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SOME SPECTRAL PROPERTIES OF THE BANACH ALGEBRA $\mathcal{A} \times_d \mathcal{B}$ WITH THE DIRECT-SUM PRODUCT

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ABSTRACT. Let $\mathcal{A}$ be a commutative Banach algebra and $\mathcal{B}$ be a closed subalgebra of $\mathcal{A}$. Then $\mathcal{A} \times \mathcal{B}$ is a commutative algebra with co-ordinatewise linear operations and the direct-sum product: $(a,b)(c,d) = (ac + ad + bc, bd)$ $(a, c \in \mathcal{A}; b, d \in \mathcal{B})$. In fact, it is a Banach algebra with a suitable norm; it is denoted by $\mathcal{A} \times_d \mathcal{B}$. Here, we study some important spectral properties of this algebra.

1. Introduction

Throughout let $\mathcal{A}$ be a commutative algebra and $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. Then $\mathcal{A} \times \mathcal{B}$ is a commutative algebra with co-ordinatewise linear operations and the *direct-sum product* defined as

$$(a,b)(c,d) = (ac + ad + bc, bd) \quad ((a,b),(c,d) \in \mathcal{A} \times_d \mathcal{B}).$$

This algebra will be denoted by $\mathcal{A} \times_d \mathcal{B}$. Further, if $\mathcal{A}$ is a Banach algebra and $\mathcal{B}$ is a closed subalgebra of $\mathcal{A}$, then $\mathcal{A} \times_d \mathcal{B}$ is a Banach algebra with respect to the norm $\|(a,b)\|_1 = \|a\| + \|b\|$ $((a,b) \in \mathcal{A} \times_d \mathcal{B})$. Some basic properties, uniqueness properties, regularity properties, and the Gel'fand theory of the Banach algebra $\mathcal{A} \times_d \mathcal{B}$ have been studied in [3]. In this paper, we further explore this Banach algebra to study its some spectral properties. These properties are spectral extension property, topological divisor of zero, multiplicative Hahn-Banach property, Quasi divisor of zero, topological annihilator condition, Ditkin's condition, and Tauberian condition.

Let $\sigma_{\mathcal{A}}(a)$ and $r_{\mathcal{A}}(a)$ denote the spectrum and the spectral radius of a in $\mathcal{A}$. Let $\Delta(\mathcal{A})$ denote the set of all non-zero, multiplicative, linear functionals on a commutative Banach algebra $\mathcal{A}$. For $a \in \mathcal{A}$, the map $\widehat{a} : \Delta(\mathcal{A}) \longrightarrow \mathbb{C}$ is defined as $\widehat{a}(\varphi) = \varphi(a)$. The topology on $\Delta(\mathcal{A})$ is the smallest topology such that $\widehat{a}$ is continuous for each $a \in \mathcal{A}$. Let $\varphi \in \Delta(\mathcal{A})$ and S be a non-empty subset $\mathcal{A}$. Define $\varphi_\diamond : \mathcal{A} \times S \longrightarrow \mathbb{C}$ as $\varphi_\diamond((a,x)) := \varphi(x)$. Now let $\mathcal{I}$ be an ideal in $\mathcal{A}$, let $\varphi \in \Delta(\mathcal{I})$, and $u \in \mathcal{I}$ such that $\varphi(u) = 1$. Define $\varphi^+ : \mathcal{A} \times \mathcal{I} \longrightarrow \mathbb{C}$ as $\varphi^+((a,x)) := \varphi(au) + \varphi(x)$. Next, for $F \subset \Delta(\mathcal{A})$, define $F^+ := \{\varphi^+ : \varphi \in F\}$ and $F_\diamond := \{\varphi_\diamond : \varphi \in F\}$. In the case $F = \Delta(\mathcal{A})$, we shall write $\Delta^+(\mathcal{A})$ and $\Delta_\diamond(\mathcal{A})$ for F^+ and $F_\diamond$, respectively. We shall need the following result in proofs.

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Lemma 1.1. [4, Chapter-3] *The Gel'fand space $\Delta(\mathcal{A} \times_d \mathcal{B})$ is homeomorphic to $\Delta^+(\mathcal{A}) \biguplus \Delta_\diamond(\mathcal{B})$ equipped with the sum-topology. Moreover, the Shilov boundary $\partial(\mathcal{A} \times_d \mathcal{B})$ is homeomorphic to $\partial^+(\mathcal{A}) \biguplus \partial_\diamond(\mathcal{B})$ equipped with the sum-topology.*

2. Spectral Properties in Commutative Banach Algebras

Definition 2.1. [7] A norm $|\cdot|$ on $\mathcal{A}$ is a *spectral norm* if $r_{\mathcal{A}}(a) \leq |a|$ $(a \in \mathcal{A})$. The Banach algebra $\mathcal{A}$ has *spectral extension property (SEP)* if every norm on $\mathcal{A}$ is a spectral norm.

Theorem 2.2. *If $\mathcal{A} \times_d \mathcal{B}$ has SEP, then $\mathcal{A}$ and $\mathcal{B}$ have SEP.*

Proof. Let $|\cdot|$ be a norm on $\mathcal{A}$. Define $|(a,b)|_1 = |a| + |b|$. Then $|\cdot|_1$ is an algebra norm on $\mathcal{A} \times_d \mathcal{B}$. Since $\mathcal{A} \times_d \mathcal{B}$ has SEP, we have

$$r_{\mathcal{A}}(a) = r_{\mathcal{A} \times_d \mathcal{B}}(a,0) \leq |(a,0)|_1 = |a| \quad (a \in \mathcal{A}).$$

Thus $|\cdot|$ is a spectral norm on $\mathcal{A}$, and so $\mathcal{A}$ has SEP. Next suppose that $|\cdot|$ is a norm on $\mathcal{B}$. Define $|(a,b)| = \|a+b\| + |b|$ on $\mathcal{A} \times_d \mathcal{B}$, where $\|\cdot\|$ is the Banach algebra norm on $\mathcal{A}$. Then, by [4, Lemma 3.2.2], $|\cdot|$ is an algebra norm on $\mathcal{A} \times_d \mathcal{B}$. Since $\mathcal{A} \times_d \mathcal{B}$ has SEP,

$$r_{\mathcal{B}}(b) = r_{\mathcal{A} \times_d \mathcal{B}}(-b,b) \leq |(-b,b)| = |b| \quad (b \in \mathcal{B}).$$

Hence, $|\cdot|$ is a spectral norm on $\mathcal{B}$. Therefore, $\mathcal{B}$ has SEP. $\square$

Definition 2.3. [4, Definition 1.4.18] A non-zero element $a \in \mathcal{A}$ is a *topological divisor of zero (TDZ)* if there is a sequence (a_n) in $\mathcal{A}$ such that $\|a_n\| = 1$ $(n \in \mathbb{N})$ and either $a_n a \longrightarrow 0$ as $n \longrightarrow \infty$. The Banach algebra $\mathcal{A}$ has *topological divisor of zero (TDZ) property* if every element of $\mathcal{A}$ is a topological divisor of zero.

Theorem 2.4. *If $\mathcal{A}$ and $\mathcal{B}$ have TDZ property, then $\mathcal{A} \times_d \mathcal{B}$ has TDZ property.*

Proof. Suppose that $\mathcal{A}$ and $\mathcal{B}$ have TDZ property. Let $(a,b) \in \mathcal{A} \times_d \mathcal{B}$. Then $a + b \in \mathcal{A}$. Suppose that $a + b \neq 0$. Since $\mathcal{A}$ has TDZ property, there exists a sequence $(a_n) \subset \mathcal{A}$ such that $\|a_n\| = 1$ $(n \in \mathbb{N})$ and $a_n(a+b) \longrightarrow 0$ as $n \longrightarrow \infty$. Then $((a_n, 0))$ is a sequence in $\mathcal{A} \times_d \mathcal{B}$ such that $\|(a_n,0)\|_1 = \|a_n\| = 1$ $(n \in \mathbb{N})$ and $(a_n,0)(a,b) = (a_n a + a_n b, 0) \longrightarrow (0,0)$ as $n \longrightarrow \infty$. Therefore, (a,b) is a TDZ. If $a + b = 0$, then $a = -b \neq 0$. Since $\mathcal{B}$ has TDZ property, there exists a sequence (b_n) in $\mathcal{B}$ such that $\|b_n\| = 1$ and $b_n b \longrightarrow 0$ as $n \longrightarrow \infty$. In this case, $((0,b_n))$ is a sequence in $\mathcal{A} \times_d \mathcal{B}$ such that $\|(0,b_n)\|_1 = \|b_n\| = 1$ and $(0,b_n)(a,b) = (-b_n b, b_n b) \longrightarrow (0,0)$ as $n \longrightarrow \infty$. Thus, in all cases, (a,b) is a TDZ in $\mathcal{A} \times_d \mathcal{B}$. Hence $\mathcal{A} \times_d \mathcal{B}$ has TDZ property. $\square$

Definition 2.5. [7] A commutative Banach algebra $\mathcal{A}$ has *Multiplicative Hahn-Banach Property (MHBP)* if, for every commutative extension $\mathcal{B}$ of $\mathcal{A}$, every $\varphi \in \Delta(\mathcal{A})$ can be extended to some element of $\Delta(\mathcal{B})$.

Theorem 2.6. *$\mathcal{A} \times_d \mathcal{B}$ has MHBP if and only if both $\mathcal{A}$ and $\mathcal{B}$ have MHBP.*

Proof. Let $\mathcal{A} \times_d \mathcal{B}$ have MHBP. Let $\mathcal{C}$ be a commutative extension of $\mathcal{A}$, then $\mathcal{C} \times_d \mathcal{B}$ is a commutative extension of $\mathcal{A} \times_d \mathcal{B}$. Let $\varphi \in \Delta(\mathcal{A})$. Then $\varphi^+ \in \Delta^+(\mathcal{A}) \uplus \Delta_\diamond(\mathcal{B})$. Since $\mathcal{A} \times_d \mathcal{B}$ has MHBP, there exists $\widetilde{\eta} \in \Delta(\mathcal{C} \times_d \mathcal{B}) = \Delta^+(\mathcal{C}) \uplus \Delta_\diamond(\mathcal{B})$ such that $\widetilde{\eta} = \varphi^+$ on $\mathcal{A} \times_d \mathcal{B}$. Now, if $\widetilde{\eta} \in \Delta_\diamond(\mathcal{B})$, then we get $\varphi(a) = \varphi^+((a,0)) = \widetilde{\eta}((a,0)) = 0$ on $\mathcal{A}$. This is not possible. Hence, $\widetilde{\eta}$ must be in $\Delta^+(\mathcal{C})$. Therefore, there exists $\widetilde{\varphi} \in \Delta(\mathcal{C})$ such that $\widetilde{\eta} = \widetilde{\varphi}^+$ on $\mathcal{C} \times_d \mathcal{B}$. Also, $\widetilde{\eta} = (\widetilde{\varphi})^+ = \varphi^+$ on $\mathcal{A} \times_d \mathcal{B}$, implies

$\widetilde{\varphi} = \varphi$ on $\mathcal{A}$. Thus $\widetilde{\varphi}$ is an extension of φ. Hence $\mathcal{A}$ has MHBP. Similarly, it can be proved that $\mathcal{B}$ has MHBP.

Conversely, assume that $\mathcal{A}$ and $\mathcal{B}$ have MHBP. Let $\mathcal{C}$ be any extension of $\mathcal{A} \times_d \mathcal{B}$. Then $\mathcal{C}$ is an extension of both $\mathcal{A} \times_d \{0\}$ and $\{0\} \times_d \mathcal{B}$. Let $\widetilde{\eta} \in \Delta^+(\mathcal{A}) \uplus \Delta_\diamond(\mathcal{B})$. Then either $\widetilde{\eta} \in \Delta^+(\mathcal{A})$ or $\widetilde{\eta} \in \Delta_\diamond(\mathcal{B})$. Suppose that $\widetilde{\eta} \in \Delta^+(\mathcal{A})$. Then $\widetilde{\eta} = \varphi^+$ for some $\varphi \in \Delta(\mathcal{A})$. Since $\mathcal{C}$ is an extension of $\mathcal{A}$, by the hypothesis, φ can be extended to some element $\widetilde{\varphi}$ of $\Delta(\mathcal{C})$. Then $\widetilde{\eta} = \widetilde{\varphi}^+ \in \Delta(\mathcal{C})$ and $\widetilde{\eta} = \widetilde{\varphi}^+ = \varphi^+$ on $\mathcal{A} \times_d \mathcal{B}$. Similarly, if $\widetilde{\eta} \in \Delta_\diamond(\mathcal{B})$, then also it can be extended to some element of $\Delta(\mathcal{C})$. Thus $\mathcal{A} \times_d \mathcal{B}$ has MHBP. □

Definition 2.7. [6] A commutative Banach algebra $\mathcal{A}$ has *Quasi Divisor of Zero (QDZ) property* if there exists an open subset G of $\Delta(\mathcal{A})$ such that

(1) $\partial(\mathcal{A}) \subset \overline{G}$;
(2) For every open subset U of G, there exist $a \in \mathcal{A}$ and a non-empty, open set $V \subset U$ such that

$$\widehat{a}(\varphi) = \begin{cases} 0 & \text{if } \varphi \in U^c \\ 1 & \text{if } \varphi \in V. \end{cases}$$

Theorem 2.8. *$\mathcal{A} \times_d \mathcal{B}$ has QDZ property iff both $\mathcal{A}$ and $\mathcal{B}$ have QDZ property.*

Proof. Let $\mathcal{A} \times_d \mathcal{B}$ has QDZ property. Then there exists an open set $\widetilde{G} \subset \Delta(\mathcal{A} \times_d \mathcal{B})$ which satisfies the following properties.

(1) $\partial^+(\mathcal{A}) \uplus \partial_\diamond(\mathcal{B}) \subset \overline{(\widetilde{G})}$.
(2) For every open subset $\widetilde{U}$ of $\widetilde{G}$, there exists $(a, b) \in \mathcal{A} \times_d \mathcal{B}$ and a non-empty open subset $\widetilde{V}$ of $\widetilde{U}$ such that

$$(a, b)\widehat{}(\varphi) = \begin{cases} 0 & (\varphi \in \widetilde{U}^c); \\ 1 & (\varphi \in \widetilde{V}). \end{cases}$$

Let $G_\mathcal{A} = \{\varphi \in \Delta(\mathcal{A}) : \varphi^+ \in \widetilde{G}\}$ and $G_\mathcal{B} = \{\varphi \in \Delta(\mathcal{B}) : \varphi_\diamond \in \widetilde{G}\}$. Then $G_\mathcal{A}$ and $G_\mathcal{B}$ are open sets in $\Delta(\mathcal{A})$ and $\Delta(\mathcal{B})$, respectively as such that $G_\mathcal{A}^+ \cup G_{\mathcal{B}\diamond} = \widetilde{G}$. Also, from (1) above, we get $\partial\mathcal{A} \subset \overline{G_\mathcal{A}}$ and $\partial\mathcal{B} \subset \overline{G_\mathcal{B}}$. Now, let $U \subset G_\mathcal{A}$ be open. Then U^+ will be open in $\widetilde{G}$. Hence, by (2) above, there exist $(a, b) \in \mathcal{A} \times_d \mathcal{B}$ and a non-empty open set $V^+ \subset U^+$ such that $(a, b)^\wedge = 0$ on $(U^+)^c$ and $(a, b)^\wedge = 1$ on V^+. Now, if $\varphi \in U^c$, then $\varphi^+ \in (U^+)^c$ and $(a + b)^\wedge(\varphi) = \varphi(a + b) = \varphi^+((a, b)) = 0$ on U^c. If $\varphi \in V$, then $\varphi^+ \in V^+$ and $(a + b)^\wedge(\varphi) = \varphi^+((a, b)) = 1$. Hence $\mathcal{A}$ has QDZ property. By similar arguments, it follows that $\mathcal{B}$ has QDZ property.

Conversely, suppose $\mathcal{A}$ and $\mathcal{B}$ have QDZ property. Then there exist open subsets $G_\mathcal{A} \subset \Delta(\mathcal{A})$ and $G_\mathcal{B} \subset \Delta(\mathcal{B})$ satisfying the properties in the definition of QDZ. Then $\widetilde{G} = G_\mathcal{A}^+ \cup G_{\mathcal{B}\diamond}$ and

$$\partial(\mathcal{A} \times_d \mathcal{B}) = \partial^+(\mathcal{A}) \uplus \partial_\diamond(\mathcal{B}) \subset \overline{G_\mathcal{A}^+} \cup \overline{G_{\mathcal{B}\diamond}} = \overline{G_\mathcal{A}^+ \cup G_{\mathcal{B}\diamond}} = \overline{\widetilde{G}}.$$

Let $\widetilde{U} \subset \widetilde{G}$ be open. Then the corresponding sets $U_\mathcal{A}$ and $U_\mathcal{B}$ are open in $G_\mathcal{A}$ and $G_\mathcal{B}$, respectively. Hence, there exist $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $\widehat{a} = 0$ outside $U_\mathcal{A}, \widehat{a} = 1$ on some non-empty open subset $V_\mathcal{A}$ of $U_\mathcal{A}$, $\widehat{b} = 0$ outside $U_\mathcal{B}$ and $\widehat{b} = 1$ on some non-empty open subset $V_\mathcal{B}$ of $U_\mathcal{B}$. Then $(a - b, b)^\wedge = 0$ on $\widetilde{U}^c = (U_\mathcal{A}^+)^c \cup (U_{\mathcal{B}\diamond})^c$ and $(a - b, b)^\wedge = 1$ on $\widetilde{V} = V_\mathcal{A}^+ \cup V_{\mathcal{B}\diamond} \subset \widetilde{U}$. Hence $\mathcal{A} \times_d \mathcal{B}$ has QDZ property. □

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Definition 2.9. [6] Let $\mathcal{I}$ be an ideal of a commutative semisimple Banach algebra $\mathcal{A}$. A *separating net* for $\mathcal{I}$ is a net $(q_\lambda)_{\lambda\in\Lambda}$ of quasi divisors of zero in $\mathcal{A}$ such that

(1) $\sup\{r_{\mathcal{A}}(q_\lambda) : \lambda \in \Lambda\} < \infty$;
(2) $\lim\limits_{\lambda\to\infty} r_{\mathcal{A}}(aq_\lambda) = 0 \quad (a \in \mathcal{I})$;
(3) There exists an element $b \in \mathcal{A}$ such that $q_\lambda b = q_\lambda$ $(\lambda \in \Lambda)$.

Definition 2.10. [6] A commutative Banach algebra $\mathcal{A}$ satisfies *Topological Annihilator (TAN) condition* if there exists a dense set $D \subset \partial\mathcal{A}$ such that, for every $\varphi \in D$, the kerφ admits a separating net.

Theorem 2.11. *$\mathcal{A} \times_d \mathcal{B}$ has TAN property iff both $\mathcal{A}$ and $\mathcal{B}$ have TAN property.*

Proof. Let $\mathcal{A}\times_d\mathcal{B}$ has TAN property. Then there exists dense subset $\widetilde{D}$ of $\partial(\mathcal{A}\times_d\mathcal{B})$ such that $\ker\widetilde{\eta}$ $(\widetilde{\eta}\in\widetilde{D})$ admits a separating net. Let $D_{\mathcal{A}} = \{\varphi \in \Delta(\mathcal{A}) : \varphi^+ \in \widetilde{D}\}$. Then $D_{\mathcal{A}}$ is a dense subset of $\partial\mathcal{A}$. Let $\varphi \in D_{\mathcal{A}}$. Then $\varphi^+ \in \widetilde{D}$. Hence $\ker\varphi^+$ admits a separating net say $((a_\lambda, b_\lambda))_{\lambda\in\Lambda}$. Then $(a_\lambda + b_\lambda)_{\lambda\in\Lambda}$ is a separating net for $\ker\varphi$. Thus $\mathcal{A}$ has TAN property. By similar arguments it follows that $\mathcal{B}$ has TAN property.

Conversely, assume that $\mathcal{A}$ and $\mathcal{B}$ have TAN property. Then there exist dense subsets $D_{\mathcal{A}} \subset \partial\mathcal{A}$ and $D_{\mathcal{B}} \subset \partial\mathcal{B}$ such that $\ker\varphi$ $(\varphi \in D_{\mathcal{A}} \cup D_{\mathcal{B}})$ admits a separating net. Let $\widetilde{D} = D^+_{\mathcal{A}} \cup D_{\mathcal{B}\diamond}$. Then $\widetilde{D}$ is a dense subset of $\partial^+(\mathcal{A}) \cup \partial_\diamond(\mathcal{B})$. Let $\widetilde{\eta} \in \widetilde{D}$. Then either $\widetilde{\eta} = \varphi^+$ for some $\varphi \in D_{\mathcal{A}}$ or $\widetilde{\eta} = \psi_\diamond$ for some $\psi \in D_{\mathcal{B}}$. If $\widetilde{\eta} = \varphi^+$, then $\ker\varphi$ admits a separating net $(a_\lambda)_{\lambda\in\Lambda}$. Hence $((a_\lambda, 0))_{\lambda\in\Lambda}$ is a separating net for $\ker\widetilde{\eta}$. Similarly, if $\widetilde{\eta} = \psi_\diamond$, then $\ker\psi$ admits a separating net $(b_\lambda)_{\lambda\in\Lambda}$. In this case, $((-b_\lambda, b_\lambda))_{\lambda\in\Lambda}$ is a separating net for $\ker\widetilde{\eta}$. Hence $\mathcal{A} \times_d \mathcal{B}$ has TAN property. □

Lemma 2.12. *Let $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Then $(a, b)^\wedge \in C_c(\Delta(\mathcal{A} \times_d \mathcal{B}))$ if and only if $(a + b)^\wedge \in C_c(\Delta(\mathcal{A}))$ and $\widehat{b} \in C_c(\Delta(\mathcal{B}))$.*

Proof. This follows from the definition of the support. □

Definition 2.13. [1, Definition 4.1.31] Let $\mathcal{A}$ be a commutative Banach algebra. Then $\mathcal{A}$ satisfies

(1) *Ditkin's condition at* $\varphi \in \Delta(\mathcal{A})$ if for every $a \in \ker(\varphi)$, there exists a sequence (a_n) in $\mathcal{A}$ such that $\widehat{a_n} \in C_c(\Delta(\mathcal{A}))$, $\varphi \notin supp\widehat{a_n}$ and $a_n a \longrightarrow a$ as $n \longrightarrow \infty$.
(2) *Ditkin's condition at infinity* if for $a \in \mathcal{A}$, there exists a sequence (a_n) in $\mathcal{A}$ such that $\widehat{a_n} \in C_c(\Delta(\mathcal{A}))$ and $a_n a \longrightarrow a$ as $n \longrightarrow \infty$.
(3) *Ditkin's condition* if it satisfies Ditkin's condition at every $\varphi \in \Delta(\mathcal{A})$ and at infinity.

Theorem 2.14. *$\mathcal{A}\times_d\mathcal{B}$ satisfies Ditkin's condition iff both $\mathcal{A}$ and $\mathcal{B}$ satisfy Ditkin's condition.*

Proof. Let $\mathcal{A} \times_d \mathcal{B}$ satisfies Ditkin's condition. Let $\varphi \in \Delta(\mathcal{A})$ and $a \in \ker\varphi$. Then $(a, 0) \in \ker\varphi^+$. Since $\mathcal{A} \times_d \mathcal{B}$ satisfies Ditkin's condition, there exists a sequence $((a_n, b_n))$ in $\mathcal{A} \times_d \mathcal{B}$ such that $(a_n, b_n)^\wedge \in C_c(\Delta(\mathcal{A} \times_d \mathcal{B}))$ $(n \in \mathbb{N})$, $\varphi^+ \notin supp(a_n, b_n)^\wedge$ and $(a_n, b_n)(a, 0) \longrightarrow (a, 0)$ as $n \longrightarrow \infty$. Then $(a_n + b_n)$ is a sequence in $\mathcal{A}$ such that $(a_n + b_n)^\wedge \in C_c(\Delta(\mathcal{A}))$ $(n \in \mathbb{N})$, due to Lemma 2.12, $\varphi \notin supp(a_n + b_n)^\wedge$ and $(a_n + b_n)a \longrightarrow a$ as $n \longrightarrow \infty$. Thus $\mathcal{A}$ satisfies Ditkin's condition at every $\varphi \in \Delta(\mathcal{A})$. By similar arguments, it follows that $\mathcal{B}$ satisfies Ditkin's condition at every $\psi \in \Delta(\mathcal{B})$.

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Next we show that $\mathcal{A}$ and $\mathcal{B}$ satisfy Ditkin's condition at infinity. Let $a \in \mathcal{A}$. Since $\mathcal{A} \times_d \mathcal{B}$ satisfies Ditkin's condition at infinity, there exists a sequence $((a_n, b_n))$ in $\mathcal{A} \times_d \mathcal{B}$ such that $(a_n, b_n)^\wedge \in C_c(\Delta(\mathcal{A} \times_d \mathcal{B}))$ and $(a_n, b_n)(a, 0)$ converges to $(a, 0)$ as $n \longrightarrow \infty$. Then $(a_n + b_n)$ is a sequence in $\mathcal{A}$ such that $(a_n + b_n)^\wedge \in C_c(\Delta(\mathcal{A}))$ due to Lemma 2.12 and $(a_n + b_n)a \longrightarrow a$ as $n \longrightarrow \infty$. Therefore, $\mathcal{A}$ satisfies Ditkin's condition at infinity. By Similar arguments, it follows that $\mathcal{B}$ satisfies Ditkin's condition at infinity.

Conversely, assume that both $\mathcal{A}$ and $\mathcal{B}$ satisfy Ditkin's condition. Let $\varphi^+ \in \Delta^+(\mathcal{A})$ and $(a, b) \in \ker \varphi^+$. Then $a+b \in \ker(\varphi)$. Since $\mathcal{A}$ satisfies Ditkin's condition at φ, there exists a sequence (a_n) in $\mathcal{A}$ such that $(\widehat{a_n}) \subset C_c(\Delta(\mathcal{A}))$, $\varphi \notin supp\widehat{a_n}$ and $a_n(a + b) \longrightarrow a + b$ as $n \longrightarrow \infty$. Since $\mathcal{B}$ satisfy ditkin's condition at infinity, there exists a sequence (b_n) in $\mathcal{B}$ such that $(\widehat{b_n}) \in C_c(\Delta(\mathcal{B}))$ and $b_n b \longrightarrow b$ as $n \longrightarrow \infty$. Then $((a_n - b_n, b_n))$ is a sequence in $\mathcal{A} \times_d \mathcal{B}$ such that $(a_n - b_n, b_n)^\wedge \in C_c(\Delta(\mathcal{A} \times_d \mathcal{B}))$ due to Lemma 2.12, $\varphi^+ \notin supp(a_n - b_n, b_n)^\wedge$ and $(a_n - b_n, b_n)(a, b) \longrightarrow (a, b)$ as $n \longrightarrow \infty$. Hence, $\mathcal{A} \times_d \mathcal{B}$ satisfies Ditkin's condition at every $\varphi^+ \in \Delta^+(\mathcal{A})$. Similarly, it follows that $\mathcal{A} \times_d \mathcal{B}$ satisfies Ditkin's condition at every $\psi_\diamond \in \Delta_\diamond(\mathcal{B})$.

Next we show that $\mathcal{A} \times_d \mathcal{B}$ satisfies Ditkin's condition at infinity. Fix an arbitrary element $(a, b) \in \mathcal{A} \times_d \mathcal{B}$. Since both $\mathcal{A}$ and $\mathcal{B}$ satisfy Ditkin's condition at infinity, there exist sequences (a_n) in $\mathcal{A}$ and $(b_n) \in \mathcal{B}$ such that $(\widehat{a_n}) \in C_c(\Delta(\mathcal{A}))$, $(\widehat{b_n}) \in C_c(\Delta(\mathcal{B}))$, $(a+b)a_n \longrightarrow a+b$ and $b_n b \longrightarrow b$. Therefore $(a_n - b_n, b_n)^\wedge \in C_c(\Delta(\mathcal{A} \times \mathcal{B})$ and $(a_n - b_n, b_n)(a, b) \longrightarrow (a, b)$. Hence $\mathcal{A} \times_d \mathcal{B}$ satisfies Ditkin's condition at infinity. □

Definition 2.15. [5, Definition 8.1.2] A commutative Banach algebra $\mathcal{A}$ is said to be *a Tauberian algebra* if the set $\{a \in \mathcal{A} : \widehat{a} \in C_c(\Delta(\mathcal{A}))\}$ is dense in $\mathcal{A}$.

Theorem 2.16. *$\mathcal{A} \times_d \mathcal{B}$ is Tauberian iff both $\mathcal{A}$ and $\mathcal{B}$ are Tauberian.*

Proof. Let $\mathcal{A} \times_d \mathcal{B}$ be a Tauberian algebra. Let $a \in \mathcal{A}$ and $\epsilon > 0$. Since $\mathcal{A} \times_d \mathcal{B}$ is Tauberian, there exits $(a_0, b_0) \in \mathcal{A} \times_d \mathcal{B}$ such that $(a_0, b_0)^\wedge \in C_c(\Delta(\mathcal{A} \times_d \mathcal{B}))$ and $\|(a, 0) - (a_0, b_0)\|_1 = \|a - a_0\| + \|b_0\| < \epsilon$. Then $(a_0 + b_0)^\wedge \in C_c(\Delta(\mathcal{A}))$ and $\|(a_0 + b_0) - a\| \leq \|a - a_0\| + \|b_0\| < \epsilon$. Therefore, $\mathcal{A}$ is Tauberian. Now, let $b \in \mathcal{B}$ and $\epsilon > 0$. Since $\mathcal{A} \times_d \mathcal{B}$ is Tauberian, there exits $(a_1, b_1) \in \mathcal{A} \times_d \mathcal{B}$ such that $(a_1, b_1)^\wedge \in C_c(\Delta(\mathcal{A} \times_d \mathcal{B}))$ and $\|(0, b) - (a_1, b_1)\|_1 = \|a_1\| + \|b - b_1\| < \epsilon$. Since $(a_1, b_1)^\wedge \in C_c(\Delta(\mathcal{A} \times_d \mathcal{B}))$, by Lemma 2.12, $\widehat{b_1} \in C_c(\Delta(\mathcal{B}))$ and $\|b - b_1\| < \epsilon$. Therefore, $\mathcal{B}$ is Tauberian.

Conversely, suppose that $\mathcal{A}$ and $\mathcal{B}$ are Tauberian algebra. Let $(a, b) \in \mathcal{A} \times_d \mathcal{B}$ and $\epsilon > 0$. Since $\mathcal{A}$ and $\mathcal{B}$ are Tauberian, there exist $a_0 \in \mathcal{A}$ and $b_0 \in \mathcal{B}$ such that $\widehat{a_0} \in C_c(\Delta(\mathcal{A})), \widehat{b_0} \in C_c(\Delta(\mathcal{B})), \|(a + b) - a_0\| < \epsilon/3$ and $\|b - b_0\| < \epsilon/3$. Therefore, by Lemma 2.12, $(a_0 - b_0, b_0)^\wedge \in C_c(\Delta(\mathcal{A} \times_d \mathcal{B}))$. Also

$$\begin{aligned} \|(a, b) - (a_0 - b_0, b_0)\|_1 &= \|a - a_0 + b_0\| + \|b - b_0\| \\ &\leq \|(a + b) - a_0\| + \|b_0 - b\| + \|b - b_0\| \ < \ \epsilon. \end{aligned}$$

Therefore $\mathcal{A} \times_d \mathcal{B}$ is a Tauberian algebra. □

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