \in A$ and $b \in B$. Then $(a, b)^\wedge \in C_c(\Delta(A \times_d B))$ if and only if $(a + b)^\wedge \in C_c(\Delta(A))$ and $b \in C_c(\Delta(B))$.

Proof. This follows from the definition of the support.

Definition 2.13. [1, Definition 4.1.31] Let $A$ be a commutative Banach algebra. Then $A$ satisfies

1. Ditkin’s condition at $\varphi \in \Delta(A)$ if for every $a \in \ker(\varphi)$, there exists a sequence $(a_n)$ in $A$ such that $\hat{a_n} \in C_c(\Delta(A))$, $\varphi \notin \text{supp} \hat{a_n}$ and $a_n a \to a$ as $n \to \infty$.
2. Ditkin’s condition at infinity if for $a \in A$, there exists a sequence $(a_n)$ in $A$ such that $\hat{a_n} \in C_c(\Delta(A))$ and $a_n a \to a$ as $n \to \infty$.
3. Ditkin’s condition if it satisfies Ditkin’s condition at every $\varphi \in \Delta(A)$ and at infinity.

Theorem 2.14. $A \times_d B$ satisfies Ditkin’s condition iff both $A$ and $B$ satisfy Ditkin’s condition.

Proof. Let $A \times_d B$ satisfies Ditkin’s condition. Let $\varphi \in \Delta(A)$ and $a \in \ker \varphi$. Then $(a, 0) \in \ker \varphi^+$. Since $A \times_d B$ satisfies Ditkin’s condition, there exists a sequence $((a_n, b_n))$ in $A \times_d B$ such that $(a_n, b_n)^\wedge \in C_c(\Delta(A \times_d B))$ ($n \in \mathbb{N}$), $\varphi^+ \notin \text{supp}(a_n, b_n)^\wedge$ and $(a_n, b_n)(a, 0) \to (a, 0)$ as $n \to \infty$. Then $(a_n + b_n)$ is a sequence in $A$ such that $(a_n + b_n)^\wedge \in C_c(\Delta(A))$ ($n \in \mathbb{N}$), due to Lemma 2.12, $\varphi \notin \text{supp}(a_n + b_n)^\wedge$ and $(a_n + b_n)a \to a$ as $n \to \infty$. Thus $A$ satisfies Ditkin’s condition at every $\varphi \in \Delta(A)$. By similar arguments, it follows that $B$ satisfies Ditkin’s condition at every $\psi \in \Delta(B)$. 

www.ijsr.net

Volume 8 Issue 8, August 2019

Paper ID: ART2020185
10.21275/ART2020185

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Next we show that $A$ and $B$ satisfy Ditkin’s condition at infinity. Let $a \in A$. Since $A \times_d B$ satisfies Ditkin’s condition at infinity, there exists a sequence $(a_n, b_n)$ in $A \times_d B$ such that $(a_n, b_n) \in C_c(\Delta(A \times_d B))$ and $(a_n, b_n)(a, 0)$ converges to $(a, 0)$ as $n \to \infty$. Then $(a_n + b_n)$ is a sequence in $A$ such that $(a_n + b_n) \in C_c(\Delta(A))$ due to Lemma 2.12 and $(a_n + b_n)a \to a$ as $n \to \infty$. Therefore, $A$ satisfies Ditkin’s condition at infinity. By similar arguments, it follows that $B$ satisfies Ditkin’s condition at infinity.

Conversely, assume that both $A$ and $B$ satisfy Ditkin’s condition. Let $\varphi^+ \in \Delta^+(A)$ and $(a, b) \in \ker \varphi^+$. Then $a + b \in \ker(\varphi)$. Since $A$ satisfies Ditkin’s condition at $\varphi$, there exists a sequence $(a_n)$ in $A$ such that $(\Delta(a_n)) \subset C_c(\Delta(A))$, $\varphi \notin \text{supp}(a_n)$ and $a_n(a + b) \to a + b$ as $n \to \infty$. Since $B$ satisfies Ditkin’s condition at infinity, there exists a sequence $(b_n)$ in $B$ such that $(b_n) \in C_c(\Delta(B))$ and $b_n b \to b$ as $n \to \infty$. Then $((a_n - b_n, b_n))$ is a sequence in $A \times_d B$ such that $(a_n - b_n, b_n) \in C_c(\Delta(A \times_d B))$ due to Lemma 2.12, $\varphi^+ \notin \text{supp}(a_n - b_n, b_n)$ and $(a_n - b_n, b_n)(a, b) \to (a, b)$ as $n \to \infty$. Hence, $A \times_d B$ satisfies Ditkin’s condition at every $\varphi^+ \in \Delta^+(A)$. Similarly, it follows that $A \times_d B$ satisfies Ditkin’s condition at every $\psi_0 \in \Delta_0(B)$.

Next we show that $A \times_d B$ satisfies Ditkin’s condition at infinity. Fix an arbitrary element $(a, b) \in A \times_d B$. Since both $A$ and $B$ satisfy Ditkin’s condition at infinity, there exist sequences $(a_n)$ in $A$ and $(b_n)$ in $B$ such that $(\Delta(a_n)) \subset C_c(\Delta(A))$, $(\Delta(b_n)) \subset C_c(\Delta(B))$, $(a + b) a_n \to a + b$ and $(a + b) b_n \to b$. Therefore $(a_n - b_n, b_n) (a, b) \to (a, b)$. Hence $A \times_d B$ satisfies Ditkin’s condition at infinity. \hfill \Box

**Definition 2.15.** [5, Definition 8.1.2] A commutative Banach algebra $A$ is said to be a Tauberian algebra if the set $\{a \in A : \hat{a} \in C_c(\Delta(A))\}$ is dense in $A$.

**Theorem 2.16.** $A \times_d B$ is Tauberian iff both $A$ and $B$ are Tauberian.

**Proof.** Let $A \times_d B$ be a Tauberian algebra. Let $a \in A$ and $\epsilon > 0$. Since $A \times_d B$ is Tauberian, there exists $(a_0, b_0) \in A \times_d B$ such that $(a_0, b_0) \in C_c(\Delta(A \times_d B))$ and $\|(a, 0) - (a_0, b_0)\|_1 = \|a - a_0\| + \|b_0\| < \epsilon$. Then $(a_0 + b_0) \in C_c(\Delta(A))$ and $\|(a_0 + b_0) - (a, b)\|_1 \leq \|a - a_0\| + \|b - b_0\| < \epsilon$. Therefore, $A$ is Tauberian. Now, let $b \in B$ and $\epsilon > 0$. Since $A \times_d B$ is Tauberian, there exists $(a_1, b_1) \in A \times_d B$ such that $(a_1, b_1) \in C_c(\Delta(A \times_d B))$ and $\|(0, b) - (a_1, b_1)\|_1 = \|a_1\| + \|b - b_1\| < \epsilon$. Since $(a_1, b_1) \in C_c(\Delta(A \times_d B))$, by Lemma 2.12, $b_1 \in C_c(\Delta(B))$ and $\|b - b_1\| < \epsilon$. Therefore, $B$ is Tauberian.

Conversely, suppose that $A$ and $B$ are Tauberian algebra. Let $(a, b) \in A \times_d B$ and $\epsilon > 0$. Since $A$ and $B$ are Tauberian, there exist $a_0 \in A$ and $b_0 \in B$ such that $\hat{a}_0 \in C_c(\Delta(A))$, $\hat{b}_0 \in C_c(\Delta(B))$, $(a + b - a_0) < \epsilon/3$ and $(b - b_0) < \epsilon/3$. Therefore, by Lemma 2.12, $(a_0 - a_0, b_0) \in C_c(\Delta(A \times_d B))$. Also

$$\|(a, b) - (a_0, b_0)\|_1 = \|a - a_0 + b_0\| + \|b - b_0\| \leq \|(a + b) - a_0\| + \|b_1 - b\| + \|b - b_0\| < \epsilon.$$

Therefore $A \times_d B$ is a Tauberian algebra. \hfill \Box

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